INVARIANT MEASURES FOR THE CONTINUAL CARTAN SUBGROUP

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To Professor Malliavin with deep respect

Abstract

We construct and study the one-parameter semigroup of σ-finite measures $\mathcal{L}^\theta$, $\theta > 0$, on the space of Schwartz distributions that have an infinite-dimensional abelian group of linear symmetries; this group is a continual analog of the classical Cartan subgroup of diagonal positive matrices of the group $SL(n, R)$. The parameter θ is the degree of homogeneity with respect to homotheties of the space, we prove uniqueness theorem for measures with given degree of homogeneity, and call the measure with degree of homogeneity equal to one the infinite-dimensional Lebesgue measure $\mathcal{L}$. The structure of these measures is very closely related to the so-called Poisson–Dirichlet measures $PD(\theta)$, and to the well-known gamma process. The nontrivial properties of the Lebesgue measure are related to the superstructure of the measure $PD(1)$, which is called the conic Poisson–Dirichlet measure – CPD. This is the most interesting σ-finite measure on the set of positive convergent monotonic real series.

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1 Introduction

The Lebesgue measure on the cone of positive vectors of $\mathbb{R}^n$ can be described as the unique (up to a multiplicative constant) σ-finite measure that is invariant with respect to the group $SDiag_+(n, \mathbb{R})$ (the positive part of the Cartan subgroup) and homogeneous of degree n with respect to homotheties. What is the infinite-dimensional analog of this fact? First of all, what is the continual analog of the Cartan subgroup? And does there exist, in an infinite-dimensional space, a measure that is invariant with respect to the Cartan subgroup and has a finite degree of homogeneity with respect to homotheties?

The goal of this paper is to introduce a family of σ-finite measures in the space of Schwartz distributions on the interval (or on a manifold) that is invariant with respect to a continual group of multiplicators and has a finite degree of homogeneity with respect to homotheties. The measure with degree of homogeneity equal to one is called the *infinite-dimensional Lebesgue measure.*

More exactly, we consider σ-finite Borel measures, in the space of Schwartz distributions on a manifold X with measure m, that are invariant under the action of the group $\mathcal{M}_0$ of operators of multiplication by positive test functions $a(\cdot)$ satisfying the condition

$$\int_X \ln a(x)dx = 0$$

. This group is a direct continual analog of the positive Cartan subgroup

$SDiag_+(n, \mathbb{R})$ of the group $SL(n, \mathbb{R})$. The above condition is an analog of the condition $\det A = 1$ for matrices.

It turns out that this invariance implies another invariance: these measures are invariant under the action of the group of measure-preserving transformations of the manifold (this is an analog of the invariance of the ordinary Lebesgue measure under the action of the symmetric group of permutations of coordinates); this group plays role of Weyl group in $SL(n, \mathbb{R})$.

In fact, we define a remarkable one-parameter family of σ-finite measures $\mathcal{L}^\theta$, $\theta > 0$, enjoying this property, and prove a uniqueness theorem which says that $\mathcal{L}^\theta$ is the unique ergodic, positive, σ-finite measure on the cone of positive Schwartz distributions that is finite on compact subsets, satisfies this invariance property, and has a fixed degree of homogeneity.

The measures $\mathcal{L}^\theta$ are closely related to the so-called Poisson–Dirichlet measures $PD(\theta)$ on the infinite-dimensional simplex and, consequently, to many combinatorial problems (see [22]).

One of the main applications of our measures is in the representation theory of current groups. The group of symmetries of the Lebesgue measure $\mathcal{L}$ (the Cartan subgroup) allows us to use the space $L^2(\mathcal{L})$ for constructing a realization of representations of the current group G^X, where G is a semisimple Lie group of rank 1. This measure was discovered in a series of papers by ([6, 7, 8]), and later ([24, 11]), and applied for constructing a new model of these representations which, instead of the Fock space with infinite-dimensional range of the one-particle subspace, uses $L^2(\mathcal{L})$. More generally, this gives a *new look on the construction of the continuous tensor product of Hilbert spaces.* We can say that our construction is a contribution to the continuing discussion of what is integration in infinite-dimensional spaces.

Another aspect, which is discussed in the papers [29, 19], is a deep parallelism between the Gaussian measure (white noise) and our Lebesgue measure. These two processes, the first one being a generalized stochastic process and the second one being a quasi-stochastic process (the term "quasi-stochastic" means that instead of a probability measure on the space of the process we use a σ-finite measure) correspond to the two endpoints of the interval $[0, 2]$ which is the set of parameters of stable Lévy processes (or stable distributions on the real line). The point 2 of the interval $(0, 2]$ corresponds to the white noise (or to the Wiener process, if one prefers to consider a process with independent increments instead of a generalized process with independent values); the point 0, as the parameter of a stable distribution, has no direct sense: *at this point, we must consider the derivative of the char-*

acteristic functional, and this gives our Lebesgue measure (see [19] for the so-called notion of zero-stable law and zero-stable distribution). The Gaussian measure (white noise) corresponding to the point 2 is invariant under the action of the infinite-dimensional orthogonal group; at the point 0, we have the invariance under the infinite-dimensional (abelian) linear group of multiplicators. The interval $(0, 2)$ can be regarded as a deformation of one measure into the other one with unknown nonlinear group of symmetries.

This paper is closely related to our previous papers [22, 20] and devoted to the same subject. In this paper, we put an end to the discussion concerning approximation of these measures with finite-dimensional invariant measures on the hyperspheres, which was started in [22] (see also [20, 25]). In that paper we insisted on an analogy between the situation with approximation of the infinite-dimensional Gaussian measure and the infinite-dimensional Lebesgue measure. In this paper we explain why there is such a drastic difference between the case of the white noise (the Maxwell–Poincaré lemma about the limit of measures on Euclidean spheres) and our case with noncompact homogeneous spaces (hyperspheres). The key difference is the difference between the σ-finiteness and finiteness of the orbital measures, or, shortly, between the noncompactness and compactness of the group of finite-dimensional symmetries (the Cartan and orthogonal groups). In a more physical language, this means the absence of equivalence between the grand and small canonical ensembles in the Lebesgue case, in contrast to the equivalence of both ensembles in the Gaussian case, see Section 4.

We want to emphasize the similarity between the construction of the measures $\mathcal{L}^\theta$ and the classical method of constructing stochastic Lévy processes as Gelfand–Itô processes. It is based on infinitely divisible measures and the corresponding characteristic functionals (in the theory of stochastic processes, there is an equivalent language of "random measures"). Unfortunately, there is no (at least to the author's knowledge) general theory of "σ-finite Lévy processes" and no Lévy–Khintchin formula for the Fourier (or Laplace) transform of infinitely divisible σ-finite measures on the real line and on $\mathbb{R}^n$. Undoubtedly, the needed definition must be very similar to the classical one, and they must be unified in a general theory of finite and σ-finite Lévy processes. But since there is no such a theory, we use special considerations in our construction, see Section 2.

Let us briefly mention the history of what we called the infinite-dimensional Lebesgue measure. It seems that this measure first appeared in the papers [7, 8], as a measure whose characteristic functional is the restriction of the canonical state on the group $SL(2,\mathbb{R})$ to the unipotent subgroup. Later it became clear that there is a link between the gamma process and the Lebesgue measure: the latter is absolutely continuous with respect to the law of the gamma process, see [19] and references therein. This link was discussed in connection with the Markov–Krein transformation [18], stable laws [22], isomorphisms with the Fock space [28]. The connections between the representation theory of current groups and the Lebesgue measure was recently considered in the joint paper of the author with M. I. Graev [24].

