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THE ADIC REALIZATION OF THE MORSE TRANSFORMATION AND THE EXTENSION OF ITS ACTION TO THE SOLENOID

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We consider the adic realization of the Morse transformation on the additive group of integer dyadic numbers. We discuss the arithmetic properties of this action. Then we extend this action to an action of the group of rational dyadic numbers on the solenoid. Bibliography: 14 *titles.*

To the memory of Alexander Livshits

Sasha Livshits (1950–2008) was the author of one of the most important theorems of modern dynamics, which is well-known now, — the theorem about the cohomology of hyperbolic systems. He proved this theorem when he was a student. Later he worked on many other problems of symbolic dynamics, ergodic theory, and combinatorics. His deep and important ideas made a great impression on those who interacted with him (this includes the second author). The first author considers him the best of his students.

1. Introduction

The Morse dynamical system was discovered by Morse and popularized by Hedlund and Gottshalk. Later it was studied by many authors (see [12, 11] and references therein) as a simplest nontrivial substitution. Moreover, it was historically the first example of a substitution. It is generated by the Thue–Morse sequence, which was extensively studied from the point of view of the combinatorics of words and symbolic complexity (see [5. 10], and references therein). The new approach to symbolic dynamics and ergodic transformations (based on the notion of adic transformation), which was suggested by the first author [2], can also be applied to substitutions (so-called stationary adic transformations). This idea was realized in the paper by A. Livshits and the first author [13]. Later, other authors developed this connection in the context of topological dynamics (see [8, 7]), but here our focus is on measure-preserving transformations. The adic realization of a substitution dynamical system allows one to consider simultaneously not only the substitution itself, but also the one-sided shift which accompanies any substitution. The idea that was advocated by the first author in [3] is to consider the natural extension of that shift and correspondingly extend the substitution system in order to make an essential link between the theory of substitutions and hyperbolic dynamics. In this paper we consider the "two-sided extension" of the Morse system which yields *the Morse action of the countable group* Q_2 (the group of dyadic rational numbers) on the group of its characters — the solenoid $\widehat{Q}_2$, reworking more carefully and correcting some details of [13]. We obtain also some new properties of the adic realization of the Morse transformation. One of the corollaries of the adic approach is an explicit calculation that shows how to obtain the Morse system as a time change of the dyadic odometer. The operation of differentiation of dyadic sequences plays an important role in our constructions. The spectral theory of the Morse system, which goes back to Kakutani [9] (see also [12, 11] and references therein) is also becoming more transparent under these considerations, but we do not address it in this paper.

In Sec. 2, we collect a series of well-known and new results on the Morse system using its adic realization. In particular, we discuss in more detail (than in [3]) the so-called "Morse arithmetic." Section 3 describes the two-sided extension of the Morse transformation and its embedding into the Morse action of the group Q_2 on the group $\widehat{Q}_2$. We also formulate some open problems.

One should consider this article as an attempt, looking at the special case of the Morse transformation, to attack the general problem of defining a two-sided extension of a substitution system, and a corresponding embedding of this system into an action of a larger group. The final goal of the constructions is to show the link between the theory of substitutions and of hyperbolic systems.

2. Definitions and the adic realization of the Morse system

2.1. The Morse transformation as a substitution system. Consider the alphabet $\{0,1\}$. The Morse substitution is defined by $\zeta(0) = 01$, $\zeta(1) = 10$; it is extended to all words in the alphabet $\{0,1\}$ by concatenation.

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The Thue–Morse sequence (sometimes also called the Prouhet–Thue–Morse sequence) is a fixed point of this substitution:

$$u = u_0 u_1 u_2 \ldots = \lim_{n\to\infty} \zeta^n(0) = 0110100110010110\ldots. \tag{1}$$

This sequence has many remarkable features (see, e.g., [5] and [11, Chaps. 2 and 5]). It is easy to see that

$$u[0, 2^{n+1}-1] = u[0, 2^n - 1]\,\overline{u[0, 2^n-1]} \quad \text{for } n \ge 0,$$

where we denote $u[i,j] = u_i \ldots u_j$ and $\overline{w}$ is the "flip" of a word w in the alphabet $\{0,1\}$, that is, the word obtained from w by interchanging $0 \leftrightarrow 1$. The sequence u is nonperiodic, but uniformly recurrent, with well-defined uniform frequencies of subwords. It is also known that u_n is the sum of the digits (mod 2) in the binary representation of n.

Let σ be the left shift on the spaces $\{0,1\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{Z}}$ with the product topology. The substitution dynamical system is sometimes considered on the space of one-sided sequences, and sometimes on the space of two-sided sequences.[1]

The "one-sided" substitution space is defined as the orbit closure of u under the shift:

$$X_\zeta^+ = \operatorname{clos}\{\sigma^n u : n \ge 0\}.$$

The "two-sided" substitution space is defined as the set X_ζ of all bi-infinite sequences in $\{0,1\}^{\mathbb{Z}}$ whose every block (subword) occurs in u. The substitution dynamical systems are (X_ζ^+, σ) (one-sided) and (X_ζ, σ) (two-sided). The advantage of the two-sided system is that it is a homeomorphism, whereas the one-sided system is not. Measure-theoretically, these two systems are isomorphic: in fact, both are minimal and uniquely ergodic, and the one-sided system is a.e. invertible. The sequence consisting of the nonnegative coordinates of any point in X_ζ lies in X_ζ^+, and all but countably many elements of X_ζ^+ have a unique extension to elements in X_ζ. The exceptions are u, which extends to $u\,.\,u$ and $\overline{u}\,.\,u$, and its "flip" $\overline{u}$, as well as their orbits.

