Invariant measures on the set of graphs and homogeneous uncountable universal graphs

F. V. Petrov, A. M. Vershik*

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Abstract

We describe the set of all invariant measures on the spaces of universal countable graphs and on the spaces of universal countable triangles-free graphs. The construction uses the description of the $\mathfrak{S}_\infty$-invariant measure on the space of infinite matrices in terms of measurable function of two variables on some special space. In its turn that space is nothing more than the universal continuous (Borel, topological) homogeneous graphs — general or triangle free, — existence of which we establish.

1 Introduction

1.1 Problem and results.

Fix a countable set V and consider the set $\mathcal{G}$ of all graphs (non-directed, without loops and without multiple edges) with the set V as a set of vertices. The infinite symmetric group $\mathfrak{S}^V$ of all permutations of V acts naturally on the set of graphs $\mathcal{G}$. Equip $\mathcal{G}$ with the natural weak topology, i.e. topology of pointwise convergence (stabilization) of the edges, the action of the group $\mathfrak{S}^V$ is continuous with respect to weak topology on $\mathcal{G}$, and to the weak topology on the group $\mathfrak{S}^V$ itself.

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We are interested in the Borel probabilistic $\mathfrak{S}^V$-invariant measures on the set of graphs $\mathcal{G}$. More exactly, we consider $\mathfrak{S}^V$-invariant classes of graphs - $\mathcal{I} \subset \mathcal{G}$ (for example — universal graphs) and want to describe the set of all probabilistic Borel $\mathfrak{S}^V$-invariant measures μ on this class. The stable subgroup of a graph under this action is a group of automorphism of the graph; and the condition of the invariance means that the measure does not depend on possible additional structures on graph (ordering and so on).

If the class I is the class of the universal graphs in the sense Rado-Erdos-Renji, then there are well-known examples of such measures - this is so called random graph ([10], also see [9]): the probability that pair of vertices v_1, v_2 is an edge equals to $p, 0 < p < 1$ for all pairs of vertices and edges as random (0-1)-variables are independent. The corresponding measure on the space of graphs is of course $\mathfrak{S}^V$-invariant measures and evidently (this was remark of Erdos and Renji in [10]) with probability one such graphs are isomorphic to the countable universal graph. The new question arises - are those measures exhaust the list of invariant measures on the class of universal graphs? As we will prove, besides Bernoulli measures *there are many other examples of $\mathfrak{S}^V$-invariant ergodic measures on the universal graphs*, and its description reduces to the description of uncountable Borel universal graphs.

It is well-known that there exist a universal triangle free graph (see below). But the problem about existence of invariant measure on the set of universal triangle free graphs was still unsolved. It was even doubt if such a measure exists. The corollary of the results of the papers [14, 15] was that the limit of the uniform measures on the finite triangle free graphs with n vertices when n tends to infinite is the measure on the space of infinite matrices concentrated on the set universal bipartite graph, which are not universal triangle free graph.[1]. This means that uniform measure as approximation tool is too rough for obtaining needed measure. But *we will prove that invariant measures on the set of universal triangle free graphs do exist (Theorem 5).* The main technical result of this paper (Theorem 4) is existence of topological universal (or triangle free universal) graphs with the real line $\mathbb{R}$, as the set of vertices, and with shift invariant graph structure.

Because each graph with the set of vertices V can be identifies with $0-1$ adjacent matrix on $V \times V$, our problem is the problem of the construction of the measures on the space of $0-1$- matrices or the problem of the definition of set of *random matrices*. For example, in the terms of adjacent matrix Erdos-

[1]We are grateful to Professor G.Cherlin for the reference on those papers

Renji measure on the graph is a Bernoulli measure on the space of symmetric 0-1-matrices with zero diagonal. So our problem can be reformulated in terms of the theory of the $\mathfrak{S}_\infty$-invariant measures on the space of matrices or tensors, This theory was started with the papers of D.Aldous [1] who gave the description of these measures (see also [11]). The second author of the paper connected this question with the classification of measurable functions of several arguments, and had proved that the measures are complete invariant of the classification in the generic case [2]. This gives new proof of Aldous's theorem using so called ergodic method. For our examples we use the simplest part of the theorem, but for the description of all measures we need the classification theorem in whole generality. It is interesting that those classes of the universal graphs give the new examples of the Kolmogorov's effect (see [13]): the action of the group is *transitive* on the set of vertices, edges etc. but there are a uncountable many of ergodic invariant measures.

But for solution of our problem we must construct the "measurable universal graph". This notion, perhaps has a proper interest in the theory of models and "continuous combinatorics".[2]. It appears step by step. Firstly we must switch the definitions of universality from countable models to uncountable graphs to the standard Borel space with measure, then introduce the topological structure on the set of vertices and finally group-topological structures on the set of vertices. The reason is that the notion of Borel universality is not constructive, and it is difficult to check the conditions of universality. But it is better to construct concrete topological universal graph using group structure (we use $\mathbb{R}$). This allows us to construct needed open sets and to check the conditions of universality and triangle free universality for continuous graph. This reminds us the construction of the (complete) universal Urysohn space as a completion of the universal rational space (see [3, 13]), our construction in the last section looks very similar.