Here are brief comments on the contents of the paper. In the second section we recall the definitions and basic properties of some important measures and processes (Dirichlet, gamma) and introduce the measures $\mathcal{L}^\theta$, first as generalized processes or random measures. We use the link between these measures and the gamma processes. It is possible to generalize the classical theory of extension of cylinder measures in linear spaces, including the definition of Gelfand–Itô generalized processes, to the σ-finite case. The reference to the properties of the gamma processes immediately allows us to establish that the support of our measures $\mathcal{L}^\theta$ is the set of discrete positive finite measures on the manifold (the simplest type of Schwartz distributions)

$$\xi = \sum_k c_k \cdot \delta_{x_k}, \quad c_k > 0,\ x_k \in X,\ k = 1, 2, \dots .$$

In the third section we describe the properties of the measures $\mathcal{L}^\theta$ and their connection with the Poisson–Dirichlet measures $PD(\theta)$. We define the Laplace transform of $\mathcal{L}^\theta$:

$$\Psi_{\mathcal{L}_\theta}(f) \equiv \Psi(f) = e^{-\theta \int \ln f(x) dm(x)}, \quad m(X) = \theta > 0.$$

In the previous paper [22] we took the formula for the Laplace transform as a definition of these measures. We establish the invariance of our measures with respect to the continual Cartan group $\mathcal{M}_0$ (see above) and prove the uniqueness of an invariant measure with a given degree of homogeneity.

There are many papers about the Poisson–Dirichlet measures $PD(\theta)$ (see [15, 2] and the references therein, and also the earlier papers [27]). The measure $PD(\theta)$ is concentrated on the infinite-dimensional simplex of monotonic positive series with sum equal to one. It is covered by a measure on the larger simplex of all positive series with sum equal to one, which is usually called the GEM-measure (we do not mention it in the paper). It is worth mentioning that the GEM-measure was perhaps first introduced, in a different form, by W. Eberlein [5] (even for nonpositive series).

We also consider the conic measures $CPD(\theta)$ and prove the characteristic properties of these measures. The measure $CPD(\theta)$ is maybe the most interesting object. In a sense, it is more natural than the measure $PD(\theta)$. This measure is concentrated on the cone of convergent monotonic positive series and has a large group of symmetries, which comes from the symmetries of the measure $\mathcal{L}^\theta$. It is interesting to obtain a characterization of the measure $CPD(\theta)$ directly.

The fourth section is devoted to the approximation theory of the measures $\mathcal{L}^\theta$ and comparison with white noise - we show that there is the new phenomenon of the absence of approximation and absence of equivalence of grand and small canonical ensembles in the case of the Lebesgue measure. We introduce in the forth section the new function (L) (see also [23]) which is similar to free energy and must have important role in this theory. We only mention the two important generalizations of this measure: to absolutely convergent (nonpositive) series and to two-sided series; this will be done later.

2 Old and new definitions; construction of the measures $\mathcal{L}^\theta$

2.1 The spaces

Let (X, m) be a manifold with a fixed continuous finite Borel measure m. In fact, in what follows we may assume that (X, m) is the interval $[0, \theta]$, $\theta > 0$, or the unit circle, with the Lebesgue measure m: this case does not essentially differ from the case of a general manifold and even an arbitrary standard Borel space. The measure m is continuous, nonnegative, and finite, but not necessarily normalized; we put $m(X) = \theta > 0$, so that $\tilde{m}(\cdot) = \frac{m(\cdot)}{m(X)}$ is a normalized measure. We will consider measures as elements of the space

$$l^1(X) = \{\eta = \sum_k c_k \cdot \delta_{x_k} : \quad \sum_{k=1}^{\infty} |c_k| < \infty,\ x_k \in X,\ k = 1, 2, \dots\}$$

of discrete (countable) finite signed measures, and in the most part of the paper we deal only with the positive cone

$$l^1_+(X) = \{\eta = \sum_{k=1}^{\infty} c_k \cdot \delta_{x_k} : \quad \sum_{k=1}^{\infty} c_k < \infty,\ c_k > 0,\ x_k \in X,\ k = 1, 2, \dots\}$$

of the linear space $l^1(X)$; we also need to use its convex subset $s^1(X) \subset l^1_+(X)$ defined as

$$s^1(X) = \{\eta = \sum_{k=1}^{\infty} c_k \cdot \delta_{x_k} : \quad \sum_{k=1}^{\infty} c_k = 1,\ c_k > 0,\ x_k \in X,\ k = 1, 2, \dots\}.$$

We have the following obvious decomposition: $l^1_+(X) \setminus \{\mathbf{0}\} = s^1(X) \times (0, \infty)$.

For some reason, we need to consider the space of Schwartz distributions (generalized functions) $D(X)$, which contains $l^1(X)$, but we can avoid this by using an *a priori* knowledge about the structure of our measures.

We will equip the space $l^1(X)$ and the cone $l^1_+(X)$ with the weak topology that comes from the natural duality between the space $l^1(X)$ and the space of bounded continuous, or even measurable, functions on the space X:

$$< f, \eta >= \sum c_k \cdot f(x_k), \quad f \in C(X),\ \eta = \sum_{k=1}^{\infty} c_k \cdot \delta_{x_k} \in l^1(X).$$

Functions f from the dual space will be called test functions. We fix the Borel structure on the cone $l^1_+(X)$.[1] Our goal is to define and study distinguished positive Borel measures on the cone $l^1_+(X)$.

2.2 The Dirichlet and gamma distributions

We recall the definitions of several finite-dimensional measures. Let $\Sigma_n = \{(x_1, \dots, x_n) : \Sigma_{i=1}^n x_i = 1,\ x_i \geq 0,\ i = 1, \dots, n\}$ be the standard $(n-1)$-dimensional simplex. The Dirichlet distribution $D^{\bar\theta}$ on the simplex Σ_n with parameter $\bar\theta = (\theta_1, \dots, \theta_n)$ is the probability measure whose density with respect to the Lebesgue measure[2] is given by the formula

$$\Gamma(\theta) \cdot \prod_{i=1}^{n} \frac{x_i^{\theta_i - 1}}{\Gamma(\theta_i)},$$

where $\theta = \sum_1^n \theta_i$ (see [15]).

[1] This structure does not coincide with the one that can appear if one considers $l^1(x)$ as a nonseparable Banach space.

[2] When we speak about the Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$, we always mean that it is normalized so that the measure of the unit interval or cube is equal to 1.

The well-known gamma measure on the half-line $\mathbb{R}_+$ with parameter θ has the following density with respect to the Lebesgue measure:

$$\lambda^\theta(x) = \frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)}, \quad x > 0.$$

The gamma measure on the orthant $\mathbb{R}_+^n$ with parameter $\bar\theta = (\theta_1, \dots, \theta_n)$ is defined as the product of n one-dimensional gamma measures with parameters $\theta_1, \dots, \theta_n$ and has the density

$$\lambda^{\bar\theta}(x) = \prod_{i=1}^n \frac{x_i^{\theta_i-1}e^{-x_i}}{\Gamma(\theta_i)}.$$

The Dirichlet distribution with parameter $\bar\theta$ is nothing more than the projection of the gamma measure with the same parameter from the orthant to the simplex. The gamma distribution λ^θ on the half-line is an infinitely divisible probability measure which generates a Lévy process called the gamma process.

The following simple fact is a consequence of the definitions, but it plays a very important role. Let us represent the orthant $\mathbb{R}_+^n$ as the product of the simplex and the ray $(0, \infty)$; then, with respect to this decomposition, we have the following decomposition of probability measures:

Lemma 1. *Let* $\bar\theta = (\theta_1, \dots, \theta_n)$, $\theta = \theta_1 + \dots + \theta_n$. *Then*

$$\lambda^{\bar\theta} = D^{\bar\theta} \times \lambda^\theta.$$

This is a characteristic property of the gamma distributions. The same is true for the law of the gamma subordinator (see below): this measure is the direct product of its projection onto the simplex and the gamma distribution on the half-line, and this fact characterizes the gamma processes among all Lévy processes (Lukacs' theorem, see [19]).

