The Pascal automorphism has a continuous spectrum

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In memory of V. I. Arnold, a troublemaker

> Mathematicians who are only mathematicians have exact minds, provided all things are explained to them by means of definitions and axioms; otherwise they are inaccurate and insufferable, for they are only right when the principles are quite clear.
>
> Blaise Pascal, *Thoughts*, English translation by W. F. Trotter

Dedication. Dima Arnold (as well as me) was very fond of B. Pascal, and disliked R. Descartes, seeing him as a forerunner of Bourbakism he hated so much. As to me, in my youth I had great respect for Bourbaki, a very high opinion of his 5th volume, and even once wrote to him (N. Bourbaki) a long eulogistic letter, of which Dima did not approve. In reply, N. Bourbaki (impersonated by J. Dieudonné) presented me the next volume *Integration*, which had just appeared; the topic was close to my interests, but the volume turned out to be a failure. I was distressed and started to believe that perhaps Arnold was right.

The keen interest to combinatorics and asymptotic problems, which appeared in the last years of V. I. Arnold's life and made us even closer, was, I believe, another manifestation of the fact that his mind revolted at any limits and prohibitions, he always violated canons, or, better to say, introduced new canons; and he was able to do this, because he was (according to Pascal) not only a mathematician.

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Abstract

In this paper, we describe the Pascal automorphism and present a sketch of the proof that its spectrum is continuous in the orthogonal complement of the constants.

1 Introduction: the definition of the adic Pascal automorphism

The transformations generated by classical graded graphs, such as the ordinary and multidimensional Pascal graphs, the Young graph, the graph of walks in Weyl chambers, etc., provide examples of combinatorial origin of the new, very interesting class of *adic transformations*, introduced back in [3].

In this paper, we study the Pascal automorphism. It is a natural transformation in the path space of the Pascal graph (= the infinite-dimensional Pascal triangle), i.e., in the infinite-dimensional cube. If we realize this automorphism as a shift in the space of sequences of zeros and ones, then a stationary measure arises, which was called the Pascal measure; we study the properties of this measure. In particular, it turns out that the set of Besicovitch–Hamming almost periodic sequences has zero Pascal measure, which eventually implies the continuity at least of the spectrum of the corresponding operator.

A crucial role in the sketched proof of the continuity of a part of the spectrum of the Pascal automorphism is played by combinatorial considerations related to the structure of repeated occurrences of growing self-similar words. These considerations are universal for a wide class of adic automorphisms. Our exposition follows the tradition maintained by V. I. Arnold (see, e.g., [1, 2]), which is to reveal the nature of a phenomenon rather than to formally describe it.

1.1 A linear order and other structures on the set of vertices of the cube

The definition of the Pascal automorphism, which is an example of an adic transformation, was given in [3, 4]; in the same papers, the problem of finding its spectrum was suggested. The definition is based on introducing a lexicographic (linear) order on the set of paths of the finite or infinite Pascal graph.[1] In turn, introducing this lexicographic order reduces to introducing a natural linear order on the set of all subsets of given cardinality in the set of n elements, or, in the geometric language, on the set of vertices of the n-dimensional cube with a given number of nonzero coordinates. We begin with several versions of the definition of this order.

1. Let $I_n = \{0,1\}^n$ be the set of vertices of the unit n-dimensional cube. Consider the hyperplanes that contain vertices with the sum of coordinates equal to k, where $k = 0, 1, \dots, n$, and denote by $C_{n,k}$ the set of vertices on the kth hyperplane; these vertices have $m = n - k$ coordinates equal to 0 and $k = n - m$ coordinates equal to 1. We define a linear order on all $C_{n,k}$ by induction. For $n = 2$, the order on the one-point sets $C_{2,0}, C_{2,2}$ is trivial, and on $C_{2,1}$ it is defined as follows: $(0,1) \succ (1,0)$. Now assume that the order is defined on $C_{n,k}$ for all $k = 0, 1, \dots, n$; then on $C_{n+1,k}$ we define it by the following rule. Given a pair of vertices with the same last coordinate, they are ordered in the same way as the pair of vertices from $C_{n,k}$ obtained from them by deleting the common last coordinate; if the last coordinates are different, then the greater vertex is the one for which it is equal to 1. With this order, the smallest vertex in $C_{m+k,k}$ is $(\underbrace{1,\dots,1}_{k}, \underbrace{0,\dots,0}_{m})$, and the greatest one is $(\underbrace{0,\dots,0}_{m}, \underbrace{1,\dots,1}_{k})$.