Professor T.Tao informed the second author that the close ideas already appeared in the recent papers of L.Lovasz and coauthors [5, 6] where the authors define the analog of the continuous weighted graph which is roughly speaking measurable function of two variables on the unit interval with Lebesgue measure; in [7] this notion was associated with Aldous's theorem. But our goals are different, and constructions also differ from the construction in [5, 6]: we consider non-weight graph but with group-invariance structure.

[2]The notion of the continuous graphs in general must be very useful in the variational calculus and geometrical optimal control etc.

For example we do not know if it is possible to use for our construction a compact group G as the set of vertices with G-invariant graph structure.

Let us shortly describe the content of the paper. After reminding the classical examples in this section, we formulate the results of the theory of invariant measures on the matrices (section 2). In the section 3 we define the notions of the Borel universal graphs with measure: the problem of the description of invariant measure can be reformulated in the terms of those graphs.

The next step is a new definition of topological universal object: we omit a measure as a ingredient of the definition, and instead use the assumption about the openness which guarantees that each non-degenerated measure on the topological universal space gives the Borel universal space with measure. So the problem reduces to construction of the topological universal graphs. The main results are contained in the section 4 where we formulate the group topological universality and give the concrete examples of the structure of invariant universal graph on the real line. *In fact we construct the universal countable graph (a universal countable triangle free graph) on the set* $\mathbb{Q}$ *-of rational numbers and invariant under the action of the group of translations on* $\mathbb{Q}$ *and then extend the graph structure on the whole real line* $\mathbb{R}$. This is similar to the construction of Urysohn of the universal metric space ([3]). All those considerations we give simultaneously for the universal and triangle free universal graphs. It is very plausible that it is possible to formulate the analog for general category in framework of Fraisse theory (see [8]). Some comments and questions we formulate in the section 5.

Shortly speaking our scheme looks like the following transitions:

universal Borel graph with measures → universal topological graph → universal homogeneous topological graph → invariant measures on the countable universal graphs.

1.2 Universal graph

The question above is especially important for the class of *universal graphs* and we start with it. The universal graph Γ_u was defined by R.Rado (see[12]).

This graph is characterized by two properties:

(i) any finite graph γ may be embed to Γ_u

(ii) for any two isomorphic finite induced subgraphs γ_1, γ_2 of Γ_u any isomorphism between them may be extended to the isomorphism of the whole graph Γ_u.

Equivalent property is the following:

(iii) for any finite subgraph $\gamma \subset \Gamma_u$, any 2-coloring of vertices of γ in black and white there exists a vertex $v \in \Gamma_u$, which is joined with the white vertices of γ and is not joined with the black.

It is known that the universal graph is unique up to isomorphisms and so the set of such subgraphs of $\mathcal{G}$ is one orbit of the action of the group $\mathfrak{S}^V$ on the space of graphs $\mathcal{G}$.

The remark of Erdos and Renji [10] consists in the simple observation, that if for any pair of vertices a, b, draw edge $a - b$ with probability $p_{a,b} = p \in (0, 1)$ independently on other edges, then with probability 1 the obtained random graph is the *universal graph* Γ_u of Rado. This is why universal graph is sometimes called (incorrectly) "random graph".

The measures above evidently are invariant under the group $\mathfrak{S}^V$ of all permutations of vertices.

Is it possible to define $\mathfrak{S}^V$-invariant measures on the set of universal graphs of the different type? We answer on this question below in positive.

We can put the same question for other type of universal objects.

1.3 Universal triangle free graph.

Consider more complicated class of graphs, for example, triangles-free graph (see [14]). In this case also it is possible to define universal graph because the Fraisse axioms [8] fulfill.

A graph $\Gamma \in \mathcal{G}$ is called a universal triangles-free graph, if Γ does not contain triangles and satisfies property (ii). Equivalent formulation is that

(iii') graph Γ does not contain triangles and satisfies the following property: for any finite subgraph $\gamma \subset \Gamma_u$ and any 2-coloring of vertices of γ in black and white such that there are no edges between two white vertices there exists a vertex $v \in \Gamma_u$, which is joined with the white vertices of γ and is not joined with the black.

The question was how to define invariant measure which is concentrated on the universal triangle free graphs - the trick which is good for the Erdos-Renji random graph evidently does not work in this case.