2.3 Weak distributions, random measures, and generalized processes

Now we introduce infinite-dimensional measures. In the theory of stochastic processes, real-valued Lévy processes, as well as other types of stochastic processes, are traditionally defined with the help of so-called "random measures," i.e., measurable maps from the σ-field of measurable sets of the parametric space to the space of measurable functions (random variables) with real values (see [4, 15] and references therein). We interpret the notion of a random measure in the spirit of Gelfand–Itô generalized stochastic processes. Namely, the general definition of a generalized process, or weak distribution, is as follows. It is a self-consistent system of finite-dimensional distributions defined for each finite collection of linear functionals from a total set of linear functionals on the space $D(X)$ (i.e., a set of functionals whose linear closed hull is the whole space), for example, the set of all characteristic functions of measurable subsets of X. In our case, we will associate with an arbitrary finite measurable partition of the parametric space X a finite-dimensional distribution in $\mathbb{R}^n$ that is a self-consistent system of finite-dimensional distributions and is continuous with respect to the partitions. This is just a random additive measure in the usual sense from the point of view of the theory of generalized processes and weak distributions on the space of distributions $D(X)$. According to Minlos' theorem, such a distribution determines a true probability measure on $D(X)$ (the law of the process), see, e.g., [9]. We do not know if such a theory exists for the case of σ-finite measures, so we use direct arguments for the description of σ-finite measures (see [8, 19]).

2.4 Three self-consistent systems of distributions and the definition of the Lebesgue measure

We will simultaneously give the definitions of three one-parameter families of "random measures," or generalized stochastic (in the third case, quasi-stochastic) processes. They determine the corresponding families of measures on the cone $l^1_+(X)$. Two of them are classical and well-known: the *Dirichlet process* and its measure D^θ; and the *gamma process* and its measure (law) Λ^θ. The third one is a "quasi-stochastic" process, which produces a σ-finite measure $\mathcal{L}^\theta$; in particular, in this way we obtain the *Lebesgue measure* $\mathcal{L}$.

Fix a positive number $\theta > 0$ as a parameter and choose any vector $\bar\theta = (\theta_1, \theta_2, \dots, \theta_n)$ with $\sum_i \theta_i = \theta = m(X)$.

Assume that we fix a finite sub-σ-field of the σ-field of all measurable sets of X, or simply a partition $\xi_{\bar\theta}$, $X = \bigcup_{i=1}^n E_i$, of the space X into measurable sets E_i, $i = 1, \dots, n$, with measures $m(E_i) = \theta_i$, $i = 1, \dots, n$. The finite-dimensional distributions corresponding to this partition have the following densities with respect to the Lebesgue measure:

1. *The Dirichlet process.* The distribution corresponding to the partition $\xi_{\bar\theta}$ is the Dirichlet distribution with parameter $\bar\theta = (\theta_1, \dots, \theta_n)$:

$$D^{\bar\theta}(x_1, x_2, \dots, x_n) = \frac{\Gamma(\theta)}{\prod_1^n \Gamma(\theta_i)} \prod_1^n x_i^{\theta_i - 1} dx_i.$$

This is a distribution on the simplex.

2. *The gamma process.* The distribution corresponding to the partition $\xi_{\bar\theta}$ is the gamma distribution:

$$\lambda^{\bar\theta}(x_1, x_2, \dots, x_n) = \prod_{i=1}^n \frac{x_i^{\theta_i - 1}}{\Gamma(\theta_i)} \cdot e^{-\theta x_i}.$$

In both cases 1 and 2, the definitions yield true probability processes; in case 2 we obtain a Lévy process.

3. *The Lebesgue case.* The distribution corresponding to the partition $\xi_{\bar\theta}$ is the σ-finite measure on the orthant $\mathbb{R}_+^n$ with the following density:

$$L_{\bar\theta}(x_1, x_2, \dots, x_n) = \prod_{i=1}^n \frac{{x_i}^{\theta_i - 1}}{\Gamma(\theta_i)}.$$

This density is the same, up to a scalar, as in case 1, but here we consider it not on the simplex, but on the orthant $\mathbb{R}_+^n$. Note that in cases 2 and 3, the density is the product of one-dimensional densities, so we can say that in case 3 we also have a *"quasi-stochastic" process with independent values*;[3] and the family $\{L_\theta\}$ of one-dimensional distributions with densities $dL_\theta = \frac{x^{\theta-1}}{\Gamma(\theta)} dx$ is a *multiplicative semigroup* with parameter $\theta \in (0, \infty)$ with respect to the convolution on the half-line $(0, \infty)$: $L_{\theta_1} * L_{\theta_2} = L_{\theta_1 + \theta_2}$ (the δ-measure at 0 can be regarded as the identity element of the semigroup).

The self-consistency and continuity of all the systems above can be checked directly. In all three examples there is one parameter, $\theta = m(X)$. The first two cases are well-known, and we can conclude the existence of true probability measures: the law of the Dirichlet process D^θ and the law of the gamma process Λ^θ. It is well known that the Dirichlet measure is concentrated on the set $s^1(X)$ of discrete probability measures, and the gamma measure is concentrated on the cone $l_+^1(X)$ of finite discrete measures. For case 3, we prove the following theorem-definition.

[3] With an appropriate definition of independence.

Theorem 1. 1. *For every $\theta > 0$, the system $\{L_{\bar\theta}\}$*[4] *is a self-consistent system of finite-dimensional distributions and defines a σ-finite measure $\mathcal{L}^\theta$ on the space of Schwartz distributions $D(X)$.*

2. *All the measures $\mathcal{L}^\theta$ are concentrated on the cone $l^1_+(X)$ and take finite values on compact subsets of $l^1_+(X)$.*

3. *The measure $\mathcal{L}^\theta$ is a σ-finite measure absolutely continuous with respect to the measure Λ^θ with a nonintegrable density*

$$\frac{d\mathcal{L}^\theta}{d\Lambda^\theta}\left(\sum_k c_k \cdot \delta_{x_k}\right) = e^{\sum_k c_k}.$$

4. *In the decomposition $l^1_+(X) \setminus \{\mathbf{0}\} = s^1(X) \times (0, \infty)$ above, the measure $\mathcal{L}^\theta$ is the direct product of the Dirichlet measure and the measure L^θ on $(0, \infty)$: $\mathcal{L}^\theta = D^\theta \times L^\theta$; in particular, for the Lebesgue measure we have $\mathcal{L} = D^1 \times L$, where L is the Lebesgue measure on $(0, \infty)$.*

5. *The family of measures $\{\mathcal{L}^\theta : \theta > 0\}$ with additive parameter θ is a semigroup of measures on $l^1_+(X)$ with respect to the convolution.*

Proof. 1, 2, 3. Consider the σ-finite measure that has the following density, regarded as a function on the cone $l^1_+(X)$, with respect to the law Λ^θ of the gamma process:

$$\frac{d\mathcal{L}^\theta}{d\Lambda^\theta}\left(\sum_k c_k \cdot \delta_{x_k}\right) = e^{\sum_k c_k}$$

(this function is well defined for Λ^θ-almost all points). Denote this measure by $\mathcal{L}^\theta$. It is clear from the definition that this measure produces the same joint distributions for partitions $\xi_{\bar\theta}$ as in the definition of Λ^θ. This gives the existence of a measure with the given projections. The uniqueness of a σ-finite measure with the given proper (finite) distributions follows from the fact that the set of characteristic functions is total as a set of functionals on the space $D(X)$[5]. The finiteness of the values of the measure $\mathcal{L}^\theta$ on compact subsets follows from the boundedness of the density on compact subsets.

4. This decomposition is, of course, a consequence of the decomposition $\Lambda^\theta = D^\theta \times \lambda^\theta$ mentioned above.

5. This is an obvious consequence of the multiplicative formula for the corresponding finite-dimensional distributions. □

[4]Here θ is fixed and $\bar\theta$ ranges over all vectors $\bar\theta$ with the given sum θ of the coordinates.

[5]In the case of σ-finite measures, it may happen that the projections to some (or all) finite-dimensional factors are totally infinite measures.

We call $\mathcal{L}^1 \equiv \mathcal{L}$ the infinite-dimensional Lebesgue measure on the cone $l^1_+(X)$. Claim 4 of the theorem says that the Lebesgue measure $\mathcal{L}$ is a *"conic superstructure" of the Dirichlet measure* D^1.