2.2. The adic realization. Here we follow the general definition of the adic transformation from [2] and [13], but focus only on the Morse system, as it was done in [3]. Consider $\mathbf{Z}_2 \cong \{0,1\}^{\mathbb{N}}$, the compact additive group of 2-adic integers, and the odometer ("adding machine") transformation T, which is an adic transformation by definition — this is the group translation on $\mathbf{Z}_2$ (see below). We obtain the adic realization of the Morse transformation by *changing the order of symbols* $0,1$ *depending on the next symbol.* Namely, consider the lexicographic order on $\mathbf{Z}_2$ induced by the relation

$$0 \prec_0 1, \ \ 1 \prec_1 0$$

as follows:

$$\{x_i\} \prec \{y_i\} \iff \exists\, j : \ x_i = y_i \text{ for } i > j \text{ and } x_j \prec_z y_j, \text{ where } z = x_{j+1} = y_{j+1}.$$

This is a partial order; two sequences are comparable if they are cofinal (i.e., agree except in finitely many places). The set of maximal points is $\mathtt{Max} = \{(01)^\infty, (10)^\infty\}$, and the set of minimal points is $\mathtt{Min} = \{(0)^\infty, (1)^\infty\}$.[2] Let M be the immediate successor transformation in the order $\prec$ on $\mathbf{Z}_2$. Here we write down the formulas for the action of M explicitly. If $x \notin \mathtt{Max}$, then x starts with $(01)^n 00$, or $(01)^n 1$, or $(10)^n 0$, or $(10)^n 11$, where $n \ge 0$. We have

$$\begin{aligned} &M((01)^n 00*) = (1^{2n+1}0*), \qquad && M((01)^n 1*) = (0^{2n}1*), \\ &M((10)^n 0*) = (1^{2n}0*), \qquad && M((10)^n 11*) = (0^{2n+1}1*). \end{aligned} \tag{2}$$

Note that M is well defined everywhere except on the two maximal points, i.e., the elements of the set $\mathtt{Max} = \{(01)^\infty, (10)^\infty\}$. It is easy to see that M is continuous on $\mathbf{Z}_2 \setminus \mathtt{Max}$. But one cannot extend M to these points by continuity: there are no well-defined limits $\lim_{n\to\infty} M((01)^n*)$ and $\lim_{n\to\infty} M((10)^n*)$, because

$$\lim_{n\to\infty} M((01)^n 00*) = (1)^\infty,$$

[1] Above we have used the terms "one-sided" and "two-sided" in a completely different sense; see also below.

[2] We denote the infinite periodic sequence with period $(ab\ldots c)$ by $(ab\ldots c)^\infty$.

but

$$\lim_{n\to\infty} M((01)^n 1*) = (0)^\infty.$$

Analogously,

$$\lim_{n\to\infty} M((10)^n 0*) = (1)^\infty,$$

but

$$\lim_{n\to\infty} M((10)^n 11*) = (0)^\infty.$$

Since we also have two minimal points, we can extend M to a bijection arbitrarily, setting

$$M((01)^\infty) = (1)^\infty, \quad M((10)^\infty) = (0)^\infty \tag{3}$$

or vice versa. But this extension is not continuous at these points.

The obvious corollary of the definition of M is that it commutes with "flips," that is,

$$M(\overline{x}) = \overline{M(x)} \text{ for any } x \in \mathbf{Z}_2. \tag{4}$$

The action of M on $\mathbf{Z}_2 \setminus \mathtt{Max}$ may be expressed as follows: we scan the sequence x from left to right until we see two identical symbols aa, and replace the beginning of the sequence by $\overline{a}\dots\overline{a}a$, keeping the second occurrence of a and everything that follows unchanged.

2.3. The relation of the adic model to the traditional representation. Now we indicate the relation between the dynamical systems $(\mathbf{Z}_2, M)$ and (X_ζ, σ). Let

$$g:\ \mathbf{Z}_2 \to X_\zeta, \qquad g(x) = \{(M^{n-1}x)_0\}_{n\in\mathbb{Z}}.$$

We have the following diagram:

$$\begin{array}{ccc} \mathbf{Z}_2 & \xrightarrow{M} & \mathbf{Z}_2 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ X_\zeta & \xrightarrow{\sigma} & X_\zeta. \end{array}$$

It is obvious from the definition that the diagram commutes. It is also easy to see that g is surjective, continuous on $\mathbf{Z}_2 \setminus \widetilde{\mathtt{Max}}$, and $g(0^\infty) = \overline{u}\,.\,u$. Here we denoted by $\widetilde{\mathtt{Max}}$ the set of points in $\mathbf{Z}_2$ that are cofinal with the points in $\mathtt{Max}$ (or, equivalently, the left semiorbits of both points from $\widetilde{\mathtt{Max}}$).