Moreover, in the papers [14, **?**] very interesting effect have been found: in our terms it means that the weak limit of the sequence of uniform measures on the set of all finite triangle-free graphs with n vertices when n tends to infinity does exist, and is an $\mathfrak{S}^V$-invariant measure, but it is concentrated on the 2-colored (=bipartite) graphs, so the support of the limit is not the set of universal triangles-free graphs, as one could guess, but this last set has zero limit measure. Roughly speaking the explanation of this effect can be done easily: the set of bipartite graphs (=set of graphs without odd cycles) is asymptotically of measure one for uniform measure on the finite triangle free graphs. Thus, the limit of the uniform measures on the set of triangle free graphs does not give the answer on our question about existence of $\mathfrak{S}^V$-invariant measures on the set universal triangle-free graphs, up to now it was not known if such measure exists. Below we give the examples. Remark that in the Erdos-Renji case the Bernoulli measure with probability of edges 1/2 is just the weak limit when n tends to infinity of the sequence of the uniform measures on sets of all graphs with n-vertices. But in the triangle-free case the uniform measure occurred to be too rough and does not reflect the difference between triangles-free and 2-colored graphs.

2 Description of the invariant measures on the set of matrices

2.1 Standard Borel sets and standard measure spaces

Recall that standard (uncountable) Borel space X is a space with fixed sigma-field of the subsets, which is Borel isomorphic to the interval $[0, 1]$ equipped with sigma-field of Borel subsets.

We will consider the measures on the standard Borel sets or standard (Lebesgue) measure spaces. This is the measure space with probabilistic, continuous totally additive measure which is isomorphic in the sense of measure theory (up to set of zero measure) to interval $[0, 1]$ with Lebesgue measure.

2.2 Description of $\mathfrak{S}^{\mathbb{N}}$-invariant measures on the space of matrices.

Let us prepare the description of the measures on the set of graphs. It is convenient to realize any countable graph (element of $\mathcal{G}$) as its adjacent matrix: suppose that the set of vertices V is the set of the natural numbers - $\mathbb{N}$, and the entries of adjacent matrix corresponding to the graph γ are defined as follows: $\varepsilon_{i,j} = 1$ iff (i, j) is an edge and $\varepsilon_{i,j} = 0$ vice versa. The action of the group $\mathfrak{S}^V = \mathfrak{S}^{\mathbb{N}}$ is the simultaneous permutations of the rows and columns. So we must define a Borel measure on the set of matrices

$M_{\mathbb{N}}(\{0;1\})$ which is invariant under that action. We quote a description of such measures.

Our method of description of measure is based on the result from [1, 2] about the structure of the random invariantly distributed matrices. More exactly, we consider the measures on space of matrices $M_{\mathbb{N}}(K)$, where K is an arbitrary Borel space - in our case we have $K = \{0;1\}$; the measures are invariant under simultaneous permutations of rows and columns e.g under the action of the group $\mathfrak{S}_{\mathbb{N}}$. The following fact was proved in [1],(see also [2]):

Theorem 1. *Each Borel probability measure μ on the space $M_{\mathbb{N}}(0;1)$ of symmetric $0;1$-matrices which is $\mathfrak{S}^{\mathbb{N}}$- invariant and ergodic with respect to the action of this group, has the following feature:*

there are the standard Borel (Lebesgue) space (Y,m) with probability measure m, and auxiliary standard Lebesgue space (Ω,ν),

and

the Borel measurable function with values $\{0;1\}$ of three variables $y_1,y_2 \in Y, \omega \in \Omega; (y_1,y_2,\omega) \mapsto f(y_1,y_2,\omega)$ (eg. characteristic function of measurable set $E \subset Y \times Y$), symmetric in the first two variables [3]*;*

such that the measure $\mu \equiv \mu(m,\omega;f)$ is the image of the product of Bernoulli measure $m^{\mathbb{N}}$ in the space $Y^{\mathbb{N}}$ and Bernoulli measure on $(\Omega^{\mathbb{N}\times\mathbb{N}},\nu^{\mathbb{N}\times\mathbb{N}})$ under the map

$$D : Y^{\mathbb{N}} \times \Omega^{\mathbb{N}\times\mathbb{N}} \to M_{\mathbb{N}}(0;1)$$

$$D(\{\{y_i\}_{i=}^{\infty}, \{\omega_{i,j}\}_{i,j=1}^{\infty}; \quad i > j, i,j = 1\dots\}) \rightarrowtail \{f(y_i,y_j,\omega_{i,j})\}_{i,j},$$

where $\{y_i\}_{i=1}^{\infty}, \{\omega_{i,j}\}_{i,j=1}^{\infty}$ are all independent random variables, - y_i i.i.d. with the distribution m on Y and with distribution ν to a measure ν on space (Ω,ν).

There are two extremal cases of the conclusion of the theorem:

1. The function f (or the set E) does not depend on y_1,y_2 (more exactly, Y is one-point space), and is a function of argument $\omega \in \Omega$,

2. The function f (set E) does not depend on ω (more exactly Ω is one-point space), and is function of y_1,y_2; - in this case all entries $e_{i,j}$ of the

[3]it is also convenient to describe the function f of three arguments as actually the random symmetric function on $Y \times Y$ of two variables, depending on $\omega \in \Omega$ as a random parameter which is equipped with a measure ν.

matrix $\{f(\omega_{i,j}\}_{i,j})$ -are independent; and the image of Bernoulli measure $m^{\mathbb{N}}$ under the map D on the space of matrices (in [13] this measure is called *matrix distribution* of function (set)) is *metric invariant* of the set function f (set E) under the group of simultaneous action on both arguments (e.g. on the space $Y \times Y$) of the group of all measure preserving transformations of the space (Y, m). In this case we can say more -see below.