2. The same definition in slightly different terms reads as follows. Setting $k = n - m$, consider the family $C_{n,k}$ of all C_n^k subsets of cardinality k in the linearly ordered set of $m + k$ elements $\{\mathbf{1,\dots,\ k,\ k{+}1,\ \dots,\ k{+}m}\}$, and introduce a linear lexicographic order on this set by the following rule: a subset F is greater than a subset G if the maximum index of an element from F that does not belong to G is greater that the maximum index of an element from G that does not belong to F. Thus we have linearly ordered all elements of $C_{n,k}$.

3. Finally, the "numerical" interpretation of this order is as follows. A

[1] The term "Pascal triangle" is used more widely, but it applies mainly to the finite object; at the vertices of this graph, one usually writes the binomial coefficients. Thus it is not quite natural to call the infinite Pascal graph a triangle. But there are also objections to using the term "Pascal triangle" for the finite graph: first of all, because the discovery of this graph is attributed also to the Indian Pingala (10th century), the Persian Omar Khayyam (12th century), and the Chinese Yang Hui (13th century). On the other hand, the results of the Google search "Pascal graph" are related mainly to graphics in the programming language Pascal.

vertex of a finite-dimensional unit cube (with ordered coordinates) is a 0-1 vector, which we regard as the binary expansion of an integer. Consider the set of all integers with a given (equal) number of zeros (and ones) in the binary expansion, and introduce the standard order $<$ on this set regarded just as a set of positive integers.

The equivalence of all three independent definitions of the lexicographic order is easy to verify.

Since a path in the finite Pascal triangle of height n can be identified with a 0-1 sequence, we have in fact linearly ordered each set of paths leading to a finite vertex of the triangle.

Definition 1. Given positive integers m, k, we define the *supporting word* $O(m,k)$ as the 0-1-vector consisting from the first coordinates of the ordered sequence of vertices of the set $C_{m+k,m}$; equivalently, $O(m,k)$ is the vector of parities of the set of positive integers whose binary expansion contains exactly k ones and m zeros written in increasing order. By $O(0,0)$ we understand the word (1). The length of the vector $O(m,k)$ is equal to C^m_{m+k}.

Example:

$$O(1,1) = (1,0), \qquad O(2,2) = (1,1,0,1,0,0),$$
$$O(4,3) = (1,1,1,0,1,1,0,1,0,0,1,1,0,1,0,0,1,0,0,0,\\ 1,1,0,1,0,0,1,0,0,0,1,0,0,0,0).$$

1.2 The infinite-dimensional cube and the Pascal automorphism

Now consider the set of vertices of the infinite-dimensional unit cube, i.e., the countable product of two-point sets: $I^\infty = \{0,1\}^\infty$; in short, we will call it the infinite-dimensional cube. On this remarkable object, there is a huge number of important mathematical structures, which have many useful interpretations and various applications.

First of all, we regard the infinite-dimensional cube as the compact additive group of dyadic integers $\mathbf{Z}_2$. We realize it as the group of sequences of residues modulo 2^n and use the additive notation. In our interpretation, a dyadic integer is also an infinite one-sided sequence of zeros and ones. The weak topology on I^∞ gives rise to the structure of a standard measure space. The Bernoulli measure μ is the infinite product of the measures with probabilities $(1/2, 1/2)$ on the factors $\{0,1\}$. It is simultaneously the Haar measure on the cube regarded as the group $\mathbf{Z}_2$. One can also consider other Bernoulli measures μ_p, each of which is the countable product of equal factors $(p, 1-p)$, where $0 < p < 1$; they are no longer invariant (only quasi-invariant) under the addition (but are invariant under the Pascal automorphism, see below). The infinite-dimensional cube can be naturally identified with the set of infinite paths in the Pascal graph (a zero corresponds to choosing the left direction, and a one, to choosing the right direction). It is this space that will be the phase space of the Pascal automorphism defined below. All these interpretations are identical, and the measure is the same; the choice of a convenient realization is a matter of taste. For us, it is usually convenient to use the infinite-dimensional cube; the topology of the space is not of importance.

