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Integral models of representations of current groups

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§ 1. Introduction

In this paper, by the current group G^X, where X is a manifold with a fixed continuous probability measure m and G is a locally compact group, we mean the group of all bounded continuous maps $X \to G$ with pointwise multiplication and compactly open topology.[1)] The goal of the paper is to describe a new method of constructing irreducible unitary representations of G^X invariant with respect to measure-preserving transformations of (X, m) for the case when the group G can be presented as the semidirect product $G = \mathbb{R}^*_+ \leftthreetimes G_0$ with some additional properties introduced below. The main examples of such groups G are the maximal parabolic subgroups P of rank 1 groups: $O(n, 1)$, $U(n, 1)$, and $Sp(n, 1)$; in the first two cases, our representations can be uniquely extended to unitary representations of the corresponding semisimple current groups. These representations themselves were discovered in the 70s in [4, 5], but the method of construction considered below is absolutely new, it is a development of the model, suggested in [3, 2], of unipotent diagonalization of representations of current groups in the simplest case when the group of coefficients is $G = SL(2, \mathbb{R})$. This method is based on a remarkable measure ν on the space of distributions (more exactly, on the cone of positive discrete finite measures on the manifold X), which is called the infinite-dimensional Lebesgue measure and is studied in detail in the recent paper [6] and some previous probabilistic papers. It is this measure that allows us to present the space of a representation of the current group as the Hilbert space of L^2-sections with respect to this measure in such a way that the structure of a semidirect product on the current group agrees with the structure of an integral of countable tensor products of the representation. While the ordinary models of integrals of representations (i.e., of representations of current groups) in the Fock space $EXPH$ (see, e.g., [10, 5]) resemble Lebesgue integration of representations, the construction suggested in this paper resembles that of the Riemann integral or the integral over the trajectories of some random process. This leads to new isomorphisms of filtrations of Hilbert spaces (see [8]) and a new interpretation of continuous tensor products of Hilbert spaces, which will be considered elsewhere.

§ 2. The structure of the group G

Let G be an arbitrary locally compact group that can be presented as the semidirect product $G = \mathbb{R}^*_+ \leftthreetimes G_0$. The multiplicative group $\mathbb{R}^*_+$ acts naturally on the normal divisor G_0, and hence on the set of all unitary representations of G_0, by outer automorphisms σ_r, $r \in \mathbb{R}^*_+$, sending the representation operators $T(g)$ to the operators $T_r(g) = T(\sigma_r(g))$.

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[1)]In fact, we use only the structure of a measure space and practically do not use the topology of the manifold X; however, it is convenient to use the language of Schwartz distributions.

Fix a unitary representation T of the subgroup G_0 in a Hilbert space H. With T we associate a unitary representation U of the group $G = \mathbb{R}_+^* \leftthreetimes G_0$ in the direct integral of the Hilbert spaces $H_r \equiv H$, in which the representations T_r are defined, taken over the invariant measure $r^{-1}dr = d\log r$ on the multiplicative group $\mathbb{R}_+^*$. We obtain a direct integral of the representations T_r of G_0 in a direct integral of Hilbert spaces

$$\mathcal{H} = \int_0^\infty H_r \, r^{-1} \, dr.$$

In more detail, by definition, elements of $\mathcal{H}$ are square integrable (with respect to $d\log r$) sections $f(r)$ of the fiber bundle over $\mathbb{R}_+^*$ in which the fiber over a point $r \in \mathbb{R}_+^*$ is the space H_r. The action of the group G_0 in each fiber of this fiber bundle determines a (reducible) unitary representation $U(\cdot)$ of this group in the whole space $\mathcal{H}$. We define the operators $U(r_0)$ of the group $\{\mathbb{R}_+^* \ni r_0\}$ on sections $f \in \mathcal{H}$ by the formula

$$(U(r_0)f)(r) = f(r_0 r).$$

The following assertion is obvious.