3 Further properties of the measures $\mathcal{L}^\theta$

Consider the properties of the σ-finite measures $\mathcal{L}^\theta$.

3.1 The Poisson–Dirichlet measure and the decomposition of $\mathcal{L}^\theta$.

First of all, the measures $\mathcal{L}^\theta$, as well as the gamma measures Λ^θ, are concentrated on the cone $l^1_+(X)$: $\mathcal{L}^\theta(D(X) \setminus l^1_+(X)) = 0$; and since they are equivalent to the gamma measures, we can obtain properties that are valid for $\mathcal{L}^\theta$-almost all points.

Recall some definitions concerning the Poisson–Dirichlet measures $PD(\theta)$ (see [15]). Consider the infinite-dimensional simplices of positive convergent series with sum equal to 1,

$$\Sigma_1 = \{\{c_k\}_{k=1}^\infty : \sum_k c_k = 1\},$$

and positive monotonic convergent series with sum equal to 1,

$$\bar{\Sigma}_1 = \{\{c_k\}_{k=1}^\infty : c_1 \geq c_2 \geq \dots, \ \sum_k c_k = 1\}.$$

The shortest definition of the measures $PD(\theta)$ on the simplex $\bar{\Sigma}_1$ uses a map from the infinite-dimensional cube

$$Q^\infty = \{\{y_i\}_i : 0 \leq y_i \leq 1, \ i = 1, 2, \dots\}$$

to the simplex $\bar{\Sigma}_1$. First we send the cube Q^∞ to the simplex Σ_1 by the map T given by the formula

$$T(\{y_k\}_{k=1}^\infty) = \{c_i \equiv y_i \prod_{j=1}^{i-1}(1 - y_j)\}_{i=1}^\infty,$$

and then use the map M that orders the elements of the series $\{c_i\}$ by decreasing. Consider the Bernoulli measure μ_θ, $\theta > 0$, on the cube Q^∞ that is the infinite power of the measure on$[0, 1]$ with the density $\theta x^{\theta-1}$.

Definition 1. *The Poisson–Dirichlet measure $PD(\theta)$ is the image of the measure μ_θ under the map $MT : Q^\infty \to \bar{\Sigma}_1$.*

There are many papers about the measures $PD(\theta)$ (e.g., [2, 27, 17, 15]). The deep facts about the structure of the measure $PD(1)$ that were presented in [27] still have not found enough applications.

The following theorem reduces the study of the structure of the Dirichlet measure D^θ and the law of the gamma process with parameter θ to the study of the Poisson–Dirichlet measure $PD(\theta)$. Let us decompose the continual simplex $s^1(X)$ as follows. Every element $\xi \in s^1(X)$ is a series $\xi = \sum_k c_k \cdot \delta_{x_k}$, $c_k > 0$, $c_1 \geq c_2 \geq \dots$, $\sum_k c_k = 1$. The correspondence

$$J : \xi \mapsto \{(c_k, x_k)\}_{k=1}^{\infty},$$

which sends $s^1(X)$ to $\bar{\Sigma}_1 \times \prod_{k=1}^{\infty} X$, is a bijection when restricted to the set of series ξ with distinct coefficients c_k (if some c_k has a multiplicity in ξ, then the corresponding (finite) set of points x_i with this coefficient can be enumerated in an arbitrary manner). It is obvious that the set of discrete measures $\xi = \sum_k c_k \cdot \delta_{x_k}$ with distinct coefficients ($c_i \neq c_k$ for $i \neq k$) is of full measure Λ^θ; consequently, we can say that the map J is a bijection mod0 (almost everywhere). The following fact is known.

Lemma 2. *The bijection J is a measure-preserving map from the measure space $(s^1(X), D^\theta)$ to the product of measure spaces $(\bar{\Sigma}_1, PD(\theta)) \times (X^\infty, \tilde{m}^\infty)$.*

An equivalent result was proved by J. F. C. Kingman [14]. The theorem which says that the projection of the gamma measure Λ^θ to the simplex is the Poisson–Dirichlet measure $PD(\theta)$ is also presented in [2]. In order to prove this fact (see [19]), it suffices to check that the measure $PD(\theta)$ has the required system of self-consistent distributions, i.e., the same system of distributions as D^θ; this is an easy consequence of the definitions of $PD(\theta)$ and the Dirichlet distributions.[6]

Theorem 2. *Denote by $(X^\infty, \tilde{m}^\infty)$ the infinite product of the measure spaces $(X, \tilde{m})$ (here $\tilde{m}$ is the normalization of the measure m). With respect to the decomposition $l^1_+(X) \setminus \{\mathbf{0}\} = s^1(X) \times (0, \infty)$, the* (mod 0) *bijection J is a measure-preserving homomorphism which represents the measures*

$$D^\theta,\ \Lambda^\theta,\ \mathcal{L}^\theta$$

[6]The main difficulty is to prove that the gamma measure is concentrated on the space $l^1(X)$, but this is a classical fact.

as follows:

$$D^\theta = PD(\theta) \times \tilde{m}^\infty;$$

$$\Lambda^\theta = PD(\theta) \times \tilde{m}^\infty \times \lambda^\theta;$$

$$\mathcal{L}^\theta = PD(\theta) \times \tilde{m}^\infty \times L_\theta = CPD(\theta) \times \tilde{m}^\infty,$$

where $CPD(\theta) = PD(\theta) \times L_\theta$ *is the conic Poisson–Dirichlet measure. In particular, for the infinite-dimensional Lebesgue measure* $\mathcal{L}$ *we have*

$$\mathcal{L} = CPD(1) \times \tilde{m}^\infty.$$

Consider the cone of monotonic positive convergent series

$$\bar{l}^1_+ = \{\{c_k\}_{k=1}^\infty : c_1 \geq c_2 \geq \cdots \geq 0,\ \sum_k c_k < \infty\};$$

in a natural sense, this is the product of the simplex $\bar{\Sigma}$ and the half-line, together with the vertex $\mathbf{0}$: $\bar{l}^1_+ = (\bar{\Sigma} \times (0,\infty)) \bigcup \{\mathbf{0}\}$. We have defined the measure $CPD(\theta)$ on the space $\bar{l}^1_+$, or the conic superstructure of the Poisson–Dirichlet measure, as

$$CPD(\theta) = PD(\theta) \times L_\theta,$$

where $PD(\theta)$ is a measure on the simplex $\bar{\Sigma}$ and the measure L_θ on $(0,\infty)$ was defined as $dL_\theta = \frac{x^{\theta-1}}{\Gamma(\theta)}dx$. As we will see, the measure $CPD(\theta)$ has more symmetries than $PD(\theta)$. It seems that the σ-finite conic Poisson–Dirichlet measure $CPD(1)$ on the cone $\bar{l}^1_+$ (in other words, the conic superstructure) has not been considered in the literature. This is a very interesting measure, which has many natural characterizations.

3.2 The Laplace transform of the measure $\mathcal{L}^\theta$

The dual method of defining measures in a linear space consists in using the Fourier or Laplace transforms. In the case of a probability measure μ on a linear space E, for every linear continuous (and even linear measurable) functional $f : E \to \mathbb{R}$ we can consider its Fourier transform

$$\int_E e^{<f,\xi>} d\mu(\xi) \equiv \Psi_\mu(f)$$

with respect to μ, and thus obtain a positive definite functional Ψ_μ on the dual space, which uniquely determines the measure μ. In the case of a σ-finite measure, some or all nonzero linear functionals may have no finite distributions, so we cannot use this method. But, fortunately, in our case there is a total set of linear functionals that have finite distributions, and we can define the Laplace transforms of these functionals.

Theorem 3. *Assume that a test function f on the space X is positive ($f(x) > 0$) and satisfies the condition*

$$\int_X \ln f(x)dm(x) < \infty.$$

Then

$$\int_{l^1_+(X)} e^{<f,\xi>} d\mathcal{L}^\theta(\xi) \equiv \Psi_\theta(f) = e^{-\theta \int_X \ln f(x)dm(x)}.$$

The functional Ψ_θ uniquely determines the measure, i.e., there is only one measure, $\mathcal{L}^\theta$, with this Laplace transform.