We will see later that first case (1.) gives the distribution on the adjacent matrices of Erdos-Renji type - "traditional random graph". More important for us is the second type of invariant measures on the space of adjacent matrices which gives the new examples. For the universal triangle free graphs only second type of measures is available.

The first proof of Theorem 1 was given in the paper [1], the simplification of the proof and important additional details - see in [2]. The easiest part of proof which we will use below is the claim that the procedure in the theorem indeed gives invariant measure. The second part is more delicate and uses some measure-theoretic constructions. The additional result of [2] claims that for generic (more exactly -for so called "pure) functions f of two variables y_1, y_2 the measure $\mu(f, m)$ on the space of matrices which called in [2] -distribution of the function f- is complete invariant of the functions f with respect to the simultaneous action of the group of measure preserving transformations of the space (Y, m), $T : Y \to Y, \quad Tm = m$; in other words measure μ is complete invariant under transformations of the functions: $f(\cdot, \cdot) \mapsto f(T(\cdot), T(\cdot))$.

3 Universal Borel and Universal Topological Graphs

For our goals we need to define several notions of continuous universal objects. Here we consider the case of graphs only; no doubts that the definitions can be extended to more general situation in the framework of the model theory ([8]).

3.1 Universal Borel Graph

Definition 1. *The structure of the graph (undirected, without loops) with the set of vertices X is a pair (X, E) where $E \subset X \times X$ is a symmetric set with empty intersection with the diagonal $diag(X \times X)$.*

It is easy to describe additional assumptions on the graph (like triangles-free, etc.). If set X is a standard Borel space and E is a Borel symmetric subset of $X \times X$ with empty intersection with diagonal, then we say that (X, E) is Borel graph.

The definition of the *universal Borel graph* is more delicate - we need to use a measure on the space X.

Definition 2. *Universal Borel graph with measure (correspondingly — universal triangles-free graph with measure etc.) is triple* (X, E, m) *where* X *is a standard Borel space,* (X, m) *is standard (Lebesgue) space with continuous probability measure, and pair* (X, E) *is a Borel graph, such that the following condition hold*

for almost all with respect to Bernoulli measure m^∞ *in the space* X^∞ *sequences* $\{x_k\}_{k=1}^\infty \in X^\infty$ *the induced countable graph with vertices* x_k *is universal countable graph (correspondingly universal countable triangles-free graph).*

Suppose that we already have such a universal Borel graph (universal triangles-free Borel graph) with measure in the sense of this definition; let $E \subset X \times X$ be the corresponding set of edges.

Theorem 2. *Suppose that a triple* (X, E, m) *is universal Borel graph with measure* m *(correspondingly universal triangles-free Borel graph with measure* m*) and* f *is characteristic function of the set* E*; then the measure* $\mu = \mu(m, E)$*, given under the scheme of Theorem 1 (with one-point space* $\Omega = \{\cdot\}$*) is the* $\mathfrak{S}^{\mathbb{N}}$*-invariant measure on the set of matrices* $M_{\mathbb{N}}(0, 1)$ *which is concentrated on the adjacent matrices of the universal countable graphs (correspondingly — on the triangles-free universal countable graphs).*

This theorem is a direct corollary of the easiest part of the theorem 1.

Remark 1. *From the second part of Theorem 1 we can conclude that each* $\mathfrak{S}^{\mathbb{N}}$ *invariant ergodic measure on the set of adjacent matrices which is concentrated on the adjacent matrices of universal triangles-free graphs can be obtained from the universal triangles-free Borel graph with measure* m *as in Theorem 2.*

Indeed, in the scheme of the Theorem 1, which gives all possible $\mathfrak{S}^{\mathbb{N}}$ invariant ergodic measures, the space Ω must be one-point space, because in opposite case triangles-free condition does not hold.

Thus, we reduce the construction of the needed examples of the measures on the set of universal graphs to the concrete examples of the universal Borel graph (universal triangle free Borel graph) with measure.

3.2 Universal Topological Graph.

Unfortunately it is not easy to check the conditions of the definition of universal Borel graphs with measure. For this reason we will give more restrictive definition of *topological universality* with the conditions which are easier to check.

Define the universal topological graph. For simplicity we assume that X is Polish space (=metric separable complete space), but this is not necessary.

For $Y \subset X$ denote its complement $Y' = X \setminus Y$ and the closure $\bar{Y}$.