Proposition 1. *Assume that the representation T of the subgroup G_0 is irreducible and all representations T_r are pairwise nonequivalent. Then the unitary representation U of the group G defined by this formula is also irreducible.*

§ 3. Summable families of representations

We say that a unitary representation T of the subgroup G_0 in a Hilbert space H and the corresponding family of representations $\{T_r, r \in \mathbb{R}_+^*\}$ are *summable* if there exists at least one unit vector $h \in H$ such that (up to the change $r \mapsto r^{-1}$) for every nonnegative numerical sequence $\{r_n\}$, the condition $\sum_{n=1}^\infty r_n < \infty$ implies

$$\sum_{n=1}^\infty \|T_{r_n}(g)\, h - h\| < \infty \quad \text{for every} \quad g \in G_0.$$

A vector $h \in H$ satisfying this property will be called *almost invariant* with respect to the representation T (or the family $\{T_r, r \in \mathbb{R}_+^*\}$).

Note that such a vector h may be not unique; for example, this is the case if G_0 is the Heisenberg group (see below). It is clear that every vector of the space linearly generated by vectors almost invariant with respect to a fixed summable representation T of G_0 is also a multiple of an almost invariant vector.

We will need the following sufficient condition for summability.

Proposition 2. *If there exists a unit vector $h \in H$ such that*

$$\|T_r(g)h - h\| < c(g)r \quad \textit{for every} \quad g \in G_0, \tag{1}$$

where $c(g)$ depends only on g, then the representation T of the subgroup G_0 is summable and the vector h is almost invariant with respect to T.

Indeed, it follows from (1) that $\sum_{n=1}^\infty \|T_{r_n}(g)\, h - h\| < c(g) \sum_{n=1}^\infty r_n$ for every nonnegative sequence $\{r_n\}$. Hence T is summable.

Proposition 3. *If a representation T of the subgroup G_0 in a space H satisfies the conditions of Proposition 2, then every almost invariant vector $h \in H$ satisfying* (1) *gives rise to a nontrivial* 1*-cocycle $a_h : G \to \mathcal{H}$, where $\mathcal{H}$ is the space of the representation U of the group $G = \mathbb{R}^*_+ \leftthreetimes G_0$ associated with T. Namely, $a_h(g) = U(g)f(r) - f(r)$, where $f(r) = e^{-r}\, h$.*

Proof. If $g \in G_0$, then $U(g)f(r) = e^{-r}\, T_r(g)h$, whence $\|U(g)f - f\|^2 = \int_0^\infty e^{-2r}\, \|T_r(g)h - h\|^2 r^{-1} dr$. In view of Proposition 2, it follows that $\|U(g)f - f\| < \infty$, i.e., $a_h(g) \in \mathcal{H}$ for $g \in G_0$. If $g = r_0 \in \mathbb{R}^*_+$, then $U(g)f = e^{-r_0 r} h$, whence $U(g)f - f = (e^{-r_0 r} - e^{-r})\, h$. It follows that $a_h(g) \in \mathcal{H}$ for $g \in \mathbb{R}^*_+$. Thus $a_h(g)$ is a 1-cocycle. Since $f \notin \mathcal{H}$, this cocycle is nontrivial. $\square$

Recall that an irreducible unitary representation of a locally compact group G is called *special* if the cohomology group of G with values in the space of this representation is nontrivial. By a theorem from [7], a special representation cannot be separated from the identity representation in the Fell topology. Thus Proposition 3 contains a simple and explicit construction of a special representation for the class of groups under consideration; cf. [4, 5], where a special representation for rank 1 groups is constructed as a limit of complementary series representations. In what follows, we assume that the conditions of Proposition 1 on representations T of the group G_0 are satisfied.

§ 4. Countable tensor products and representations of the groups $l^\infty(G_0)$

The following definition is widely known.

Definition 1. The countable tensor power of a Hilbert space H with unit stabilizing vector $h \in H$ is the inductive limit of the finite tensor powers $\otimes_{i=1}^n H$ of H with respect to the isometric embeddings $x \in \otimes_{i=1}^n H \mapsto x \otimes h \in \otimes_{i=1}^{n+1} H$. We denote this limit by $\widetilde{H} = \otimes_{i=1}^\infty (H, h)$ or, in short, $\otimes_{i=1}^\infty H$.