It may be useful to write down g^{-1} explicitly. Consider the substitution map on X_ζ:

$$\zeta:\ X_\zeta \to X_\zeta, \quad \zeta(\dots a_{-2}a_{-1}\,.\,a_0a_1\dots) = \dots\zeta(a_{-2})\zeta(a_{-1})\,.\,\zeta(a_0)\zeta(a_1)\dots.$$

It is well known (and easy to see) that for every $\mathbf{a} \in X_\zeta$ there is a unique $\mathbf{a}' \in X_\zeta$ such that either $\mathbf{a} = \zeta(\mathbf{a}')$ or $\mathbf{a} = \sigma\zeta(\mathbf{a}')$, and these cases are mutually exclusive. Let $\Psi:\ X_\zeta \to X_\zeta$ be given by $\Psi(\mathbf{a}) = \mathbf{a}'$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Z}_2 & \xrightarrow{\sigma} & \mathbf{Z}_2 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ X_\zeta & \xrightarrow{\Psi} & X_\zeta, \end{array}$$

where σ is the left shift on $\mathbf{Z}_2$. Therefore, to compute the nth symbol of $g^{-1}(\mathbf{a})$, we need to take $(\Psi^n(\mathbf{a}))_0$ for $n = 0, 1, 2\dots$.

Now let us explain why this model is richer than the "two-sided" model X_ζ. In the adic realization, we have the adic transformation, which is isomorphic, up to neglecting two orbits, to the substitution, AND we have the one-sided shift in the space $\mathbf{Z}_2$. The evolution under the adic transformation of the first digit x_0 of a sequence $\{x_i\} \in \mathbf{Z}_2$ gives exactly the orbit of u under the transformation σ on X_ζ. The one-sided shift in the space $\mathbf{Z}_2$, in terms of the theory of substitutions, is a *proper substitution*, i.e., the transformation that replaces, in any sequence, 0 by 01 and 1 by 10. Thus in the adic model we have a simultaneous realization of both transformations: the shift (it turned into the adic shift) and the substitution (it turned into the one-sided shift). The problem arises how to introduce into this picture the natural extension of the one-sided shift, the two-sided shift; and, at the same time, how to extend the adic transformation to the whole space. We will do this in the next section, but first we interpret a familiar property of the Morse system in our terms.

2.4. The Morse system as a 2-point extension of the odometer

Definition 2.1. *The classical* 2*-odometer is the following affine transformation* T *on the additive group* $\mathbf{Z}_2$ *of dyadic integers:*

$$T : Tx = x + 1.$$

The transformation T preserves the Haar (= Bernoulli, Lebesgue) measure on the group $\mathbf{Z}_2$. It is well known that the Morse system can be represented as a group (2-point) extension of the dyadic odometer. This is the most popular point of view on the Morse system in dynamics. The adic realization of the Morse transformation gives another way to look at this fact; the homomorphism of the Morse transformation to the odometer is, in our model, the composition of the Morse transformation with the differentiation.

Let us define an important map.

Definition 2.2. *The differentiation of sequences is the map* $D : \mathbf{Z}_2 \to \mathbf{Z}_2$ *given by*

$$D(\{x_n\}_{n=0}^{\infty}) = \{(x_{n+1} - x_n) \bmod 2,\ n = 0, 1, \dots\}.$$

This is nothing else than a 2-to-1 factorization of $\mathbf{Z}_2$ on itself. It is clear that the differentiation commutes with the "flip" defined above: $D(\overline{x}) = D(x)$.

In spite of the simplicity of the definition of the map $D : \mathbf{Z}_2 \to \mathbf{Z}_2$, there are no good and simple "arithmetic" or "analytic" expressions for the description of $D(\cdot)$. Recently, V. Arnold, for different reasons, made many experiments on the behavior of $0-1$ sequences under the iteration of differentiation [6]. But the most important thing for us is that the map D takes the Morse transformation to the odometer.

Proposition 2.3. *The following equality takes place:* $T \circ D = D \circ M$.

This is an immediate corollary of (2). Thus, in the adic realization, the Morse transformation M is a 2-covering of the odometer in its algebraic form. Let us give a precise description of the equivalence between the Morse transformation and the 2-extension of the odometer. Let $F(x) = (Dx, x_0)$ be the map from $\mathbf{Z}_2$ to $\mathbf{Z}_2 \times \{0,1\}$. This is a bijection, and we have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Z}_2 \arrow[r, "M"] \arrow[d, "F"] & \mathbf{Z}_2 \arrow[d, "F"] \\
\mathbf{Z}_2 \times \{0,1\} \arrow[r, "T(\phi)"] & \mathbf{Z}_2 \times \{0,1\}.
\end{tikzcd}$$

Here $T(\phi)$ is the 2-extension of T with the cocycle ϕ on Z_2 defined by

$$\phi(y) = \begin{cases} 0 & \text{if } y \text{ starts with an odd number of } 1\text{'s}, \\ 1 & \text{if } y \text{ starts with an even number of } 1\text{'s}. \end{cases} \tag{5}$$

To make it work on maximal elements, we also need to set $\phi(1^\infty) = \phi(0^\infty) = 1$. Recall that the group extension is defined by

$$T(\phi)(x, g) = (Tx, \phi(x) + g).$$

We have

$$M = F^{-1} T(\phi) F,$$

so M is canonically isomorphic to the 2-extension of the odometer T with the cocycle ϕ. We can identify $\mathbf{Z}_2$ with $\mathbf{Z}_2 \times \{0,1\}$ regarding the second component, i.e., an element of $\{0,1\}$, as a new digit of a sequence. Then the map F becomes a transformation of $\mathbf{Z}_2$, and we can consider this extension as a new transformation on the group $\mathbf{Z}_2$ itself. We give another interpretation of this cocycle in the next section.

