1. Vershik: Gromov's Geometry

1. In the middle of the XX century geometry and topology in Leningrad were represented by two extraordinary schools: the ones of A.D. Alexandrov and V.A. Rokhlin. Needless to say, A.D. Alexandrov - a student of B.N. Delone - obviously engaged in a geometric way of thinking. Sometimes his school was referred to as a school of visual geometry. It is certainly not quite precise. The geometric philosophy of the topologist Rokhlin is known only to those who were closely acquainted with him and his research. I remember very well a conversation I had with him in the early 60s. He stated, very emotionally (which is not typical of him), that depth and beauty of geometry cannot be compared with those of any other field of mathematics. His geometric way of thinking is clearly seen in both his topological and metric works. It may look like the triumph of algebraic topology in 50s and 60s moved the geometric and combinatorial topology to the background, and an entirely new insight emerged. However, in reality, this triumph just shifted geometric ideas to a new level. Whereas V.A. Rokhlin promoted algebra and strongly suggested that his students should adopt algebraic philosophy, for him, and I think for Gromov, algebra was only one of the many languages of geometry, not the method of thinking. It is very difficult to explain this fine difference without going into details. It is easier to refer to the classic topological work of the algebraist J.P. Sierre and algebraic work of the topologist J. Milnor.

One would engage here into lengthy explanations of the used terms such as geometry, geometric philosophy, geometric ideas, algebra, etc. However, I do not think that it is possible to do that in a productive way. It is better to stay at the level of vague understanding shared by most mathematicians, though with variations. I even think that clarifying the meaning of these terms is the responsibility not only of mathematicians but as well of philosophers and psychologists. Even a discussion of geometric motives in modern and classical music, in poetry and, needless to say, in modern art could be meaningful.

It seems to me that the coexistence of the two schools mentioned above is remarkable and perhaps unique. Let us recall that it was A.D. Alexandrov (the Rector of the Leningrad State University in the 50s 60s) who invited V.A. Rokhlin there (on the initiative of a number of friends). Due to a variety of reasons Rokhlin was in a hard situation at that time. In Soviet reality and taking into account a unique biography of V.A. Rokhlin this was not an easy task even for a Rector. M. Gromov was one of the first topology students of V.A. Rokhlin during his first years of work in Leningrad. In my opinion Gromov was also a follower of geometric ideas of Alexandrov.

2. All junior mathematicians, no matter how talented they are, need a certain period of time to accumulate a wide supply of knowledge. Perhaps it was difficult to do that in Leningrad in the 50s and the early 60s. Despite of the presence of numerous extraordinary mathematicians, Leningrad school of mathematics was suffering from obvious narrowness (functional analysis, certain areas of algebra, classical theory of partial differential equations, Alexandrovs geometry, etc.; but it lacked modern topology, representation theory, algebraic geometry and many other areas of mathematics). The situation changed after Rokhlins arrival. I remember the first topology course he taught (probably in 1961) which was audited by two or three professors, two or three freshmen (including Misha), and one or two Ph. D.

students (including myself). The matter is not only Rokhlins personal intellectual investment into the broadening of academic interests of his colleagues and students. Rokhlin graduated from the Moscow State University before the war and was a student of A. N. Kolmogorov, L. S. Pontryagin, P.C. Alexandrov, A. I. Plesner, etc. at the same time. Due to Rokhlins close relations with Moscow mathematical schools, their representatives became regular visitors of seminars in Leningrad. Rokhlins students, in turn, became popular and were frequently invited to Moscow. Misha even received the Moscow Mathematics Society Prize, probably he was the only recipient who was not from Moscow during the entire existence of the prize. Top Moscow Mathematicians - I. M. Gelfand, V. I. Arnold, S. P. Novikov were opponents at Mishas doctoral defense (the latter was substituted by N.N. Uraltseva because S. Novikov was not able to attend). It would hardly be possible to organize this without Rokhlin.

On the other hand, after acquiring certain knowledge of the current state of mathematics (at least in ones own area), the main part in the formation of a researcher is played by his own effort in understanding of what has been done before and contemplating on what is not accomplished yet. I think that Misha was very different from many of his colleagues in that way. His interests and endless curiosity, critical thinking and even lack of trust that was evident in his conversations and seminars, were indicating a constant thinking process. He continues displaying the above qualities even now. I. M. Gelfand once explained to me approximately the following: in order to avoid mistakes in his works, a mathematician must not trust himself too much, but in order to obtain significant results one must not trust others too much. The ratio of these two distrusts changes with age, and it seems like Misha finally found the perfect proportion. I, however, do not know of Mishas mistakes.