In other words, $\widetilde{H}$ is the Hilbert space obtained by completing the space $\lim\operatorname{ind}_{n\to\infty} \otimes_{i=1}^n H$ with respect to the norm.

It is natural to present the elements of the subspaces $\otimes^n H \subset \widetilde{H}$ (which form a total set in $\widetilde{H}$) as infinite products that stabilize at a finite step:

$$y_n = x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes h \otimes h \otimes \ldots \quad \text{where} \quad x_k \in H. \tag{2}$$

Lemma. *If $\sum_{n=1}^\infty \|x_n - h\| < \infty$, then a sequence $\{y_n\}$ of the form* (2) *converges in the norm of the space $\widetilde{H}$.*

We will write the limits of such sequences $\{y_n\}$ as infinite products $y = \otimes_{n=1}^\infty x_n$, where $\sum_{n=1}^\infty \|x_n - h\| < \infty$; in what follows, when describing the space $\widetilde{H}$, we will restrict ourselves only to such elements from $\widetilde{H}$ and their finite linear combinations.

Consider the group $l^\infty(G_0)$ (or, in short, l^∞) of all infinite bounded sequences $g = \{g_1, g_2, \ldots\}$ of elements of G_0 such that all elements g_n, $n = 1, 2, \ldots$, belong to some (depending on g) compact set $K(g) \subset G_0$, with coordinatewise multiplication. Starting with a summable unitary representation T of G_0 in a space H, we construct unitary representations of the group $l^\infty(G_0)$.

Theorem 1. *If a representation T of the subgroup G_0 in a space H is summable, then for every nonnegative summable numerical sequence $\{r_n\}$ and every vector $h \in H$ almost invariant with respect to T, there is a well-defined unitary representation of the group $l^\infty(G_0)$ in the space $\widetilde{H}_{\{r_n\},h} = \otimes_{i=1}^\infty (H, h)$ by operators of the form $\widetilde{T}_{\{r_n\},h}(g) = \otimes_{n=1}^\infty T_{r_n}(g_n)$ for $g = \{g_1, g_2, \dots\}$, i.e.,*

$$\widetilde{T}_{\{r_k\},h}(g)(\mathop{\otimes}_{n=1}^{\infty} f_n) = \mathop{\otimes}_{n=1}^{\infty} (T_{r_n}(g_n) f_n).$$

If the representation T of the group G_0 is irreducible and the representations T_r are pairwise nonequivalent, then the representations $\widetilde{T}_{\{r_n\},h}$ of the group $l^\infty(G_0)$ are also irreducible and pairwise nonequivalent.

Proof. The representation is well-defined, because the convergence in $\widetilde{H}_{\{r_n\},h}$ of the sequence $y_n = x_1 \otimes x_2 \otimes \dots \otimes x_n \otimes h \otimes h \otimes \dots$ implies that for every $g = \{g_1, g_2, \dots\} \in l^\infty(G_0)$ the sequence $\{T_{r_1}(g_1)x_1 \otimes T_{r_2}(g_2)x_2 \otimes \dots \otimes T_{r_n}(g_n)x_n \otimes h \otimes h \otimes \dots\}$ also converges.

The unitarity of $\widetilde{T}$ follows from the unitarity of T_{r_n}.

Further, assume that some representations $T_{\{r_n\},h}$ and $T_{\{r'_n\},h'}$ are equivalent. Then, obviously, $\{r_n\} = \{r'_n\}$, and every intertwining operator A for these representations maps all subspaces $\otimes_{i=1}^n H$ to themselves and is a multiple of the identity operator on each of them. We may assume without loss of generality that $A = \mathrm{id}$ on each subspace $\otimes_{i=1}^n H$. Then A preserves every sequence $\{y_n\}$ of the form (2) convergent in $\widetilde{H}_{\{r_n\},h}$. Obviously, such a sequence converges in $\widetilde{H}_{\{r_n\},h'}$ only for $h' = h$. □

In what follows, the constructed representations $\widetilde{H}_{\{r_n\},h}$ of the group $l^\infty(G_0)$ will be the components of the direct integral of representations of the current group G_0^X. More exactly, we can let the whole current group pass through the representation of $l^\infty(G_0)$ defined above.