Remark. The Morse system can also be realized as a 2-point extension of the odometer in the traditional substitution form, and it is interesting that the projection is again given by the differentiation map. This follows from the fact that for the Thue–Morse sequence u (see (1)), its derivative sequence $D(u) = 1011101010\dots$ is a fixed point of the substitution $0 \to 11$, $1 \to 10$ (see [5, p. 201]), which generates a measure-preserving transformation isomorphic to the 2-odometer.

Denote by S the map $x \to [x/2]$ on $\mathbf{Z}_2$. This is nothing else than the one-sided noninvertible shift, or Bernoulli endomorphism, if we represent the elements of $\mathbf{Z}_2$ as sequences of 0's and 1's. It is easy to check the following fact.

Proposition 2.4. *The 2-odometer, as well as the Morse transformation, satisfies the following equation:*

$$TS = ST^2, \quad MS = SM^2.$$

Note that under the two-sided extension of the 2-odometer T and Morse transformation M and the replacement of S by the two-sided shift, these relations turn into relations (10) and (14), which define the action of the group of dyadic rational numbers on the solenoid.

2.5. The Morse system as a time change of the odometer, and the Morse arithmetic. Since the group of rational integers $\mathbb{Z}$ is a dense invariant subgroup of the group of dyadic integers, we can consider the Morse transformation M in the adic realization as a map of the set of integers to itself. This subsection is based on [3, p. 538], but we provide more details.

Let us identify a sequence $x_0x_1x_2\dots$ with the dyadic decomposition of the number $\sum_j x_j 2^j$. Here is the list of several first values of $M(n)$:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
1	3	7	2	5	15	4	6	9	11	31	10	13	8	12	14	...

The table can easily be verified using (2).

To simplify this verification, we introduce the following sequence:

$$a_r = \begin{cases} \dfrac{2^r-1}{3} & \text{if } \ r \equiv 0 \ (\mathrm{mod}\, 2), \\ \dfrac{2^r-2}{3} & \text{if } \ r \equiv 1 \ (\mathrm{mod}\, 2). \end{cases} \tag{6}$$

Each $n \in \mathbb{N}$ can be uniquely represented in one of the following ways ($r = r(n)$):

$$n = \begin{cases} 2^r\ell + a_{r-1} & \text{(i)}, \\ 2^r\ell + 2^{r-1} + a_r & \text{(ii)}, \end{cases} \tag{7}$$

where $\ell \geq 0$ is an integer. Define a mapping $M : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ by

$$M(n) = \begin{cases} n + a_{r(n)} & \text{in the case (i)}, \\ n - a_{r(n)} & \text{in the case (ii)}. \end{cases} \tag{8}$$

Although these formulas look a bit mysterious, they easily follow from (2). In fact,

$$a_r = \frac{2^r-1}{3} = (10)^{(r-2)/2}, \quad r \equiv 0 \ (\mathrm{mod}\ 2), \qquad a_r = \frac{2^r-2}{3} = (01)^{(r-1)/2}, \quad r \equiv 1 \ (\mathrm{mod}\ 2).$$

The case (i) above occurs when the first pair aa in the binary representation of n is 00. Then $M(n)$ replaces the beginning of the sequence with 1's, which increases the number by $a_{r(n)}$ (observe that $a_{r-1} + a_r = 2^{r-1} - 1 = (1)^{r-1}$ independently of the parity of r). Similarly, the case (ii) above occurs when the first pair aa in the binary representation of n is 11. In this case $M(n)$ decreases or increases the number n by $a_{r(n)}$.

Thus we have described independently the restriction of the adic Morse system to $\mathbb{N}$; that is why we use the same symbol M. Define M for negative integers by $M(-n) = -M(n-1) - 1$. Then it is easy to check that the mapping thus defined has the property $M(\overline{x}) = \overline{M(x)}$, where $\overline{n} = -n-1$; this should be understood by identifying integers with their binary expansions. Thus we have $M : \mathbb{Z} \to \mathbb{Z} \setminus \{0, -1\}$. Note that $0 = (0)^\infty$ and $-1 = (1)^\infty$ are the two minimal points in our order on $\mathbf{Z}_2$. According to (3),

$$M(-1/3) \equiv M((10)^\infty) = (0)^\infty \equiv 0 \ \text{ and } \ M(-2/3) \equiv M((01)^\infty) = (1)^\infty \equiv -1.$$

2.6. The orbit equivalence of the Morse system and 2-odometer. The orbit of a point $x \in X$ with respect to an invertible transformation S of X is the set $\{S^n x,\ n \in \mathbb{Z}\}$. Obviously, the T-orbit of any point $x \in \mathbf{Z}_2$ that has infinitely many 0's and 1's is the set of all points that eventually coincide with x. The set of all points that have finitely many 0's or 1's makes one orbit (this is the common T-orbit of $(0)^\infty$ and $(1)^\infty$). Let us describe the orbit partition of the Morse transformation, which follows directly from definition (2).

Proposition 2.5. *If a point $x \in \mathbf{Z}_2$ has infinitely many subwords* 00 *and infinitely many subwords* 11*, then the M-orbit of x is the set of all points that eventually coincide x. The remaining countable set of points that have finitely many subwords* 00 *or* 11 *is exactly the union of four semiorbits of M: two positive M-semiorbits – of the point $(0)^\infty$ and of the point $(1)^\infty$, and two negative M-semiorbits – of the point $(10)^\infty$ and of the point $(01)^\infty$.*

Note that the negative M-semiorbit of $(10)^\infty$ (respectively, $(01)^\infty$) consists of the points that eventually coincide with $(10)^\infty$ (respectively, $(01)^\infty$) and have an initial *even* word.