It is not appropriate to mention any personal relations here. Let me just note that I was always impressed by 100% independence of Mishas thoughts and deeds, which illustrates the strength of his character. The Soviet regime did not favor independent and courageous ones, so many research talents were not able to develop fully. However, sometimes life's circumstances and especially personal qualities of some turn out to be stronger than the routine, and then the talent obtains the full freedom of expression to the great benefit of science.

3. We discuss here only a limited circle of Gromov's ideas which play a modest role in his huge repertoire. The epigraph to Misha Gromov's book Metric Structures for Riemannian and Non-Riemannian Spaces, which we discuss below is remarkable in some respects: "...Meme ceux qui furent favorables a ma perception des verites que je voulais ensuite graver dans le temple, me feliciterent de les avoir decouvertes au microscope, quand je metais au contraire servi dun telescope pour apercevoir des choses, tres petites en e?et, mais parce qu elles etaient situees a une grande distance, et qui etaient chacune un monde." Marcel Proust, Le temps retrouve (Pleiade, Paris, 1954, p. 1041)

In short, slightly roughening ornamental ligature of words by Proust, this reads as follows: "Even those who commended my perception of the truths which I wanted eventually to engrave within the temple, congratulated me on having discovered them with a microscope, when on the contrary it was a telescope that I had used to observe things which were indeed very small to the naked eye, but only because they were situated at a great distance, and which were each one of them in itself a world."

And indeed, numerous plots considered and studied by Gromov (not exclusively in this book) are not focused on the details (possibly even crucial ones) of recognized theories. They are more about new geometric realms as well as other realms which cannot be considered totally unknown, but about which we either knew almost nothing or possessed a false knowledge.

I would not undertake studying all work of Gromov from this perspective, but I think that this point of view would be useful and even necessary for those who will study his work in the future.

Among numerous geometric projects of Gromov including a few discussed in the book in question, there is one that was especially interesting to me. However, that project perhaps was not considered especially important and interesting by mathematicians including the author himself. I am referring to concept of mm-spaces, that is measure-metric spaces - $(X; \mu; \rho)$. Anyone who worked in analysis on manifolds, geometry and dynamical systems has encountered such triples. I would like to illustrate Proust's idea of a "telescopic" view by this concrete and important example. It is unlikely that it would occur to traditional analyst or geometer to convert such situation into an abstract setup and pose a question of a categorical classification of such triples where morphisms are measure-preserving isometries. It is well-known that the classification of metric spaces up to an isometry is impossible to comprehend (recently A. Kechris gave a precise meaning to this statement) and not very useful. On the contrary, the classification of standard measure spaces is trivial (V. A. Rokhlin). So, what can we say about the triples?

Let us consider a triple $\tau = (X; \mu; \rho)$, where ρ is a metric on X which turns it into a Polish space and μ a continuous Borel probability measure which is nondegenerate in the sense that every nonempty open set has a positive measure. Obviously if we randomly and independently choose n points (x_1, x_2, x_n) distributed with probability μ, then the random matrix of distances $\{p(x_i, x_j)\}_{i,j=1}^{n}$ generates a probability measure M_n on the set of distance matrices of order n, that is, in the space of metrics on and n-point sets. It is also obvious that this measure does not change under isometries that preserve the measure. Gromov posed the following question: is it true that the combination of all those measures $M_n, n = 1$ fully determines the triple up to an isomorphism (that is, up to a measure-preserving isometry). The positive answer is proven (in a rather difficult way; actually by the method of momenta) by Gromov in his book (p. 120 123). This crucial fact is known as Gromovs Reconstruction Theorem. Approximately in 1997 or not much earlier M. Gromov asked me what I thought of that theorem and its proof. I conveyed to him my very simple proof based on passing to infinite sequences of independent random points. Both proofs are discussed in the book and then the author, not without a naughty trick, asks a reasonable question if the reader is puzzled by the fact that complicated analytical constructions can be replaced by a "spineless argument"? Certainly, the answer is very easy: the spine of the second proof in nontrivial though widely known. It is the individual Law of Large Numbers, and its fruitful usage requires passing to infinite sampling in infinite distance matrices. Here is the argument. For every metric triple $\tau = (X; \mu; \rho)$, let us define a map