Theorem 2. *Choose a countable sequence of points $\{x_k\}_{k=1}^\infty$, $x_k \in X$, and a convergent positive series $\{r_k\}$, $\sum_{k=1}^\infty r_k < \infty$. Under the conditions of Theorem 1, for every such collection $\{r_k, x_k\}$ and every vector $h \in H$ almost invariant with respect to a summable representation T of the group G_0, there is a unitary representation $\widetilde{T}_{\{r_n\},h}$ of the current group G_0^X in the space $\widetilde{H}_{\{r_n\},h} = \otimes_{i=1}^\infty (H, h)$ by operators of the form $\widetilde{T}_{\{r_n\},h}(g(\cdot)) = \otimes_{n=1}^\infty T_{r_n}(g(x_n))$, i.e.,*

$$\widetilde{T}_{\{r_n\},h}(g(\cdot))(\mathop{\otimes}_{n=1}^{\infty} f_n) = \mathop{\otimes}_{n=1}^{\infty} (T_{r_n}(g(x_n)) f_n).$$

If the representation T of the group G_0 is irreducible and the representations T_r are pairwise nonequivalent, then the representations $\widetilde{T}_{\{r_n\},h}$ of the group G_0^X are also irreducible and pairwise nonequivalent.

Indeed, it suffices to observe that the restriction of G_0^X to the chosen set of points $\{x_k\}$ is an element of the group $l^\infty(G_0)$. Therefore the obtained representations of G_0^X depend substantially on the collection of points $\{x_n\}$ and the sequence $\{r_n\}$. Note that the Hilbert space in which they act depends only on the almost invariant vector $h \in H$.

§ 5. The Lebesgue measure in the space of distributions

We proceed to the central part of our construction, involving the description of a σ-finite measure invariant with respect to a remarkable infinite-dimensional group. The measure itself was first discovered in [5] and partially studied in [9]. However, in those papers it was defined by the density with respect to some probability measure; only later it became clear that this σ-finite measure is related to the gamma process and stable Lévy processes (see also [8]). A new approximative approach to this measure was suggested in the recent paper [6], in which its relations to other measures (to the Poisson–Dirichlet measures) and close problems are described. Note that the role of this measure in the problems under consideration, combinatorics, number theory is comparable to the role of the Gaussian measure in mathematical physics and infinite-dimensional analysis. In this paper we will need only the existence of this measure and some its properties.

Consider the cone of real positive finite atomic measures on X:

$$D_+(X) = \{\sum_{k=1}^{\infty} r_k \delta_{x_k}, r_k > 0, \sum_k r_k < \infty\}.$$

Elements of $D_+(X)$ will be denoted in short by $\xi = \{r_k, x_k\}_{k=1}^{\infty}$.

Theorem-definition. *There exists a σ-finite (infinite) measure ν on the cone $D_+(X)$ that is finite on every compact set and is uniquely determined by its Laplace transform*

$$F(f) \equiv \int_{D_+(X)} \exp(\sum_k r_k f(x_k)) d\nu(\xi) = \exp(-\int_X \log f(x) dm(x)),$$

where f is an arbitrary nonnegative measurable function on (X, m) such that the integral of $\log f$ is finite.

Remark. The Laplace transform is well-defined for a wide class of σ-finite measures, and there is a uniqueness theorem for measures with a given Laplace transform. The proof of the existence of such a measure can be carried out in several different ways: a direct construction of the gamma process with the subsequent introduction of a density [9], an approximative construction and the definition via the Poisson–Dirichlet measures [6].

There is a characteristic description of this measure which is especially important for our purposes. Let a manifold X and a normalized nonnegative measure m on X be fixed.