Corollary 2.6. *The orbit partitions of the 2-odometer and the adic realization of the Morse transformation coincide* (mod 0) *with respect to the Haar (Lebesgue) measure on $\mathbf{Z}_2$.*

As we have seen, these partitions coincide on the complement of a countable set. We will refine this claim below.

Using our extension of M defined by (3), we can make an additional remark about those four semiorbits; we do not use it later. Note that two positive M-semiorbits generate one T-orbit, and each negative M-semiorbit is a full T-orbit. Thus in our definition (3) we cut the common T-orbit of $(0)^\infty$ and $(1)^\infty$ and glue the T-semiorbit of 0^∞ with the M-semiorbit of $(10)^\infty$, and the T-semiorbit of 1^∞ with the M-semiorbit of $(01)^\infty$.

If $x \in \mathbb{N} \subset \mathbf{Z}_2$, then we have the tautology

$$M(n) = T^{M(n)-n}(n),$$

where in the left-hand side $M(n)$ is the image of n under the transformation M, and in the right-hand side $M(n)$ is a natural number. Now observe that, by definition (8) of the action of the Morse automorphism M on the set of integers, we have

$$M(n) - n = (-1)^{r(n)} \cdot a_{r(n)}.$$

It is worth mentioning that the value of the cocycle $\phi(n)$ from the previous subsection is exactly $M(n) - n$ (mod 2), i.e., it is equal to 0 if and only if n and $M(n)$ have the same parity.

Denote

$$\theta(n) = (-1)^{r(n)} \cdot a_{r(n)}.$$

Then we have

$$M(n) = T^{\theta(n)} n$$

for each rational integer n. It is clear that the function $r(\cdot)$ and, consequently, the function $\theta(\cdot)$ can be extended from the set of positive integers $\mathbb{N}$ to the group of all dyadic integers $\mathbf{Z}_2$ as follows: formula (6) with some $r \in \mathbb{N}$ and $\ell \in \mathbf{Z}_2$ makes sense for all $x \in \mathbf{Z}_2$, not only for integers x. Hence we may consider infinite sequences of x_n as well. Thus $\theta(\cdot)$ becomes a function on $\mathbf{Z}_2$ with integer values; we may say that this is simply the extension of $\theta(\cdot)$ by continuity in the pro-2-topology.

We have proved the following theorem.

Theorem 2.7. *Let M be the adic realization of the Morse transformation in the space $\mathbf{Z}_2$. Let $\widetilde{\mathtt{Max}} \cup \widetilde{\mathtt{Min}}$ be the countable set that is the union of the M-semiorbits of the four points of $\mathbf{Z}_2$:*

$$(0)^\infty, \quad (1)^\infty, \quad (01)^\infty, \quad (10)^\infty.$$

Then on the M-invariant set $\mathbf{Z}_2 \setminus (\widetilde{\mathtt{Max}} \cup \widetilde{\mathtt{Min}})$, the odometer $T: \ Tx = x + 1$ and the Morse transformation M have the same orbit partition, and, moreover,

$$Mx = T^{\theta(x)} x \quad for \quad x \in \mathbf{Z}_2 \setminus (\widetilde{\mathtt{Max}} \cup \widetilde{\mathtt{Min}}),$$

where $\theta(x)$ is the function defined above.

The formula above gives an independent definition of the Morse transformation using a time change of the odometer.

Dye's theorem asserts that any ergodic automorphism S is isomorphic (mod 0) to an automorphism that is a time change of the odometer T (or any other given ergodic automorphism): $Sx = T^{\theta(x)}(x)$. Nevertheless, there are few examples of an explicit formula for such a time change function $\theta(\cdot)$. The theorem above is just of this type: the Morse automorphism is represented as a time change of the dyadic odometer. It is also known (see [1, Theorem 3.8] that if the ergodic automorphisms have the same orbits, then the time change integer-valued function $\theta(\cdot)$ cannot be integrable, unless $T = S$ or $T = S^{-1}$. It is easy to check that our function θ is indeed nonintegrable, because it has exactly two singularities on the space $\mathbf{Z}_2$ at the points $(01)^\infty$ ($\equiv -1/3$), and $(10)^\infty$ ($\equiv -2/3$), the measure of the cylinder on which the values of $\theta(x)$ are equal to a_r being of order $C2^{-r}$, so that the singularities are simple poles and the function in their neighborhoods is equivalent to $1/t$. The weakness (closeness to integrability) of these singularities shows that the Morse automorphism is, in a sense, very close to the odometer, i.e., to an automorphism with discrete spectrum.

Question. What is the group generated by two transformations of $\mathbf{Z}_2$ — the odometer T and the Morse transformation M? Is it a free group?

3. Extension of the Morse transformation up to an action of the group Q_2 on the solenoid

In this section, we define the so-called two-sided extension of the Morse transformation, which acts on the group of characters of dyadic rational numbers. This is an elaboration of [3, p. 539], with important changes and additions.

3.1. Preliminary facts about the dyadic groups Q_2, $\widehat{Q}_2$, etc. Consider the exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow Q_2 \longrightarrow Q_2/\mathbb{Z} \longrightarrow 1,$$

where Q_2 is the countable additive group of real dyadic rational numbers ($r/2^m$, $r \in \mathbb{Z}$, $m \geq 0$), the subgroup $\mathbb{Z} \subset Q_2$ is the group of rational integers, and the quotient group $Q_2/\mathbb{Z}$ is the group of all roots of the unity of orders 2^n, $n = 0, 1, \ldots$ (a subgroup of the rotation group).