It is convenient to write dyadic integers in the form

$$\underbrace{0,\dots,0}_{m_1}\underbrace{1,\dots,1}_{k_1}** = 0^{m_1}1^{k_1}**.$$

Obviously, the translation T on the additive group $\mathbf{Z}_2$ defined by the formula $Tx = x+1$ preserves the Haar–Bernoulli measure; in dynamical systems, it is called the *odometer*, or the *dyadic automorphism*. This is one of the simplest ergodic automorphisms; its spectrum (= the set of eigenvalues) is the group of all roots of unity of order 2^n, $n = 1, 2, \dots$. The orbits of the odometer are the cosets of the (dense) subgroup $\mathbb{Z} \subset \mathbf{Z}_2$. The general adic model of measure-preserving transformations is a far-reaching generalization of the odometer (see [3, 4]).

Consider the dyadic metric ρ on the additive group $\mathbf{Z}_2$ of dyadic integers; it induces the weak topology on I^∞. The metric ρ looks as follows: $\rho(x,y) = \|x - x'\|$, where $\|g\| = 2^{-t(g)}$ is the *canonical normalization*; here $t(g)$ is the index of the first nonzero coordinate of g. The metric ρ is obviously invariant with respect to the odometer; however, as we will see, it is not invariant with respect to the Pascal automorphism.

Now we introduce an order on $\mathbf{Z}_2 \sim I^\infty = \{0,1\}^\infty$ as follows. We say that two vertices (points) of the infinite-dimensional cube are comparable if their coordinates coincide from some index on (i.e., they have "the same tail") and the number of ones among the coordinates with smaller indices in both sequences is the same. Given two comparable sequences, the greater one is, by definition, the sequence whose initial finite segment is greater in the sense of the order defined above. This is the lexicographic order on the infinite-dimensional cube we need; denote it by $\prec$. One can easily extend all descriptions of the order on the finite-dimensional cube given above to the infinite-dimensional cube.

But every vertex of the infinite-dimensional cube can be regarded as an infinite path in the Pascal graph. In the order we have introduced, paths are comparable if they have the same "tail," i.e., their coordinates coincide from some index on. Thus we have defined a linear order on the set of paths in the Pascal graph.

Definition 2. The order introduced on the set of vertices of the (finite- or infinite-dimensional) cube will be called the *adic order*.

The order type of the class of comparable paths is that of the one-sided ($\mathbb{N}$) or two-sided ($\mathbb{Z}$) infinite chain; it is infinite to the left if the corresponding vertex of the cube has only finitely many zeros, and infinite to the right if it has finitely many ones. For all other points (paths), the order type is that of $\mathbb{Z}$; they constitute a set of full Bernoulli measure. One may say that we have redefined the order on the cosets of the subgroup $\mathbb{Z}$; each coset breaks into countably many linearly ordered subsets.

Definition 3 ([4])**.** The *Pascal automorphism* is the map P from the infinite-dimensional cube (in any interpretation) to itself that sends every point to its immediate successor in the sense of the adic order ($\prec$).

The immediate successor, as well as the immediate predecessor, exists for all points except for countably many. Thus the Pascal automorphism and its inverse are defined everywhere except for countably many points (more exactly, except for the elements of the group $\mathbb{Z}$ regarded as a subgroup in $\mathbf{Z}_2$). It is easy to see that the transformation P is measurable and even continuous in the weak topology everywhere apart from the above-mentioned exceptions.

Proposition 1. *The Pascal automorphism preserves the Bernoulli measures* $\mu_p = \prod_1^\infty \{p, 1-p\}$, $0 < p < 1$, *on the infinite-dimensional cube and is ergodic with respect to all* μ_p, $0 < p < 1$.

The ergodicity follows from the Hewitt–Savage 0-1 law, or, alternatively, from the ergodicity of the action of the group S_∞ on the same space. For a more detailed analysis, see below.