Theorem 3. *The measure ν on the cone $D_+(X)$ is completely characterized by the following two properties:*

1) (*Projective invariance with respect to the group of multiplicators.*) *If $a(x)$ is a nonnegative measurable bounded function on X such that the integral $\int_X \log a(x) dm = c \geq 0$ converges, then the operator M_a that multiplies elements $\xi \in D_+(X)$ by $a(x)$ (i.e., $M_a \sum_k r_k \delta_{x_k} = \sum_k a(x_k) r_k \delta_{x_k}$) multiplies the measure ν by $\exp c$. In particular, if $c = 0$, then ν is invariant with respect to this multiplicator.*

2) *Invariance and ergodicity with respect to the group of all measure-preserving transformations of (X, m).*

The fact that the measure ν satisfies these properties follows from the above formula for the Laplace transform of ν.

Note that if we replace the constant $\exp c$ in condition 1) with the constant $\exp \theta c$, $\theta > 0$, then we obtain an important one-parameter family of measures $\nu_\theta \theta > 0$. These measures are in a natural correspondence with the Poisson–Dirichlet measures $PD(\theta)$ on the infinite-dimensional simplex (see [6]). If we do not impose the normalizing condition on the measure m, then the value $m(X) \equiv \delta$ adds another positive parameter δ, which is not essential.

It is natural to call ν the infinite-dimensional Lebesgue measure, because its properties generalize the invariance properties of the finite-dimensional Lebesgue measure on the nonnegative octant. The novelty of the infinite-dimensional case is the ergodicity with respect to the group of multiplicators, which is worth mentioning separately.

Theorem 4. *The action of the group $\mathcal{M}$ of multiplicators on the cone $D_+(X)$ endowed with the measure ν is ergodic.*

§ 6. Construction of a representation of the current group

Now we proceed to the construction of an irreducible representation of the current group $G^X = (\mathbb{R}^*_+ \leftthreetimes G_0)^X$ invariant with respect to the group of transformations of the manifold X preserving the measure m. Fix a summable unitary representation T of the group G_0 in a Hilbert space H and a vector $h \in H$ almost invariant with respect to T. By Theorem 2, for every element $\xi = \{r_k, x_k\}_{k=1}^\infty \in D_+(X)$ there is a representation $\widetilde{T}_{\{r_n\},h}r$ of the current group G_0^X in the space $\widetilde{H}_{\{r_n\},h} = \otimes_{i=1}^\infty (H, h)$. The support of this representation is the sequence $\{x_k\}$, $x_k \in X$, and the representation itself depends on the sequence $\{r_k\}$. In other words, we considered a representation passed through the homomorphism of restricting G_0^X to the points $\{x_k\}$, $x_k \in X$:

$$G_0^X \to l^\infty(G_0).$$

Of course, these representations do not use the measure ν and are not invariant with respect to the group of m-preserving transformations of X. Besides, they are still representations of ${G_0}^X$ rather than G^X.

Now we proceed to the description of the unitary representation U of the group $G^X = (\mathbb{R}^*_+ \leftthreetimes G_0)^X$ associated with T.

Take the direct integral of the Hilbert spaces $H_{\{r_k\},h} = \bigotimes_{i=1}^\infty (H, h)$ with respect to the measure ν,

$$\mathcal{H} = \int_{D_+} \widetilde{H}_{\xi,h}\, d\nu(\xi), \quad \widetilde{H}_{\xi,h} = \bigotimes_{i=1}^\infty (H, h),$$

i.e., the space of sections $F(\xi) = F(\{r_k, x_k\})$ of the fiber bundle over the space $D_+(X)$ of discrete measures in which the fiber over a point $\xi = \{r_k, x_k\}$ is the countable tensor product $\widetilde{H}_{\{r_k\},h} = \otimes_{k=1}^\infty H_{r_k}$, $H_{r_k} = H$, with stabilizing vector h. (This fiber, regarded as a Hilbert space, does not depend on the sequence $\{x_k\}$, but the representation acting in this fiber does depend on $\{x_k\}$.) An action of the group G_0^X is already defined in each fiber of this fiber bundle, inducing a unitary representation of G_0^X on the whole space $\mathcal{H}$. It remains to define the operators $U(r_0(\cdot))$ of the group $\mathbb{R}^X$. We define them by the formula

$$(U(r(\cdot))F)(\{r_k, x_k\}) = e^{\frac{1}{2}\int_X \log r_0(x)\, dm(x)}\, F(\{r_0(x_k) r_k, x_k\}) \quad \text{for every} \quad r(\cdot) \in \mathbb{R}^X_+;$$

that is, the operators $U(r_0(\cdot))$ permute the fibers of the fiber bundle and thus induce a representation of the group $\mathbb{R}^X$ on the whole space $\mathcal{H}$. The unitarity of these operators follows from the invariance of the measure ν with respect to multiplicators. Together with the operators of G_0^X, they generate the required unitary representation of G^X.