The group Q_2 can be presented as the inductive limit

$$\varinjlim_n (\mathbb{Z}, w_n)$$

of the groups $\mathbb{Z}$, with the embedding of the nth group given by

$$w_n(x) = 2x, \quad n = 0, 1 \ldots .$$

Consider the corresponding dual exact sequence for the groups of characters of the groups above:

$$1 \longleftarrow \mathbb{R}/\mathbb{Z} \longleftarrow \widehat{Q}_2 \longleftarrow \mathbf{Z}_2 \longleftarrow 1.$$

The group of characters of the group $Q_2/\mathbb{Z}$ is just the additive group $\mathbf{Z}_2$ of dyadic integers, which we considered in the previous sections and which is the inverse limit of the 2^n-cyclic groups:

$$\mathbf{Z}_2 = \varprojlim (\mathbb{Z}/2^n, p_n),$$

with the maps $p_n : \mathbb{Z}/2^n \to \mathbb{Z}/2^{n-1}$, $p_n(x) = x \bmod 2^{n-1}$. The group of characters of the group $\mathbb{Z}$ is the rotation group $S^1 = \mathbb{R}/\mathbb{Z}$ (or the unit circle).

Our main object, the group $\widehat{Q}_2$ of characters of the group Q_2, is the so-called *2-solenoid* and can be presented as an inverse limit of the rotation groups:

$$\widehat{Q}_2 = \varprojlim_n (\mathbb{R}/\mathbb{Z}, v_{n+1}), \; n = 0, 1, \ldots,$$

where the homomorphisms are

$$v_n : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \quad v_n(u) = 2u, \; n = 1, 2, \ldots .$$

The group $\mathbf{Z}_2$ is a closed subgroup of the group $\widehat{Q}_2$ consisting of those elements that have zero projection to $\mathbb{R}/\mathbb{Z}$.

The additive group Q_2 of dyadic rational numbers is naturally embedded into $\widehat{Q}_2$ as a dense subgroup. It consists of those characters that send elements of the group Q_2 to roots of the unity of degree 2^n.

Note that the additive group of the locally compact field $\mathbf{Q}_2$ of all 2-adic numbers is naturally embedded into the solenoid $\widehat{Q}_2$ as a dense subgroup consisting of those elements of $\widehat{Q}_2$ whose projection under the map $\widehat{Q}_2 \to \mathbb{R}/\mathbb{Z}$ is a root of the unity of degree 2^n for some n:

$$\mathbf{Q}_2 \subset \widehat{Q}_2.$$

Being a compact group, $\widehat{Q}_2$ has a normalized Haar measure, which is the product of the Haar measures on the groups $\mathbf{Z}_2$ and $\mathbb{R}/\mathbb{Z}$.

The group Q_2 is a subgroup of the direct product of the groups $\mathbb{Z}$ and $Q_2/\mathbb{Z}$ (the generator $z \in \mathbb{Z}$ is the square of the generator $h_1 \in Q_2/\mathbb{Z}$). Consequently, the group $\hat{Q}_2$ is the quotient group of the direct product of the groups of characters S^1 and $\mathbf{Z}_2$ over the subgroup that consists of characters that are identically equal to 1 on Q_2. More precisely, there is an exact sequence that describes the group of characters $\hat{Q}_2$:

$$1 \to \operatorname{diag}(\Delta) \to S^1 \times \mathbf{Z}_2 \to \hat{Q}_2 \to 1,$$

where Δ is the subgroup of all roots of the unity whose degree does not contain twos, and diag is its natural embedding into the direct product of the factors. For details, see [6].

But we will use the coordinates on the group $\hat{Q}_2$ and represent its elements (nonuniquely) as pairs of elements of the groups S^1 and $\mathbf{Z}_2$.

Since the group Q_2 of dyadic rational numbers can be represented as the group of all finite (on both sides) two-sided sequences of 0's and 1's with the usual binary expansion, one may think that the analog of this decomposition for the group $\widehat{Q}_2$ is also true. Moreover, we have used one-sided sequences with positive indices for parametrization of the elements of the subgroup $\mathbf{Z}_2$, and that parametrization agrees with the group structure of 2-adic integers. Thus it is tempting to consider the whole group $\widehat{Q}_2$ of characters of the group Q_2 as the compact space of all two-sided infinite $\{0,1\}$-sequences: $\mathbf{X} = \prod_{-\infty}^{+\infty}\{0,1\} = \{0,1\}^{\mathbb{Z}}$. But this is not correct, because there is no required group structure on the space $\mathbf{X}$. Nevertheless, we can define a map $\pi : \mathbf{X} \to \widehat{Q}_2$ with the help of the usual dyadic decomposition of points of the unit interval $(0,1)$ as follows. Let $\{x_n\}$, $n \in \mathbb{Z}$, be a point of $\mathbf{X}$; consider the pair (y,λ), where $y = (x_0, x_1, \ldots) \in \mathbf{Z}_2$ (see Sec. 2) and

$$\lambda = \sum_{n=1}^{\infty} x_{-n}2^{-n}.$$

Denote this map by π:

$$\pi : \prod_{-\infty}^{+\infty}\{0,1\} \longrightarrow \widehat{Q}_2, \quad \pi : \{x_n\} \mapsto (y,\lambda). \tag{9}$$