Definition 3. *Let X be a Polish space and $E \subset X \times X$ be a Borel symmetric set with empty intersection with diagonal; denote for $x \in X$ the section: $E_x = \{y \in X : (x,y) \in X\}$. The pair (X,E) will be called topological universal graph (correspondingly, topological universal triangles-free graph) if the set E has the following property:*

(U) For any two disjoint finite sets $\{a_1, \dots a_n\} \in X$ and $\{b_1 \dots b_m\} \in X$, the intersection

$$\bigcap_{i,j} (E_{a_i} \cap E'_{b_j})$$

is nonempty subset in X and has non-empty interior;

correspondingly, for triangle free case instead of the condition 2 we have the condition:

(UTF) For any two points $a, b \in X$ such, that $(a,b) \in \bar{E}$ and any $c \in X$, either $(a,c) \notin \bar{E}$ or $(b,c) \notin \bar{E}$) (absence of triangles), and for any two disjoint finite sets $\{a_1, \dots a_n\} \in X$ and $\{b_1, \dots b_m\} \in X$ such that $a_j \notin \overline{E(a_i)}$ for $1 \leq i \leq j \leq n$, the intersection

$$\bigcap_{i,j} (E_{a_i} \cap E'_{b_j})$$

has non-empty interior in X).

As it is seen from the definition, in triangles-free case it is convenient to take closed set of edges.

Recall that Borel measure on the Polish space is called non-degenerated if it is positive on all nonempty open sets,

Theorem 3. *Let (X, E) be topological universal graph (corr. topological universal triangle free graph); then for each non-degenerated Borel probability measure m the triple (X, E, m) is Borel universal graph (corr. Borel universal triangle free graph) in the sense of the definition of the previous section.*

Proof. Let m be non-degenerated measure on X. Then, almost all sequences $\{x_k\}$ with respect to Bernoulli measure m^∞ are everywhere dense on X. We have to check that the property (iii) (correspondingly (iii')) of the section 1.2 and 1.3 is valid for almost all (with respect to Bernoulli measure m^∞) sequences $\{x_k\}$. For each n vertices of randomly chosen vertices in X this condition holds because of openness of E and density of the sequence of the set E. Then it suffices to note that the intersection (over n) of a countable number of events each holding with probability 1 also holds with probability 1.

The proof for the case of triangle free graphs is the same. □

Comparing Theorems 2 and 3 we obtain

Corollary 1. *Each non-degenerated measure on the topological universal (correspondingly triangle free) graph generates the $\mathfrak{S}^{\mathbb{N}}$- invariant measure on the universal (correspondingly — universal triangle free) graphs.*

Now we must prove the existence of topological universal (correspondingly — universal triangle free) graphs.

4 Construction of the continuous homogeneous graphs.

We will describe the construction of a universal continuous graph.

In our examples the space X of the vertices of the graph subtracted more serious restrictions, — we will equip this space with a group structure — X will be the real line $\mathbb{R}$. In general if G is a group, we will say that graph with G as set of vertices has (left) G-invariant graph-structure if the set of the edges $E \subset G \times G$ will have a form $E = \{(g, h) : g^{-1}h \in A\}$, where $A \subset G$ and for the topological group A is Borel set. In the same way we can consider more generally X the homogeneous G-space of the group as the set of the vertices. We will not define the notion of universality for topological group case in whole generality and consider the case $G = \mathbb{R}$. Group invariant graph structure as well as group invariant model of universal Urysohn space had been considered in the papers ([17, 16, 18]) — homogeneous Cayley objects in terminology [17]). Here we will give new examples.

We will consider the additive group $X = \mathbb{R}$, and define a subset $E \subset \mathbb{R}^2; E = \{(x, y) : |x - y| \in Z\}$ where set $Z \subset (0, +\infty)$ is constructed by induction.

We begin with the construction of shift-invariant countable universal graph (corresp. universal triangle free graph) *on the group of rational numbers* and later will extend the definition on the set of all real numbers.

It is easy to formulate the conditions of the universality in terms of the set $Z \subset \mathbb{R}_+$:

For universal graph:

$\overline{U}$. For each pair of disjoint finite sets of the real numbers $\{a_1, \dots, a_n\}, \{b_1 \dots b_m\}$ there exists a real number c such, that $|c - a_i| \in Z, i = 1,\ 2, \dots, n;\ |c - b_j| \notin Z, j = 1,\ 2, \dots, m$. Moreover, the set of such c has non-empty interior.

For universal triangle free graph:

$\overline{UTF}$.

a) "Triangle free" condition transforms into the following *sum-free condition*: there are no solutions of the equation $a + b = c$, where $a, b, c \in \bar{Z}$.

b) for each pair of disjoint finite sets of the real numbers $\{a_1, \dots, a_n\}, \{b_1, \dots, b_m\}$ such that $|a_i - a_k| \notin \bar{Z}, \quad 1 \le i < k \le n$, there exists real number c such that $|c - a_i| \in Z, i = 1, 2, \dots, n;\ |c - b_j| \notin Z, j = 1, 2, \dots, m$. Moreover, the set of such c has non-empty interior.