Theorem 5. *If the representation T of the group G_0 is irreducible and the representations T_r are pairwise nonequivalent, then the above-defined unitary representation U of the group G^X is irreducible. It is invariant with respect to the group of transformations of the base X preserving the measure m.*

The fact that the representation is irreducible follows from the ergodicity of the measure ν with respect to the group of multiplicators (see above). The other properties are also determined by the properties of ν.

Thus the parameters of our construction for the class of groups under study are a summable irreducible unitary representation T of the group G_0 and an almost invariant vector in the space of this representation. We will see that in some classical cases these parameters are uniquely determined.

§ 7. Example: the one-dimensional extension of the Heisenberg group

Let $G = \mathbb{R}_+^* \leftthreetimes N$, where N is the Heisenberg group of order $2n-1$ realized as the group of pairs (t,z), $t \in \mathbb{R}$, $z \in \mathbb{C}^{n-1}$, with multiplication law $(t_1,z_1)(t_2,z_2) = (t_1+t_2-\operatorname{Im} z_1 z_2^*,\ z_1+z_2)$. Elements $r \in \mathbb{R}_+^*$ and $(t,z) \in N$ are related by $r(t,z)r^{-1} = (r^2 t, rz)$.

Essentially, there are two infinite-dimensional summable irreducible unitary representations of the group N (see [11]). The first one acts in the Hilbert space H^+ of entire analytic functions $f(z)$ on $\mathbb{C}^{n-1}$ with the norm

$$\|f\|^2 = \int_{\mathbb{C}^{n-1}} |f(z)|^2\, e^{-zz^*}\, d\mu(z). \tag{3}$$

The operators of this representations are given by the formula

$$(T(t_0,z_0)f)(z) = e^{\zeta_0 - zz_0^*}\, f(z+z_0) \quad \text{where} \quad \zeta_0 = it_0 - \frac{1}{2} z_0 z_0^*. \tag{4}$$

The second representation acts in the space H^- of entire anti-analytic functions on $\mathbb{C}^{n-1}$ with the same norm (3) by operators of the form $(T(t_0,z_0)f)(z) = e^{\overline{\zeta}_0 - z_0 z^*}\, f(z+z_0)$.

The invariance condition, and even the stronger condition (1), are satisfied by all, up to a factor, monomials $z^k = z_1^{k_1} \dots z_{n-1}^{k_{n-1}}$ and $\overline{z}^k = \overline{z}_1^{k_1} \dots \overline{z}_{n-1}^{k_{n-1}}$ in the spaces H^+ and H^-, respectively. Hence the almost invariant vectors linearly generate everywhere dense subsets in these spaces.

Thus the irreducible unitary representations of the current group N^X are indexed by pairs (ξ, h) where $\xi \in D_+(X)$ and h is an almost invariant vector in H^+ or H^-, and all these representations are pairwise nonequivalent. Correspondingly, every irreducible unitary representation of the group $G^X = (\mathbb{R}_+^* \leftthreetimes N)^X$ of currents of the one-dimensional extension of the Heisenberg group is determined, according to our scheme, by an almost invariant vector in H^+ or H^-, and all these representations are pairwise nonequivalent.

§ 8. **The main example: the parabolic subgroups of rank 1 groups**

Consider the case $G = P$ where P is the maximal parabolic subgroup of the group $O(n,1)$ or $U(n,1)$. In both cases, P is the semidirect product of the group of homotheties and a subgroup P_0: $P = \mathbb{R}_+^* \leftthreetimes P_0$. Thus the description of irreducible representations of the current group P^X reduces to the description of summable irreducible representations of the subgroup P_0. Let us give this description.