Proof. Let us begin with positive step functions of the type $f = \sum_{i=1}^n d_i \chi_{E_i}$, $d_i > 0$, with equal measures of steps: $m(E_i) = \frac{\theta}{n}$, $\theta = m(X)$; put $\bar\theta = (\frac{\theta}{n}, \dots, \frac{\theta}{n})$. Then, using the definition of the finite-dimensional projections of $\mathcal{L}^\theta$, we obtain

$$\int_{\mathbb{R}^n_+} e^{-\sum <f,\xi>} dL_{\bar\theta} = \prod_i \int e^{-d_i x_i} dL_\theta(x_i)$$

$$= \prod_i d_i^{-\frac{\theta}{n}} = e^{-\frac{\theta}{n}\sum_i \ln d_i} = e^{-\theta \int_X \ln f(x)dm(x)}.$$

Then this formula can be extended by continuity to all positive functions with $\int_X \ln f(x)dm(x) < \infty$. The uniqueness follows from the totality of the set of positive functions. □

The Laplace transform of the Lebesgue measure $\mathcal{L}$ is given by the formula

$$\Psi_{\mathcal{L}}(f) \equiv \Psi(f) = e^{-\int \ln f(x)dm(x)}, \quad m(X) = \theta = 1.$$

This formula can be taken as a definition of the measure $\mathcal{L}$: this is the unique σ-finite measure whose Laplace transform is the functional Ψ (see [22]).

Remark. What are the distributions of linear functionals? It is easy to check that if a function is nonpositive on a set of positive measure, then the corresponding linear functional on the space $l^1_+(X)$ has no finite distribution, because the preimage of every measurable set (i.e., every Lebesgue set) on the real line is of infinite measure. If the condition $\int \ln f(x)dm(x) = c(f) < \infty$ is satisfied, then the functional $< f, \cdot >$ has a proper distribution with respect to the measure $\mathcal{L}^\theta$, which is simply the measure $e^{c(f)} \cdot L_\theta$. In particular, the distribution of the functional $< f, \cdot >$ with respect to the Lebesgue measure $\mathcal{L}$ is the Lebesgue measure $e^{c(f)} \cdot L$ on the half-line.

This is a big contrast with the case of the gamma measure and Dirichlet measure, for which the distribution of the linear functional on $D(X)$ generated by a function f can be very various, and the relation between this distribution and the distribution of f as a function on X is very nontrivial (this question is related to the so-called Markov–Krein transform, see [13]).

3.3 The invariance property of the measures $\mathcal{L}^\theta$

The main property of the measures $\mathcal{L}^\theta$, $\theta > 0$, and, in particular, of the Lebesgue measure is the multiplicative invariance.

Definition 2. *Denote by $\mathcal{M}(X)$ and $\mathcal{M}_0(X)$ the following groups, with respect to pointwise multiplication, of continuous bounded functions on (X, m):*

$$\mathcal{M}_0(X) = \{a : a(x) \geq 0, \int_X \ln a(x)dm(x) = 0\}$$

and

$$\mathcal{M}(X) = \{a : a(x) \geq 0 \quad |\int_X \ln a(x)dm(x)| < \infty\}.$$

The topology is defined by the following system of neighborhoods of the identity: $V_\epsilon(\mathbf{1}) = \{a \in \mathcal{M} : \int |a(x) - 1|dm < \epsilon < 1\}$.[7]

Denote

$$\phi(a) = e^{\int_X \ln a(x)dm(x)}.$$

We call $\mathcal{M}$ the *continual Cartan group*, and $\mathcal{M}_0$, the special continual Cartan group. These groups are natural continual versions of the groups $SL(n, \mathbb{R}_+)$ and $GL(n, \mathbb{R}_+)$. The group $\mathcal{M}(X)$ acts on the space $l^1(X)$, as well as on the space of test functions. Sometimes we denote by M_a the operator of multiplication by a function a: $M_a f = a \cdot f$.

Consider the group $\mathfrak{A}(X)$ of measurable transformations of the space (X, m). This group acts on the space $D(X) \supset l^1_+(X)$ as $(U_T f)(\cdot) = f(T\cdot)$ So

[7]In this framework, one may also consider the set of measurable bounded functions satisfying the same condition, but here we restrict ourselves to continuous functions.

U_T is a linear operator that acts in the spaces $D(X)$ and $l^1_+(X)$. The group $\mathfrak{A}(X)$ can be regarded as an analog of the Weyl group in the group $GL(n,\mathbb{R})$. We can also define the cross product of the groups $\mathfrak{A}(X)$ and $\mathcal{M}$.

Theorem 4 (Invariance)**.** *Let $a \in \mathcal{M}$.*

1. *The measures $\mathcal{L}^\theta$ are invariant, up to a multiplicative constant, under the action of the group $\mathcal{M}$:*

$$\mathcal{L}^\theta(M_a^{-1}A) = \phi(a)\mathcal{L}^\theta(A), \quad a \in \mathcal{M},$$

for any measurable set $A \subset l^1_+(X)$. In particular, the measures $\mathcal{L}^\theta$ are invariant under the action of the group $\mathcal{M}_0$:

$$\mathcal{L}^\theta(M_a^{-1}A) = \mathcal{L}^\theta(A), \quad a \in \mathcal{M}_0.$$

2. *The measures $\mathcal{L}^\theta$ are invariant under the action of the group $\mathfrak{A}(X)$.*

Proof. 1. Let us consider the behavior of the Laplace transform of the measure $\mathcal{L}^\theta$ under the action of the group $\mathcal{M}$. Denote for a moment $\mathcal{L}'(\cdot) = \mathcal{L}^\theta(M_a\cdot)$, $a \in \mathcal{M}$. Then

$$\Psi_{\mathcal{L}'}(f) = \int e^{-<\eta,f>} d\mathcal{L}^\theta(M_a\eta) = \int e^{-\theta<M_{a^{-1}}\eta,f>} d\mathcal{L}^\theta(\eta)$$

$$= \int e^{-\theta<\eta,M_{a^{-1}}f>} d\mathcal{L}^\theta(\eta) = e^{-\theta\int_X \ln[a(x)^{-1}f(x)]dm(x)}$$

$$= e^{-\theta\int_X[\ln f(x)-\ln a(x)]dm(x)} = \phi(a)^\theta \cdot \Psi_{\mathcal{L}^\theta}(f).$$

By the uniqueness theorem for the Laplace transform, we have

$$\mathcal{L}^\theta(M_a\cdot) = \phi(a)^\theta\mathcal{L}^\theta(\cdot).$$

So if $a \in \mathcal{M}_0$ then $\phi(a) = 1$, and the measure $\mathcal{L}^\theta$ is invariant under the operator M_a. For $a \in \mathcal{M}$, the measures $\mathcal{L}^\theta$ are *projective invariant* under the operator M_a. Below we will prove that the action of the measure-preserving group $\mathcal{M}_0$ is ergodic.

2. The second claim follows from the fact that the Laplace transform $\Psi_{\mathcal{L}^\theta}(f)$ of the measure $\mathcal{L}^\theta$ depends only on the distribution of the function f, which does not change under measure-preserving transformations. $\square$

Remark. If $a \equiv \text{const} > 0$, then $\phi(a) = a$, and we have $\mathcal{L}^\theta(a\cdot) = a^\theta \mathcal{L}^\theta(\cdot)$. For the Lebesgue measure ($\theta = 1$), we see that the measure $\mathcal{L}$ is homogeneous of degree one. This is an important property: the degree of homogeneity of the n-dimensional Lebesgue measure is equal to n, and it is natural to believe that in the infinite-dimensional case there is infinite homogeneity; but our measure $\mathcal{L}$ has homogeneity of degree one!