2 The Pascal automorphism as the result of a time change in the odometer, and random substitutions on the group $\mathbb{Z}$

Now we proceed to a more detailed study of the Pascal adic automorphism. We will show that it is measure-preserving. Indeed, taking the immediate successor of a point results in a substitution of finitely many coordinates, so the measure-preserving property follows from the fact that the Bernoulli measure with equal factors is invariant under the action of the infinite symmetric group by substitutions of coordinates. It is clear from above that every orbit of the Pascal automorphism lies in one orbit of the odometer, namely, in the same coset of $\mathbb{Z}$. Hence an element of the group of dyadic integers of the form $x - Px$ lies in the subgroup $\mathbb{Z} \subset \mathbf{Z}_2$, and, consequently, the Pascal automorphism can be regarded as the result of a time change in the odometer.

We emphasize again that the partition into the orbits of the Pascal automorphism coincides $\bmod 0$ with respect to any Bernoulli measure with the partition into the orbits of the group of finite substitutions of positive integers, and the set of Bernoulli measures coincides with the set of ergodic invariant measures with respect to the Pascal automorphism.

If, using binary expansions, we identify (up to a set of zero measure) the group $\mathbf{Z}_2$ with the unit interval, then the Pascal automorphism turns into a transformation of the interval which belongs to the class of so-called rational countable rearrangements.

Below we will write down explicit formulas for the time change that should be made in the odometer in order to obtain the Pascal automorphism. As we will see, the Pascal automorphism reorders the points on cosets of the subgroup $\mathbb{Z}$ (i.e., on orbits of the odometer) in a quite complicated way.

The analysis below is similar to that made in [11] in a simpler case; namely, a detailed comparison of the standard order with the so-called *Morse order* arising from the study of the Morse automorphism.

It is easy to deduce from the definition of the Pascal automorphism that it is given by the following formula:

$$x \mapsto Px; \quad P(0^m 1^k \mathbf{10} * *) = 1^k 0^m \mathbf{01} * *, \quad m, k = 0, 1 \ldots.$$

It is convenient to write the automorphism P^{-1} in a similar form:

$$P^{-1}(1^k 0^m \mathbf{01} **) = 0^m 1^k \mathbf{10} **, \quad m,k = 0,1\dots.$$

The passage from P to P^{-1} swaps m and k, i.e., 0 and 1. The automorphism P and its inverse P^{-1} are defined for all x with infinitely many zeros and ones, i.e., on the set $\mathbf{Z}_2 \setminus \mathbb{Z}$. On the other hand, since $P(x)$ lies in the same coset as x, one may ask what is the difference $P(x)-x$. We summarize the answer in the following lemma.

Lemma 1. *The Pascal automorphism is given by the formula*

$$P(0^m 1^k 10 **) = 1^k 0^{m+1} 1 **, \quad m,k \geq 0,$$

or, in the numerical representation,

$$P(2^{m+k} - 2^m + r) = 2^{m+k+1} + 2^k - 1 + r, \quad m,k \geq 0,\ r \in \mathrm{Ker}(\theta_{m+k+1});$$

here θ_n is the homomorphism defined by the formula $\theta_n : \boldsymbol{Z}_2 \to \boldsymbol{Z}_2/Z_{2^n}$, and its kernel consists of the sequences with the first n coordinates equal to zero.

Correspondingly, the time change that transforms the odometer T into the Pascal automorphism P is given by the formula

$$Px = T^{n(x)}x \equiv x + n(x),$$

where $x = 0^m 1^k 10 **$ and

$$n(x) = P(x) - x = 1^{\min(k,m)} 0^{|m-k|} 10^{\infty},$$

or, in numerical form,

$$n(2^{m+k+1} - 2^m) = 2^m + 2^k - 1.$$

These formulas define the Pascal automorphism for any element $x \in \mathbf{Z}_2$ whose expansion contains a fragment of the form 10 and a fragment of the form 01, i.e., for any dyadic integer that is not of the form $0^m 1^{\infty}$, $m \geq 0$ (a negative integer) and not of the form $1^k 0^{\infty}$ (a positive integer). The above formulas for $n(x)$ are easy to verify in either of the cases $m > k$ and $m \leq k$.

Now consider the functions $n_k(x)$ defined by the formula $P^k x = T^{n_k(x)}x = n_k(x) + x$ for all positive integers $k = 0,1,2\dots$:

$$n_0(x) = 0, \quad n_1(x) = n(x), \quad n_k(x) = P^k x - x, \quad \dots.$$

A recurrence formula for $n_k(x)$ follows from the definition, as described in the lemma below.