In the case of $O(n,1)$, the subgroup P_0 is the semidirect product $P_0 = D_0 \leftthreetimes \mathbb{R}^{n-1}$ where $D_0 = \{\pm 1\} \oplus O(n-1)$. In this case, there is a unique family T_r of summable irreducible representation of P_0. It is convenient to realize it in the L^2-space of functions $f(\omega)$ on the unit sphere in $\mathbb{R}^{n-1}$. The operators of this representation are as follows: $(T(\gamma)f)(\omega) = e^{i\langle\gamma,\omega\rangle} f(\omega)$ for $\gamma \in \mathbb{R}^{n-1}$; $(T(\epsilon,u)f)(\omega) = f(\epsilon\omega u)$ for $(\epsilon,u) \in D_0$. There is a unique almost invariant vector with respect to T: $f(\omega) = \text{const}$. Thus in this case we obtain one irreducible unitary representation of the group P^X.

The case $SL(2,R) \sim SO(2,1)$ looks especially simple: in this case, the subgroup P_0 and the spaces $\widetilde{H}$ are one-dimensional. Hence the direct integral reduces to the scalar space $L^2(\nu)$; it is in this space that the representation of the group P^X of lower triangular 2×2 matrices acts in the so-called commutative model [3]. This representation can be extended to a representation of the whole current group $SL(2,R)^X$.

In the case of $U(n,1)$, the subgroup P_0 is the semidirect product $P_0 = D_0 \leftthreetimes N$ where $D_0 = U(1)\oplus U(n-1)$ and N is the Heisenberg group of order $2n-1$. Elements $(\epsilon,u) \in D_0$ and $(t,z) \in N$ are related by $(\epsilon,u)^{-1}(t,z)(\epsilon,u) = (t,\overline{\epsilon}zu)$.

In this case, there are two countable families of irreducible summable representations T_r of P_0. One of them is obtained from the decomposition of the unitary representation of P_0 in the space $\widetilde{H}$ of *all* functions $f(z)$ on $\mathbb{C}^{n-1}$ with the norm (3) into irreducible components. The operators of the subgroup N in this space are given by (4), and the operators of the subgroup D_0 have the form $(T(\epsilon,u)f)(z) = f(\overline{\epsilon}zu)$ for $(\epsilon,u) \in D_0$. The space $\widetilde{H}$ decomposes into the direct sum of irreducible pairwise nonequivalent subspaces: $\widetilde{H} = \oplus_{m=0}^{\infty} H_m$. The representation of P_0 in each subspace H_m is summable and has a unique almost invariant vector: $\varphi_m(z) = l_m^{n-2}(zz^*)$, where $l_m^{n-2}(x)$ is the Laguerre polynomial, see [12].

Another family of irreducible summable representations of P_0 is obtained from the irreducible decomposition of the same space H in which the operators of the subgroup N are given by $(T(t_0,z_0)f)(z) = e^{\overline{\zeta}_0 - z_0 z^*} f(z+z_0)$.

Note that two of the irreducible representations of P_0 defined in this way are obtained by extending to P_0 the irreducible representations of the Heisenberg group N in the spaces of entire analytic and anti-analytic functions on $\mathbb{C}^{n-1}$, respectively.

The representations $\widetilde{T}$ of the maximal parabolic subgroup P associated with these summable representations of P_0 are irreducible and pairwise nonequivalent; they have the nontrivial 1-cocycles $a(g) = \widetilde{T}(g)f(r,z) - f(r,z)$, where $f(r,z) = e^{-r}\varphi_m(z)$.

Theorem 6. *The representations $U(g(\cdot))$ of the current group P^X associated with these summable irreducible representations of the subgroup P_0 are irreducible and pairwise nonequivalent.*

A question arises about extending the irreducible unitary representations of P^X obtained in this way to the whole current group $(O(n,1))^X$ or $(U(n,1))^X$, respectively. In the case of $O(n,1)$, by the construction and the uniqueness of an irreducible unitary representation of the current group, this question is resolved in the affirmative; in the case of $U(n,1)$, an extension exists only for the two representations of P^X associated with the irreducible representations of P_0 in the spaces of entire analytic and anti-analytic functions, respectively.