3.4 Uniqueness theorem for the measures $\mathcal{L}^\theta$

We will prove the uniqueness of the family of measures $\mathcal{L}^\theta$ that are invariant under the action of the group $\mathcal{M}_0$, ergodic, and subject to some conditions. Recall that $\mathcal{M}_0$ is the multiplicative group of nonnegative functions on the interval,

$$\mathcal{M}_0 = \{a : \int_0^1 \ln a(t) dm(t) = 0\},$$

which acts on the space of test functions, as well as on the space of Schwartz distributions, as a group of multiplicators.

Theorem 5. *Let $\mathcal{L}$ be a σ-finite Borel measure on the cone $l^1_+(X)$ whose Laplace transform $\Psi_\mathcal{L}$ is finite on the cone of positive test functions and continuous on this cone:*

$$\int_{l^1_+(X)} \exp\{- < f, \eta >\} d\mathcal{L}(\eta) \equiv \Psi_\mathcal{L}(f) \equiv \Psi(f) < \infty$$

for all positive bounded test functions f. Suppose that the measure $\mathcal{L}$ is invariant under the action of the group $\mathcal{M}_0$ and is homogeneous of degree θ under the multiplication by a constant: $\mathcal{L}(cE) = c^\theta \mathcal{L}(E)$.

Then $\mathcal{L} = \mathcal{L}^\theta$.

Proof. In [19, Theorem 4.2, p. 285] it was proved that a measure that is absolutely continuous with respect to the law of a Lévy process and satisfies the multiplicative invariance is one of the measures $\mathcal{L}^\theta$. We prove that instead of the absolute continuity with respect to the law of a Lévy process it suffices to assume the multiplicativity. The first step is to prove that the invariance of the measure $\mathcal{L}$ under the group of measure-preserving transformations follows from the invariance and projective invariance under the groups $\mathcal{M}_0$ and $\mathcal{M}$. The invariance under the action of the group $\mathcal{M}_0$ means that for every function $a \in \mathcal{M}_0$ we have $\Psi(a \cdot f) = \Psi(f)$ for all positive test functions

f. If we choose $f \equiv 1$, then $\Psi(a) = \Psi(1) < \infty$. This means that the functional Ψ takes the same values on the whole group $\mathcal{M}_0$, and since the measure is σ-finite, we can normalize it so that $\Psi(1) = 1$. But $\Psi(c \cdot a) = c^\theta \Psi(a) = c^\theta \Psi(1) = c^\theta$, so that Ψ is a homomorphism of the group $\mathcal{M}$ to the group of positive numbers: $\Psi : \mathcal{M} \to \mathcal{M}/\mathcal{M}_0 = \mathbb{R}_+$. Consequently, taking into account the normalization, we obtain $\Psi(f) = \exp\{-\int_X \theta \ln a(x) dm(x)\}$. Recall once again that the measures we had constructed have finite degrees of homogeneity; in the case of the Lebesgue measure, it is equal to 1. $\square$

The invariance of the measure $\mathcal{L}^\theta$ under the group $\mathcal{M}_0$ and the uniqueness theorem above, together with the relation between $\mathcal{L}^\theta$ and $CPD(\theta)$, imply the following very important characteristic property of the measure $CPD(\theta)$:

Corollary 1 (see also [22])**.** *Consider the conic Poisson–Dirichlet measure $CPD(\theta)$ on the cone $\bar{l}^1$ of monotonic positive convergent series $\{x_k\}$. Suppose that, given a probability vector $\bar{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, we divide the set of elements of the random series $\{x_k\}$ into n parts, assuming that each element independently belongs to the ith part with probability θ_i. Calculating the sum of each part, we obtain n numbers. Then the joint distribution of these n sums is the distribution $L_{\bar{\theta}}$ in $\mathbb{R}^n_+$ (see Section 2.4). Conversely, if some measure on the cone $\bar{l}^1$ satisfies this property, then this measure coincides with $CPD(\theta)$.*

Question. Assume that a bounded sequence of positive numbers $a_1, a_2, \ldots$ has the following property:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln a_k = 0.$$

Does the map $\{c_k\} \mapsto \{a_k \cdot c_k\}$ preserve the measures $CPD(\theta)$ for all θ?

The positive answer on this question is a generalization of the claim of the previous theorem.

3.5 Ergodicity of the action of the group $\mathcal{M}_0$

On first sight, the action of the group $\mathcal{M}_0$ on the cone $\bar{l}^1_+(X)$ does not change x_k, so that it is not ergodic. But this is not true.

Theorem 6. *The measure-preserving action of the group $\mathcal{M}_0$ on the measure space $(l^1_+(X), \mathcal{L}^\theta)$ is ergodic.*

Proof. The uniqueness theorem for the measures $\mathcal{L}^\theta$ already contains the ergodicity; indeed, if the measure is not ergodic, then we can decompose it into ergodic components,[8] so there is no uniqueness. But we will give a sketch of a direct proof, because it uses the structure of the action. Let us apply the above-defined isomorphism J between the space $(l^1_+(X), \mathcal{L}^\theta)$ and the product space $(\bar{l}^1 \times X^\infty, CPD(\theta) \times \tilde{m}^\infty)$. Then the action of the group $\mathcal{M}_0$ is given by the following formula:

$$M_a(\{c_k\}, \{x_k\}) = (\{a(x_{g(k)})c_{g(k)}\}, \{x_{g(k)}\})$$

where $g = g_{a,\{x_k\}}$ is the permutation of positive integers that arranges the sequence $\{a(x_k)c_k\}$ by decreasing: $a(x_{g(1)}) \cdot c_{g(1)} \geq a(x_{g(2))} \cdot c_{g(2)} \geq \dots$. The action of the group $\mathcal{M}_0$ is not a skew product, but is similar to it. Namely, if we drop the monotonic reordering of the sequence $\{x_k\}$, then the action of the group $\mathcal{M}_0$ becomes fiberwise (i.e., it does not change the sequence $\{x_k\}$) and ergodic in the fibers. But the monotonic reordering of the sequence $\{x_k\}$ is an ergodic action on the Bernoulli product X^∞; from this we can conclude that the action of $\mathcal{M}_0$ is ergodic. ☐

Some details about the measures $\mathcal{L}^\theta$ and their links with $PD(\theta)$ and $CPD(\theta)$ can be found in [22].

4 The absence of geometric approximations of the Lebesgue measures

4.1 About approximations. The case of the Gaussian measure

A measure in an infinite-dimensional space that has a big group G of symmetries, like the measure $\mathcal{L}^\theta$, can be represented as the weak limit of finite-dimensional invariant measures. Suppose we can find a dense subgroup of G that is the union of an increasing sequence of finite-dimensional subgroups

[8] Here we can do this, because the action of the group $\mathcal{M}$ on $l^1_+(X)$ is measurable (individual). It is important to mention this fact, because the group is not locally compact.

G_n of the group G. Then it is natural to describe the measure as the weak limit of invariant measures on the orbits of G_n. This method of description of invariant measures of "big" groups was called the "ergodic method" (see [20]), because, in a sense, it uses various types of individual ergodic theorems. For example, for the Gaussian measure in the infinite-dimensional linear space, the group of symmetries is the infinite-dimensional orthogonal group, and, according to the remarkable Maxwell–Poincaré lemma, this measure is the weak limit of invariant measures on the finite-dimensional spheres of increasing radii (see below; for details, see also [22]). This also proves the Schoenberg theorem on description of $O(\infty)$-invariant measures (see [1]).