The map π is not an isomorphism of groups or even topological spaces, but trivially is an isomorphism (mod 0) of measure spaces, where the space $\mathbf{X}$ is endowed with the $(1/2,1/2)$ Bernoulli (product) measure, and the group $\widehat{Q}_2$, with the Haar measure. Thus if we ignore the group structure of $\widehat{Q}_2$ and consider it not as the solenoid but as a symbolic space with measure-preserving transformations (odometer, Morse, etc.), then it is convenient to use the canonical map $\pi : \prod_{-\infty}^{+\infty}\{0,1\} \to \widehat{Q}_2$, which identifies only countably many pairs of points. Roughly speaking, we can consider the 2-solenoid $\widehat{Q}_2$ as the space $\mathbf{X}$ of all two-sided sequences of 0's and 1's after some identifications of elements from the negative (left) side corresponding to the nonuniqueness of dyadic decompositions.

3.2. Some transformations and differentiation on the solenoid. There is a canonical automorphism $\widehat{S}$ on the group $\widehat{Q}_2$: the multiplication by 2. It is conjugate to the automorphism S^* of the group Q_2 that is the multiplication by 1/2. The transformation $\widehat{S}$ is a hyperbolic automorphism of the solenoid, and in the usual coordinatization it is just the Bernoulli 2-shift and a *natural extension* in the sense of Rokhlin [4] of the one-sided shift S of the space $\mathbf{Z}_2$ defined in Sec. 2.

Now we define the two-sided version of the 2-odometer. Let 1 be the unit of the ring $\mathbf{Z}_2$ (unity of the multiplicative group). We define the extension $\widehat{T}$ of the odometer T from Sec. 2 by the same formula

$$\widehat{T}x = x + 1,$$

where x now is an element of $\widehat{Q}_2$. It is useful to keep in mind that 1 is a character of the group Q_2 that sends integers $\mathbb{Z} \subset Q_2$ to 1.

The action of $\widehat{T}$ does not change the second (left) component in the decomposition $\widehat{Q}_2 = Z_2 \times \mathbb{R}/\mathbb{Z}$, so it is indeed an extension of T. Note that $\widehat{T}$ is not an ergodic transformation of $\widehat{Q}_2$, whereas T is ergodic on $\mathbf{Z}_2$. We can also define the family of odometer transformations

$$T_0 := \widehat{T}, \quad T_i := S^i T_0 S^{-i}, \quad i \in \mathbb{Z}.$$

It is clear that T_i and T_j commute, and the joint (over all $i \in \mathbb{Z}$) action of the T_i's determines an *action of the group* Q_2 on the solenoid $\widehat{Q}_2$. Indeed, the T_i's act as translations, so that

$$T_i^2 = T_{i+1}, \quad i \in \mathbb{Z}. \tag{10}$$

This equation is immediate for $i = 0$ and hence for all i.

Together with the shift $\widehat{S}$, the odometers T_i generate a solvable group (wreath product) $\mathbb{Z} \leftthreetimes \sum_{\mathbb{Z}} \mathbb{Z}$; the action of this group on the group $\widehat{Q}_2$ is continuous and locally transversal in the sense of [3].

We define the *differentiation* $\widehat{D}$ as the transformation of $\mathbf{X}$ that extends the map D to the space of two-sided sequences:

$$\widehat{D}(\{x_n\}_{-\infty}^{+\infty}) = \{(x_n - x_{n+1}) \pmod 2\}.$$

Of course, we may define the differentiation on the solenoid $\widehat{Q}_2$ by the formula $\widetilde{D} = \pi \circ \widehat{D} \circ \pi^{-1}$, whose right-hand side is well defined almost everywhere. Observe that $\widehat{D}$ identifies a two-sided sequence $\tilde{x}$ with its "flip" $\overline{\tilde{x}}$, and hence almost everywhere on $\widehat{Q}_2$ we have

$$\widetilde{D}(y, \lambda) = \widetilde{D}(z, \gamma) \iff (y, \lambda) = (z, \gamma) \text{ or } (y, \lambda) = (\overline{z}, 1 - \gamma).$$

As we mentioned above, it is difficult to give a precise formula for $\widetilde{D}$ in terms of characters.

3.3. Extension of the Morse transformation. Now we would like to extend the Morse transformation M from the subgroup $\mathbf{Z}_2$ to the whole group $\widehat{Q}_2$ and the space $\mathbf{X}$.

We want $\widehat{M}$ to have the following properties. First, it must be a 2-extension of the extended odometer $\widehat{T}$, namely, satisfy the relation generalizing Proposition 2.1:

$$\widehat{T} \circ \widehat{D} = \widehat{D} \circ \widehat{M}. \tag{11}$$

Second, it must be an extension of M:

$$\widehat{M}|_{\mathbf{Z}_2} = M.$$

Theorem 3.1. *There is a unique transformation of the space* $\mathbf{X}$ *that satisfies the last two equations. It defines a measure-preserving transformation* $\widetilde{M}$ *on* $\widehat{Q}_2$ *via* $\widetilde{M} = \pi \circ \widehat{M} \circ \pi^{-1}$*, where* π *is defined by* (9).