Since in order to construct such an extension, we must additionally define the representation on a unique element (involution), we would like to have an explicit formula for this involution. This turned out to be possible for the group $(U(n,1))^X$. A more detailed treatment of the theory will be published separately.

§ 9. **Relation to the Fock space**

All known representations of the current group were realized, in the papers [5, 10], in the Fock space, or the exponent of a Hilbert space. In this realization, the one-particle subspace is the Hilbert space $L^2(X;\widetilde{H})$ of square integrable functions with values in the Hilbert space $\widetilde{H}$ in which a *special* representation of the group of coefficients acts (see above).

Since the representations under study are irreducible, there is a unique (up to scalar operator) isomorphism of the integral model, suggested in this paper, of a representation of the current group and the Fock model of this representation. The explicit form of this isomorphism is rather complicated. Let us schematically describe the difference between the Fock model and the integral model.

First of all, in order to construct a realization of representations of currents, free fields, etc., we need to introduce a correct notion of continuous symmetric tensor product of Hilbert spaces, or tensor integral of Hilbert spaces over some base X. In other words, we need to give an exact meaning to the space described by the symbolic expression

$$\int_X^{\otimes} H_\alpha \, dm(\alpha),$$

where X is the base of the integral (i.e., the space of parameters of field operators). We may assume that all Hilbert components H_α are identified with the same Hilbert space H; this space is rather virtual, or "infinitesimal," and the integral is its "continuous infinite tensor power." The understanding of this integral that is generally accepted originates in quantum theory and consists in passing to the Fock model, i.e., the exponent of the one-particle space, which is most convenient to interpret as the Hilbert space $L^2_m(X;H)$ of square integrable functions with respect to m with values in some Hilbert space H:

$$\int_X^{\otimes} H \, dm = EXP[L^2_m(X;H)].$$

The most frequent analytic interpretation of the exponent is that it is the $\mathcal{L}^2$ space over a Gaussian process (white noise) and the one-particle space consists of linear functionals of trajectories of this process.

In this paper we suggest another, quite different, understanding of the continuous tensor product. Namely, we replace the base of the integral, the measure space (X,m), with the cone of positive discrete measures on the base, i.e., the cone $D_+(X)$ endowed with the σ-finite measure ν. To each point (or trajectory) $\xi = \{r_k, x_k\} \in D_+(X)$, where $x_k \in X$,

$r_k > 0$, $\sum_k r_k < \infty$, we associate the countable tensor product (over k) of the Hilbert spaces H_{r_k}, and then take the direct integral of these countable tensor products over the measure ν. One may say that our construction replaces the continuous tensor product by the direct integral of countable tensor products.

Let us reproduce this formula as applied to the class of groups under study and representations of their current groups.

Let $G = \mathbb{R}^*_+ \leftthreetimes G_0$, let T be a summable irreducible unitary representation of the subgroup G_0 in a space H, and let $\widetilde{T}$ be the associated special representation of the group G in the space $\widetilde{H}$. Then our interpretation of the continuous tensor product is as follows:

$$\int_X^{\otimes} \widetilde{H} dm \approx \int_{D_+(X)}^{\oplus} [\bigotimes_{k=1}^{\infty} H_{r_k}] d\nu(\xi).$$

Here $\xi = \{r_k, x_k\} \in D_+(X)$, and H_{r_k} are the spaces of the corresponding representations of the group G_0^X, more exactly, the representations of the group $l^\infty(G_0)$ obtained by restricting the elements of the current group to a countable set of points $\{x_k\}$, $k = 1, 2, \ldots$; the current group $(\mathbb{R}^*)^X$ acts ergodically on the base $(D_+(X), \nu)$ of the direct integral and simultaneously intertwines the representations in the fibers H_{r_k}.

Apparently, this model of tensor product has also wider applications; but it is still to be understood in more detail how the ordinary operators of the Fock model look in terms of this model.

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