One of the definitions of the white noise, regarded as a generalized process in the space $L^2(X, m)$, is in the framework of the theory of Lévy processes: the one-dimensional Gaussian measure is infinitely divisible, and the corresponding generalized process is exactly the white noise. This is parallel to the definitions of the gamma process and the "σ-finite Lévy process" from Section 2. We can also formulate this definition using a "random Gaussian measure." Another equivalent (dual) definition uses the characteristic functional, or Fourier transform. This notion makes sense not only for true σ-additive Borel measures, but also for cylinder measures, or generalized processes, which are defined only on the algebra of cylinder sets of a linear topological space with measure, or cylinder measure, μ:

$$\Psi_\mu(f) = \int_{-\infty}^{+\infty} \exp\{it\} d\mu_f(t);$$

here μ_f is the distribution of the functional f with respect to the (cylinder) measure μ. The characteristic functional is defined on the dual (conjugate) space of linear functionals on the space where the measure is defined. In the case of a Hilbert space, both spaces coincide. The white noise is defined as a true measure in the Hilbert–Schmidt extension of the space $L^2(X, m)$, but we do not need to consider this space. The white noise W is the generalized process in the space $L^2(X, m)$ (or in an arbitrary Hilbert space) with characteristic functional

$$\Psi_W(f) = \exp\{-\|f\|^2\}.$$

Let $\xi_n \equiv \xi$ be an arbitrary partition of the space X into n pieces $A_1, \dots, A_n$ of equal measure $m(A_i) = 1/n$, $i = 1, \dots, n$. Denote by E_n the n-dimensional subspace of $L^2(X, m)$ that consists of all functions that are constant on all A_i,

$i = 1, \ldots, n$, with the induced norm. Denote by ρ_ξ the normalized Lebesgue measure on the $(n-1)$-sphere $E_n \bigcap S_1$, where $S_1 = \{f \in H : \|f\| = 1\}$ is the unit sphere in H. Obviously, we can regard ρ_ξ as a cylinder measure.

Theorem 7. *The limit of the sequence of characteristic functionals of the cylinder measures ρ_{ξ_n} is the characteristic functional of the white noise.*

We will not go into the details, but a consequence of this theorem is as follows.

Corollary 2. *The measure generated by the white noise (which is defined in the Hilbert–Schmidt extension of $L^2(X, m)$) is the weak limit of the finite-dimensional measures ρ_{ξ_n}.*

Proof. We have

$$\lim_n \int_{O_n} \exp\{i < s, x >\} d\omega_n(x) = \lim_n C_n \left[\int_0^1 (1-r^2)^{\frac{n-3}{2}} \cos(\|s\|\sqrt{n} r) dr \right]$$

$$= \lim_n \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{\|s\|\sqrt{n}}\right)^{\frac{n-3}{2}} \cdot J_{\frac{n-3}{2}}(\|s\|\sqrt{n}) = \lim_n \left[-\frac{\|s\|^2}{2} + \frac{\|s\|^4}{2 \cdot 4(1+2/n)} - \ldots \right]$$

$$= \exp(-\frac{\|s\|^2}{2}),$$

where $s \in \mathbb{R}^n$; O_n is the $(n-1)$-sphere of radius $\sqrt{n}$; $x \in O_n$; ω is the normalized Lebesgue measure on the sphere O_n; and $J_m(\cdot)$ is the Bessel function. Here we have used the standard asymptotics of the Bessel function J_m and the gamma function, see [10, 1, 22]. ☐

Remarks. 1. The calculations above are equivalent to the proof of the Maxwell–Poincaré (MP) lemma, which claims that the weak limit of the normalized Lebesgue measures on the spheres $S^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^\infty$ of radius $\sqrt{n}$ is the standard Gaussian measure in the space $\mathbb{R}^\infty$. See [18] for details and history. But the conclusion of our theorem involves another space rather than $\mathbb{R}^\infty$, and approximation in our case is quite different.

2. Theorems that claim that an infinite-dimensional measure with an infinite-dimensional group of symmetries is the weak limit of measures on the orbits of finite-dimensional groups can be regarded as claims about the equivalence of the small and grand canonical ensembles in statistical physics. In the case of the Gaussian measure, this analogy can be pursued further.

4.2 The case of the infinite-dimensional Lebesgue measure

Is it possible to approximate our measures $\mathcal{L}^\theta$ in the same way using the orbits of the finite-dimensional Cartan subgroups? The similarity between the Gaussian and Lebesgue cases is obvious: both measures are obtained by the same construction of Lévy processes, the first one being generated by the semigroup of Gaussian measures on the line with density $G(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, and the second one, by the semigroup of σ-finite measures with densities $L_\theta(x) = \frac{x^{\theta-1}}{\Gamma(\theta)}$ on the half-line. Nevertheless, the difference between the infinite-dimensional Gaussian and Lebesgue measures is also big: the group of symmetries $\mathcal{M}_0$ is abelian and does not contain a dense subgroup that is the limit of compact groups, as in the case of the orthogonal group $O(\infty)$. But it has many finite-dimensional noncompact subgroups. So we can consider the orbits of those subgroups that are smooth noncompact manifolds in the cone $l^1_+(X)$.

In comparison with the Gaussian case, instead of the n-dimensional spheres $S^n_{r_n}$ of radius $r_n = c\sqrt{n}$ used in the Maxwell–Poincaré lemma, we must consider a hypersurface, the "hypersphere" in $\mathbb{R}^n$ defined as

$$M_{n,r} = \left\{(y_1, \dots, y_n) : \prod_{k=1}^{n} y_k = r^n,\ y_k > 0,\ k = 1, \dots, n\right\}.$$

The number $r = r(n)$ will be called the *radius* of the hypersphere; it depends on n. This hypersphere is a homogeneous space of the positive part $SD_+(n)$ of the Cartan subgroup of $SL(n, \mathbb{R})$. Now let $\mu_{n,r}$ be the invariant σ-finite measure on $M_{n,r}$ with some scaling (= a choice of a set of measure 1).

Now we formulate a precise statement which shows that in the Lebesgue case there is no approximation of this type. A detailed version can be found in [23].

We want to find the asymptotic properties of the invariant measure on the positive Cartan subgroup $SDiag_+$ (consisting of positive diagonal matrices) of the group $SL(N, \mathbb{R})$ as N tends to infinity. More exactly, we want to find the *Laplace transform $D_n(\cdot)$ of the invariant σ-finite measure on the hypersphere $M_{n,r}$*:

$$D_n(f) = \int_{M_{n,r}} \exp\{-\sum_{k=1}^{n} y_k \cdot f_k\} dm_n(y),$$

where $f = (f_1, \ldots, f_n)$ are the dual variables. We introduce the function F_n on the positive half-line $\mathbb{R}_+$ which is sometimes called the Mellin–Barnes function (see, e.g., [16]):

$$F_n(\lambda) = \int_{H_n} \exp\{-\lambda \sum_{k=1}^{n} \exp x_k\} dx,$$

where $H_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{k=1}^n x_k = 0\}$.

After some transformations we obtain the formula

$$D_n(f) = F(\rho_n(f) r_n),$$

where $\rho_n(f) = (\prod_{k=1}^n f_k)^{\frac{1}{n}}$.

The function F_n is the inverse Mellin transform of the nth power $\Gamma(s)^n$ of the gamma function (up to the multiplier n, which we can omit). Thus the functions $F_n(\lambda)$ and $\Gamma(s)^n$ represent a "Mellin pair" [16]. For example, for $n = 1$ the Mellin pair is $\exp s$ and $\Gamma(s)$.