We want to prove the following result.

Theorem 4. *There is a universal topological graph (correspondingly, universal topological triangle free graph) with additive group $\mathbb{R}$ as the set of vertices and with graph structure which is invariant under the additive action of the group $\mathbb{R}$ on itself.*

Proof. We define firstly the $\mathbb{Q}$-invariant universal (correspondingly - universal triangle free) graph-structure on the group $\mathbb{Q}$, it will be a special new model of countable universal (correspondingly - triangle free universal) graph, and then will check that it is possible to extend that structure on the group $\mathbb{R}$ and this will give a the *topological* $\mathbb{R}$-invariant universal (triangle free universal) uncountable graph. The construction is a little bit different for two cases.

1)The case of Universal graph. Consider the set

$$P(\mathbb{Q}) = \{(a_1 < \cdots < a_n); a_i \in \mathbb{Q}, i = 1, \ldots n, n = 1, 2 \ldots\}$$

of all finite sets of rational numbers which we will be called pre-patterns, and the set of pairs of the pre-patterns which we will be called a pattern — $P^2(\mathbb{Q})$. Let $\gamma \in P^2(\mathbb{Q})$ where

$$\gamma = (\alpha, \beta), \alpha, \beta \in P(\mathbb{Q}), \alpha = (a_1 < \cdots < a_n), \beta = (b_1 < \cdots < b_m),$$

where $a_i, b_j \in \mathbb{Q}$. We will call α (corr. β) an a-part (corr. b-part) of the pattern γ. For our goals it is enough to consider only the subset of $\gamma \in P^2(\mathbb{Q})$ which consists with the *patterns containing zero and all other points of which are positive*:

$$\mathcal{P} = P_0^2(\mathbb{Q}) = \{\alpha = (0 \leq a_1 < a_2 \cdots < a_n), \beta = (0 \leq b_1 < \cdots < b_m),$$

here $a_i \neq b_j, a_i, b_j \in \mathbb{Q}_+$,and either $a_1 = 0$, or $b_1 = 0$.

We want to define a graph with shift-invariant structure, and consequently it is enough to describe only the set of edges between the point (vertex) 0 and positive rational numbers, or to define the set Z will be just the set of rational numbers r for which $\{0, r\}$ is the edge of the graph.

Introduce two characteristics of the patterns $\gamma \in \mathcal{P}$: $\epsilon(\gamma) = \frac{1}{2} \min\{1, |a_i - a_k|, |a_i - b_j|, |b_j - b_s|\}$ (minimum over all different coordinates), and $M(\gamma) = max_{i,j}\{a_i, b_j\}$.

This set will be constructed inductively as countable union of the sets $Z = \bigcup_n Z_n$. Choose $\gamma_1 = (\alpha = (a_1 = 0), \beta = \emptyset)$ and put $T_1 = 0, T_2 = 1, Z_1 = (1/3, 2/3) \bigcap \mathbb{Q}$, So $Z_1 \subset (T_1, T_2)$.

Suppose we already have constructed the sets $Z_1, \ldots Z_{n-1}, Z_i \subset (T_i, T_{i+1}), T_i < T_{i+1}, i = 1, 2, \ldots, n-1$ of the patterns $\gamma_1, \ldots \gamma_{n-1}$ which satisfies the condition $\bar{2}$ for the patterns $\gamma_1, \ldots \gamma_{n-1}$ and each Z_i ($i = 1, 2, \ldots, n-1$)is a union of the rational intervals. We assume also that $T_k - a_i^k > T_{k-1}$ where $\gamma_k = \{(a_1^k, \ldots, a_l^k), (b_1^k, \ldots b_s^k)\}$.

Now we will define an integer number T_{n+1} and construct the sets Z_n : $Z_n \subset (T_n, T_{n+1}),\ T_n < T_{n+1}$,

Consider $\gamma_n = \{(a_1, \ldots a_l), (b_1, \ldots b_s)$ the n-th pattern of $\mathcal{P}$, and choose $T_{n+1} = T_n + M(\gamma_n)$ and

$$Z_n = \bigcup_{i=1}^{l} V_{\varepsilon(\gamma_n)}(T_{n+1} - a_i),$$

here $V_\delta(y)$ is δ-neighborhood of the point y. This means that $T_{n+1} - a_i \in Z_n, i = 1, \dots l$ and because of choice ε, we have: $T_{n+1} - b_j \notin Z_n$, and also $T_{n+1} - b_j \notin \bigcup_{i=1}^{n} Z_i$, because interval (T_n, T_{n+1}) does not intersect with $Z_i, i < n$. This completes the process of the construction of the set Z_n and consequently the set $Z = \bigcup_n Z_n$.