Proof. The uniqueness is clear, and the existence can be shown as follows. The sequences of the space $\mathbf{X}$ can be divided into the positive and negative parts: given $\hat{x} = (\ldots x_{-1}, x_0, x_1 \ldots)$, denote $x_- = (\ldots x_{-2}, x_{-1})$ and $x_+ = (x_0, x_1 \ldots)$. Now set

$$\widehat{M}(\hat{x}) \equiv \widehat{M}((x_-, x_+)) = \begin{cases} (x_-, M(x_+)) & \text{if } \phi(x_+) = 0, \\ (\overline{x_-}, M(x_+)) & \text{if } \phi(x)_+ = 1, \end{cases} \tag{12}$$

Here ϕ is the cocycle defined in (5). Equation (11) is immediate.

We obtain an explicit formula for the Morse transformation on the solenoid $\widehat{Q}_2$:

$$\widetilde{M}(y, \lambda) = (My, \lambda) \text{ if } \phi(y) = 0, \quad \widetilde{M}(y, \lambda) = (Ma, 1 - \lambda) \text{ if } \phi(y) = -1.$$

(Here we use the coordinates (y, λ) introduced above.) $\square$

Observe that $\widehat{M}$ is not "local," in the sense that it does change the negative coordinates when the cocycle does not vanish.

The extension $\widehat{M}$ is continuous on $\{x \in \widehat{Q}_2 : \ x_+ \notin \widetilde{\mathtt{Max}}\}$, which is a set of full (Haar) measure.

Denote $\widehat{M} = M_0$ and define $M_i = S^i M_0 S^{-i}$; clearly, we have

$$T_i \circ \widehat{D} = \widehat{D} \circ M_i, \tag{13}$$

because $\widehat{D}$ commutes with S.

Theorem 3.2. *The group of transformations generated by the M_i's, $i \in \mathbb{Z}$, is algebraically isomorphic to the group Q_2:*

$$M_{i+1} = M_i^2, \quad i \in \mathbb{Z}. \tag{14}$$

Thus we obtain a new (Morse) action of the group Q_2 on $\widehat{Q}_2$. For every i, the transformation M_i is a 2-point extension of T_i.

Proof. We only need to check (14); all other statements follow immediately. Using the equations $S\widehat{D} = \widehat{D}S$ and $T_{i+1} = T_i^2$, we obtain that $\widehat{D}M_{i+1} = \widehat{D}M_i^2$. It remains to observe that $M_{i+1}(\hat{x})$ and $M_i^2(\hat{x})$ are cofinal (agree sufficiently far to the right) for all $x \notin \widetilde{\mathtt{Max}}$. □

We have defined two canonical measure-preserving actions of the solvable group $\mathbb{Z} \leftthreetimes Q_2$ on $\widehat{Q}_2$. The first one is generated by the odometer (this is an algebraic action), and the second one is generated by the Morse action. Recall that the Morse action is continuous only almost everywhere.

Questions.

1. Find the cocycle that defines the Morse action as a 2-extension of the algebraic action analogously to (5).
2. Find a formula analogous to the formula of Theorem 2.7 that defines the Morse action on the solenoid as a time change of the algebraic action.
3. How can we characterize both actions of the group $\mathbb{Z} \leftthreetimes Q_2$ in an intrinsic way?

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REFERENCES

1. J.-P. Allouche and J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge Univ. Press, Cambridge (2003).
2. V. I. Arnold, “Complexity of finite sequences of zeros and ones and geometry of finite spaces of functions,” *Funct. Anal. Other Math.*, **1**, No. 1, 1–15 (2006).
3. R. M. Belinskaya, “Partitions of a Lebesgue space into trajectories defined by ergodic automorphisms,” *Funkts. Anal. Prilozh.*, **2**, No. 1, 190–199 (1968).
4. F. Durand, B. Host, and C. Skau, “Substitutional dynamical systems, Bratteli diagrams and dimension groups,” *Ergodic Theory Dynam. Syst.*, **19**, 953–993 (1999).
5. A. Forrest, “K-groups associated with substitution minimal systems,” *Israel J. Math.*, **98**, 101–139 (1997).
6. I. M. Gelfand, M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation Theory and Automorphic Functions*, W. B. Saunders Co., Philadelphia–London–Toronto (1969).
7. S. Kakutani, “Ergodic theory on shift transformations,” in: *Proceedings of the 5th Berkeley Symposium on Mathematical Statistics and Probability*, Vol. II, Univ. California Press, Berkeley (1967), pp. 405–414.
8. Y. Moshe, “On the subword complexity of Thue–Morse polynomial extractions,” *Theoret. Comput. Sci.*, **389**, No. 1-2, 318–329 (2007).
9. N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lect. Notes Math., **1794**, Springer-Verlag, Berlin (2002).
10. M. Queffelec, *Substitution Dynamical Systems — Spectral Analysis*, Lect Notes Math., **1294**, Springer-Verlag, Berlin (1987).
11. V. A. Rokhlin, “Exact endomorphisms of a Lebesgue space,” *Izv. Akad. Nauk SSSR Ser. Mat.*, **25**, 499–530 (1961).
12. A. M. Vershik, “Uniform algebraic approximations of shift and multiplication operators,” *Sov. Math. Dokl.*, **24**, 97–100 (1981).
13. A. M. Vershik, “Locally transversal symbolic dynamics,” *St.Petersburg Math. J.*, **6**, No. 3, 529–540 (1995).
14. A. M. Vershik and A. N. Livshits, “Adic models of ergodic transformations, spectral theory, and related topics,” *Adv. Sov. Math.* **9**, 185–204 (1992).