Thus our problem reduces to the calculation, by the saddle point method, of the integral

$$F_n(\lambda) = \frac{1}{2\pi i n} \int_{\gamma - i\infty}^{\gamma + i\infty} [\Gamma(s)]^n \lambda^{-ns} ds,$$

or, in the "real" form (put $s = \gamma + it$),

$$F_n(\lambda) = \frac{1}{2\pi i n} \int_{-\infty}^{+\infty} [\Gamma(\gamma + it)]^n \lambda^{-n(\gamma + it)} dt.$$

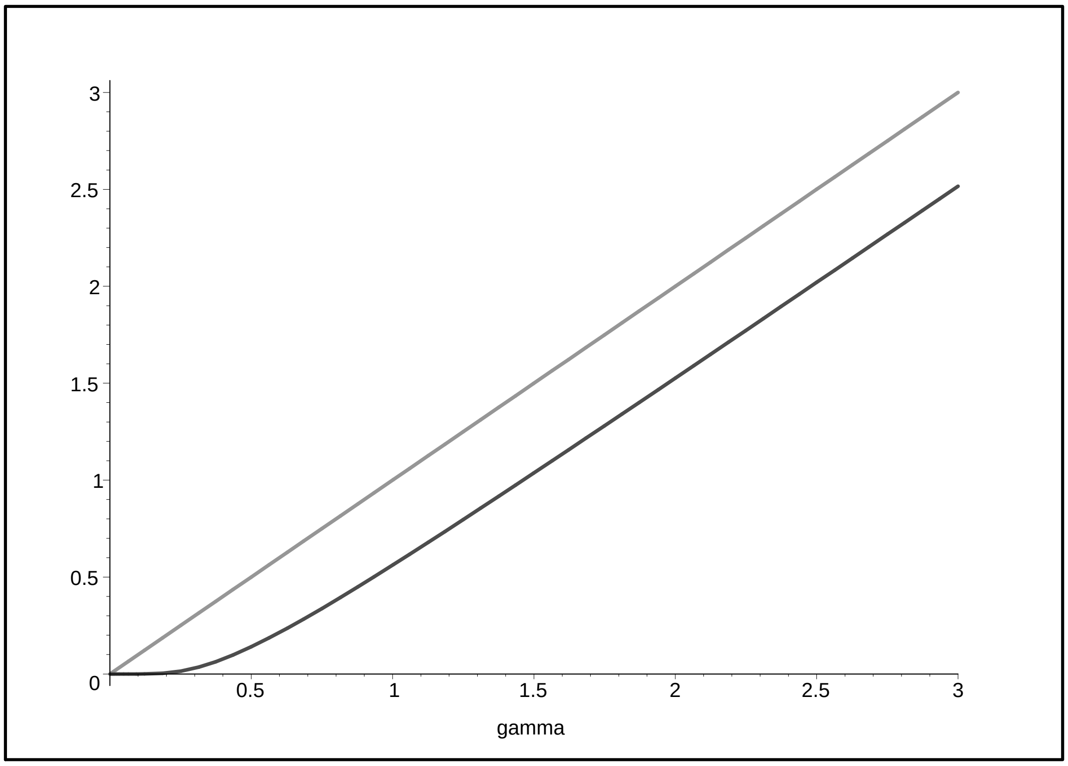
We want to find an appropriate limit of this sequence as $n \to \infty$. It happens that a suitable saddle point is $(\gamma, 0) \in \mathbb{C}$ where γ is the root of the following equation:

$$\ln \lambda = \frac{\Gamma'(\gamma)}{\Gamma(\gamma)},$$

or

$$\lambda = \exp\left\{\frac{\Gamma'(\gamma)}{\Gamma(\gamma)}\right\}.$$

The relation between these parameters is expressed by the following graph:



Consider the function

$$L(\lambda) \equiv \lim_{n} \frac{\ln F_n(\lambda)}{n}.$$

The calculation presented in detail in [23] yields the following result:

Theorem 8. *The function L satisfies the following equation:*

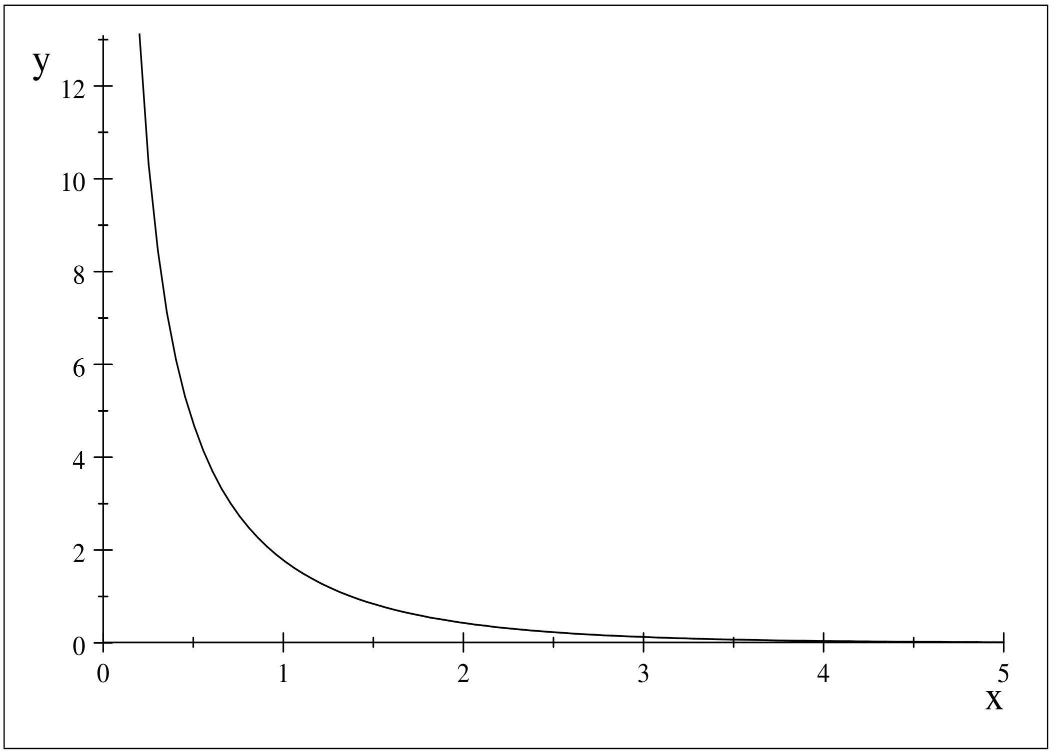
$$L(\lambda) \equiv \lim_{n} \frac{\ln F_n(\lambda)}{n} = \frac{\Gamma(\gamma(\lambda))}{\lambda^{\gamma}},$$

where γ and λ satisfy the equation

$$\lambda = \exp\left\{\frac{\Gamma'(\gamma)}{\Gamma(\gamma)}\right\}.$$

This answer proves the existence of the remarkable function L and means that there is no convergence of the functions $F_n(\lambda)$ as $n \to \infty$ and no convergence of the Laplace transforms D_n (*for any choice of the radius* r_n) of the invariant measures on the hyperspheres $M_{n,r(n)}$ to a finite limit. Thus F_n does not tend to the value of the Laplace transform of the measure $\mathcal{L}^\theta$ at the constant function identically equal to λ, which is finite and equal to $\frac{1}{\lambda^\theta}$. Consequently, there is no weak convergence of the measures m_n (in the sense of convergence of Laplace transforms). So we cannot represent the measure $\mathcal{L}^\theta$ as the weak limit of finite-dimensional $SDiag_+$-invariant measures. In turn, we can say that there is no equivalence of the grand and small canonical ensembles for the Cartan subgroups. It seems that the "tail" of the hypersphere $M_{n,r}$ carries too much part of the measure in order to preserve the finiteness of the limit and violate the equivalence of the ensembles.

The function L is a very interesting object; author does not know if it has been studied in the literature. It looks like the free energy in statistical mechanics and plays the role of the generator of the family of invariant measures on the hyperspheres. The graph of L shows that it is similar to the graph of the function $-\ln$, which is the true generator of the one-parameter semigroup of one-dimensional Lebesgue measures (L_θ):



Concluding remarks.

1. It is very easy to construct the family of Lebesgue measures on the whole space $l^1(X)$ instead of the measures $\mathcal{L}^\theta$ on the cone $l^1_+(X)$: we need only to replace the gamma subordinator in the definition of $\mathcal{L}^\theta$ with the whole (nonpositive) gamma process. The measures L_θ on the half-line must be replaced in all definitions with the measure on the whole line that is the symmetric extension of L_θ. Apparently, the results of the theory of Poisson–Dirichlet measures, including those of [26], have never been generalized to this case of nonpositive series.

2. Another aspect of the theory we discussed in this paper concerns generalization of the Lebesgue measure to the set of *two-sided series*, which will be considered elsewhere.

Acknowledgements. This work was partially supported by the grants RFBR 08-01-00379 and NSh-2460.2008.1. The author is grateful to the Erwin Schrodinger Institute (Vienna) for the possibilities to finish this paper during the semester "Statistical Physics and Asymptotic Combinatorics"; and to Natalia Tsilevich for her great help with editing this manuscript.

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