Note that the pair $\{a_i, T_{n+1}\} \in E$, e.g. the pairs of numbers $\{a_i, T_{n+1}\}$ are the edges of our graph for all $i, i = 1 \dots l$ and all the a-parts of the pattern $\gamma_n, n = 1 \dots$, in the same time because of the same reason the pairs $\{(b_j, T_{n+1}\}$ - is not edges for all b-part of the patterns γ_n, and property $\bar{2}$ is valid for all patterns. Now the arbitrary pair of the rational numbers $(r_1, r_2) \in E$ — is the edge of the graph iff $|r_1 - r_2| \in Z$; thus the graph structure is defined on the rational numbers and we have constructed the universal graph whose vertices are rational numbers $\mathbb{Q}$ and set of edges is invariant under the shift.

Now we show that the definition of the set Z allow to extend the graph structure on the set of real numbers: the set of vertices is $\mathbb{R}$, and the edges are defined as follow: consider the open set Z' which is interior of the closure of set $Z\mathbb{Q}$ and define $E' = \{(\lambda_1, \lambda_2), \lambda_1, \lambda_2 \in \mathbb{R} : |\lambda_1 - \lambda_2| \in Z'\}$.

Let us check that the graph $(\mathbb{R}, E')$ is universal topological graph with graph $\mathbb{R}$-invariant graph structure (e.g. with the set of edges which are invariant under the shift). The set E' is open by definition and we need to check property 2. Let us consider the two arbitrary finite patterns of different real numbers $\Lambda = \{(\lambda_1, \dots \lambda_n), (\mu_1, \dots \mu_m)\}$.Find rational approximation of these numbers $\gamma = \{(a_1, \dots a_n), (b_1 \dots b_m)\}$ with accuracy to $\frac{\varepsilon(\gamma)}{2}$, because of construction of the set Z and because validity of the property $\bar{2}$ for pattern γ and corresponding number T, we obtain that property 2 is valid for pattern Λ. Thus, topological $\mathbb{R}$-invariant graph is constructed.

Note that the set Z - set of the vertices which joined with the vertex 0, - as well as other points x, is the unbounded open set of $\mathbb{R}$ which is separated from 0 (x).

2) Proof for triangle free case.

We start with the following remark. Transition to the real case from rational was very simple in the previous proof, because our construction was in a sense continuous. The same method can be applied in triangle free case and it gives the construction of the universal $\mathbb{Q}$-invariant triangle free countable graph. But in opposite to the condition $\bar{2}$ the condition $\bar{2}'$ is not continuous, and we can not in the same way to extend structure to the $\mathbb{R}$: we can loose universality because the set Z could be too small. For this reason we must change the construction. Namely we can not consider the arbitrary ordering of the set of rational patterns, numeration must be agreed with sizes of patterns. But there is even more simple way to construct the set Z.

We start with enumerating not all finite sets of rational numbers, (patterns in the previous sense) as before but we numerate *all finite sets of rational intervals*. Say, that two intervals (a, b) and (c, d) are strictly disjoint, if $b < c$ or $d < a$. Define a pre-pattern as a finite union of strictly disjoint intervals with positive rational endpoints. Now define a pattern as an ordered pair of two pre-patterns, (call them a-part and b-part of the pattern) such that all intervals of both pre-patterns are strictly disjoint. So, the term "pattern" has another sense than before.

Enumerate all patterns: γ_1, γ_2, ... We construct our set Z as a union $\cap_i Z_i$, which are defined inductively. Start with $Z_1 = (2/5, 3/5)$. Suppose the sets Z_1, Z_2, ..., Z_{n-1} are already constructed, and the union $\cap_{i=1} 6n - 1 Z_i$ is sum-free set. The n-th step consist in the definition of the set Z_n. Consider the pattern γ_n and all differences of (distinct) elements of a-part - α_n of pattern γ_n. If some of difference belongs to $\cup_{i=1}^{n-1} Z_i$, then we put $Z_n = \emptyset$ and missed pattern $\gamma_n$and consider the next pattern. In opposite case, we define set $Z_n = T_n - \alpha_n$, where T_n is some integer such that $T_n > 5 \sup(\gamma_n \cup_{i=1}^{n-1} Z_i)$. Note that because of our choice of Z_n the set $\cup_{i=1}^{n} Z_i$ is still sum-free. Thus the set $Z = \cup_{i=1}^{\infty} Z_i$ is constructed and sum-free.