$$F_\tau : X^N \to Mat_N$$

by the formula $F_t(\{x_n\}) = \{p(X_i, X_j)\}(i, j)$. By D_τ we denote the F_τ-image of the Bernoulli measure M_n. We call it the matrix distribution of a metric on a space with measure μ. Obviously, this is an invariant of a triple in the sense of the equivalence in question. We show that the matrix distribution is a full invariant of the equivalence of triples and that the assertion becomes a trivial corollary of the individual law of large numbers. If the two triples $\tau = (X; \mu; \rho)$, $\tau' = (X; \mu; \rho)$ have the same matrix distributions $D_\tau = D_{\tau'}$ then there is a countable everywhere dense set (since the measures are non-degenerate) $X_0 \subset X$ which is isometric to some countable everywhere dense set $X_0 \subset X$. This isometry extends to an isometry X to X. The fact that this isometry preserves the measure follows from the observation that by the law of large numbers, the same distance matrix determines both the measure of balls of all radii centered at points in X_0 and all measures in the algebra of sets generated by the balls. Thus the measures coincide in both spaces, since they are determined by the same matrix, and therefore the measures m and m are the same. As one can see the whole argument is based on the strong law of large numbers.

This formulation demonstrates, in a somehow unexpected way, the classification of triples is "smooth", and the space of invariants is the standard Borel space, namely, the space of Borel probability measures on a set of metrics on integers such that the measures are ergodic and invariant under the group of infinite substitutions. At the same time a close connection with nowadays popular theory of infinite random matrices emerges but the distribution of matrices that arise here are not the same as the ones that are considered in that theory (Gaussian Orthogonal Ensemble, Gaussiian Unitary Ensemble etc.) This connection will undoubtedly lead to new interesting results. My work [2] proves a generalization of this theorem to a more complicated case: the matrix distribution is a complete invariant of generic measurable functions in two (and more) variables with respect to the permutation group acting by transformations that preserve measure separately in each argument.

Let me note that the concept of metric triples (Gromovs triples or mm-spaces) restores the symmetry between measure and metrics which is absent in usual analysis. Traditionally, the metric is defined first and then one studies the variety of Borel measures on this metric space. However, independently and almost simultaneously the opposite disposition of considering various admissible metrics on a given measure space was suggested ([3]). This subordination is useful in the theory of dynamical systems where the principle invariants are first defined using both the measure and metric but at the end it turns out that they are independent of the choice of the latter (for instance, in defining Kolmogorov's entropy). This opens the road for constructing new invariants. However, here it is wise to impose some conditions of compatibility of the structures of mm-spaces satisfied for the triples discussed above.

4. Another well-known and productive geometrical initiative of Gromov is the idea of a space of metric spaces with a metric on it (known as the Gromov-Hausdorff metric). It is a "nonlinear" generalization of a corresponding notion in Banach geometry, but it is a way more general. It is noteworthy that introducing the concept of the Uryson universal metric space, to which several statements in the book are dedicated, allows us to simplify the definition of the Gromov-Housdorff metric. The recent application of this idea to metric triples proved to be very useful

here. The Master Thesis of M. Gromov was dedicated to Banach geometry, or more precisely to applications of topological results to it. This work remains of interest even nowadays. For instance, Gromov's results in spirit of generalizations of the Dvoretzky theorem, Levi's principle, etc. laid a foundation for subsequent work.

A gigantic theme of Gromov "Groups as a geometric objects" has no doubt opened a new chapter in combinatorial group theory. Roughly speaking, the proposition states that the familiar word metric turns groups into metric spaces which should be approached as purely metric objects. Group properties are deeply rooted into metric structures of the spaces. This is how the initiative of introducing hyperbolic groups arises. Geometric in its nature, it allows us to speak about a reformation of our understanding of applications of group theory in dynamics, topology and analysis. Gromov's celebrated theorem stating that groups of polynomial growth are virtually nilpotent might be one of the most eminent single theorems in mathematics of the XX century; this theorem as well originates from this geometric perception of groups.

And, finally, the latest paper "In a Search for a Structure" (Arxive) conveys so far preliminary thoughts of a geometric approach to entropy. This approach is connected more not to the modern but quite the opposite, quite distant in time concepts of Boltzmann and others. Simultaneously, the latest works on this topic are being comprehended. For instance, in [1] the definition of entropy is modified in a drastic and nontrivial way so that it can be applied to non-aminable groups (unlike Kolmogorov's entropy). The concept of randomness, which plays a role in Gromov's geometric repertoire, for instance, the prominent idea of random groups, is not fully supported in further work,.

This circle of ideas would be quite enough for the work of one mathematician, but this is only a small part of Gromov's accomplishments.

References

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