Check that it satisfies $(\bar{2}')$. Fix two disjoint finite sets $A = \{a_1, \dots a_k\} \subset \mathbb{R}$ and $B = \{b_1, \dots b_m\} \subset \mathbb{R}$ such that $|a_i - a_j| \notin \bar{Z}$ for $1 \leq i \leq j \leq k$. There exists a (rational) pattern γ, such that a-part α of it as a system of open intervals contains set A, and the b-part β of is contains a set B, and such that $\bar{(}\alpha - \alpha) \cap \bar{Z} = \emptyset$. Then according to our construction we have $Z_n = T_n - \alpha$ and $T - \beta \cap \bar{Z} = \emptyset$, where n is the number of the pattern γ. So, all numbers c from some sufficiently small neighborhood of T satisfy to the condition $(\bar{2}')$. □

The next theorem sums up our construction:

Theorem 5. . *The pair* $(\mathbb{R}, E)$ *with* $E = \{(x, y) \in \mathbb{R}^2 : |x - y| \in Z\}$ *where* $Z \subset \mathbb{R}$ *is constructed above (e.g. satisfies the conditions above 1,2G (2'G)) is universal topological (universal topological triangle free) graph. For each non-degenerated Borel probability measure* m *on* $\mathbb{R}$ *the triple* $(\mathbb{R}, E, m)$ *(corresp* $(\mathbb{R}, E', m)$*) is Borel universal (corresp.universal triangle free) graph. Each* *such measure m generated the $\mathfrak{S}^{\mathbb{N}}$-invariant measure on the set of matrices $M_{\mathbb{N}}(0,1)$ which is concentrated on the adjacent matrices of the universal countable graphs (correspondingly - — on the triangles-free universal countable graph)*

Proof. . Follows from the Theorems 1,2,3. □

EXAMPLE. Now we are giving the precise example of the $\mathfrak{S}^{\mathbb{N}}$-invariant measure on the space of adjacent matrices. Let

$$dm(t) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{t^2}{2}\}dt$$

is the standard gaussian measure on $\mathbb{R}$ and $\xi_1, \dots \xi_n, \dots$ is the sequence of the independent random variables each of which is distributed with that gaussian measure. Let $E \equiv \{(t,s) : |t-s| \in Z\} \subset \mathbb{R}^2$ (corresp $E' \equiv \{(t,s) : |t-s| \in Z'\}$) where the sets Z, Z' were defined in the theorem 4. Then the random $0;1$-matrices:

$$\{\chi_Z(\xi_i - \xi_j)\}_{i,j=1}^{\infty}$$
$$(\{\chi_{Z'}(\xi_i - \xi_j)\}_{i,j=1}^{\infty}),$$

are with probability 1 adjacent matrices of the universal (universal triangle free) graphs. In other words, the distribution of those random matrices is the $\mathfrak{S}^{\mathbb{N}}$-invariant measure with concentrated on the universal (triangle free) graphs.

It follows from the theorems, that instead of gaussian measure we can take any non-degenerated measures and the choice of the set $E(E')$ is not unique. But each random matrix with the quoted property can be obtain in this way with some non-degenerates measures m and with a set E (E') which satisfied to the conditions of the theorems above.

5 Some comments.

1. **The properties of invariant measures**. The choice of the measures in the examples is very wide and we can put additional assumption on the measures. For example we can assume that the measure $m(E \times E) = \lambda$, where $\lambda \in (0,1)$ is given number (say, 1/2). It is not clear how to describe

the concrete property of the invariant measures which was constructed. Of course it is interesting to give more information about the distributions of the entries of matrices in our examples.

2. **Distribution of the entries**. The finite dimensional distribution of the measure μ is very important characteristic of measures. These are the measures on the $n \times n$ adjacent $\{0,1\}$-matrices; it is evidently invariant under the finite symmetric group, and consequently concentrated on the bunches of orbits of these groups; the structure and asymptotic size of these orbits is interesting characteristic of the measure μ. In particular for the case of triangle-free graph, our measures give more significant weight (in opposite to uniform measure, see [14]) to the triangle free but not 2-colored finite graphs.

3.**Homogeneity** Perhaps, it is not easy to express in terms of finite dimensional distributions of matrices the fact, that the graph structure on $\mathbb{R}$ is shift invariant and the measure m is quasi-invariant under the shift. In our construction the homogeneous is very convenient, but not so important.

4.**Uniqueness** Discussing the definitions of universality above the following question naturally arises: under the what conditions (which must be invariantly formulated) the set E (and universal Borel graph) is unique up to Borel isomorphisms of the space X? Equivalently, when Borel version of "back and forth" method works? The same question can be put in the categories differ from category of Borel spaces; the example of positive answer on this question in the similar situation is the category of Polish metric spaces - uniqueness of Urysohn space.

5.**Compactness** It is interesting if it is possible to construct universal topological shift-invariant graph on some compact group (say, on a circle).

6.**Link with Urysohn space** The special interest in this sense represents the Urysohn space. We will consider it from this point of view elsewhere. Here we mention only that Urysohn space $\mathbb{U}$ plays the role of "Borel universal object" (or topological universal object) for rational or integer universal metric space; and any Borel probability measure m on this space defines the $\mathfrak{S}^{\mathbb{N}}$-invariant measure μ on the space $\mathcal{R}$ *distance real matrices* which are universal with probability one (that means that completion of the $\mathbb{N}$ with respect to that random metric, is isometric to Urysohn space, see [13]). The similarity of the theory of Urysohn space and example of the section 4 above can be illustrated by the result of [16] where Urysohn space was realized as a completion of the real line with respect to universal shift-invariant metric.

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