Lemma 2. $n_{k+1} = n_1(n_k(x) + x) + n_k(x)$.

Proof. We have $P^{k+1}x \equiv T^{n_{k+1}(x)}x = n_{k+1}(x) + x$. But $P^{k+1}x = P(P^k)x = T^{n_1(P^k x)}P^k x$, so that $P^{k+1}x = T^{n_1(P^k x)+n_k(x)}x$, i.e., $x + n_{k+1}(x) = n_1(P^k x) + n_k(x) + x = n_1(x + n_k(x)) + n_k(x) + x$. □

Thus

$$\begin{aligned} P^k x &= n_1(n_{k-1}(x) + x) + n_{k-1}(x) = \dots \\ &= x + n_1(x) + n_1(n_1(x) + x) + \dots + n_1(n_{k-1}(x) + x). \end{aligned}$$

Observe that the formulas expressing the functions $n_k(x)$, $k > 1$, in terms of the function $n_1(x) = n(x)$ are, of course, universal: they hold for a time change in an arbitrary automorphism. We will use only the function $n(\cdot) = n_1(\cdot)$.

The distinguished orbit $\mathbb{Z} \subset \mathbf{Z}_2$ of the odometer breaks into countably many finite orbits of the Pascal automorphism; namely, every positive integer $x \in \mathbb{N}$ belongs to the finite orbit that ends at $2^s - 1$ where s is the number of ones in the binary expansion of x; and every negative integer belongs to the finite orbit that begins at $-2^s + 1$. All the other orbits of the automorphism P, regarded as linearly ordered sets, are of order type $\mathbb{Z}$.

In connection with the formula for Px, an important question arises which we have already mentioned above: how do the cosets of $\mathbb{Z}$, i.e., the orbits of the odometer, transform under the action of the Pascal automorphism? We introduce the following substitution on the set $\mathbb{Z}$ of all integers:

$$\sigma_x : k \mapsto n(x + k), \quad k \in \mathbb{Z}.$$

Thus the Pascal automorphism determines a random (the randomness parameter is $x \in \mathbf{Z}_2$) infinite substitution σ_x that maps $\mathbb{Z}$ (as a countable set) to itself and has infinitely many infinite cycles. The image of the Bernoulli measure on $\mathbf{Z}_2$ under the map $x \mapsto \sigma_x$ is a measure on the group $S^{\mathbb{Z}}$ of all infinite substitutions of $\mathbb{Z}$. It differs substantially from the measure arising in a similar analysis of the Morse transformation [11] (in that case, the measure is supported by one-cycle substitutions); the analysis of this measure is of considerable interest and can be used in the study of the properties of the Pascal automorphism.

A general principle says that every time change in a dynamical system with invariant measure determines a measure on the group of infinite substitutions of time, and the properties of this measure allow one to derive conclusions about the system. It is this observation that gives meaning to the statement that an action of a group with invariant measure can be regarded as an action of a random substitution on this group. But for this we should choose a *reference action*, an initial dynamical system to make a time change in. In our case (the group $\mathbf{Z}$), this reference action is that of the odometer.

3 The Pascal automorphism and a σ-finite invariant measure

It turns out that the Pascal automorphism (without mentioning either this term or the link to the Pascal triangle) was defined and used in 1972 in the paper [7] by Hajian, Ito, and Kakutani and in 1976 in the paper [8] by Kakutani.[2] The authors of [7] use the product (Bernoulli) measures with nonequal probabilities ($(p, 1-p)$, $0 < p < 1/2$) on the product of two-point sets; these measures are invariant under the Pascal automorphism, but only quasi-invariant under the odometer. Using the Radon–Nikodym cocycle, the authors construct a new automorphism R of the direct product $\mathbf{Z}_2 \times \mathbb{Z}$ with an R-invariant σ-finite measure. This automorphism R is also a special automorphism over the base where the Pascal automorphism acts, and the ceiling function coincides with the function $n(x)$ defined above. Recall that, since $\int n(x)\,dm(x) = \infty$, the global measure is σ-finite. The ergodicity of R follows from the ergodicity of the Pascal automorphism, i.e, from the 0-1 law, or from the triviality of the "substitutional" σ-algebra. This example was the first to demonstrate that some ergodic automorphisms with infinite measure may commute with non-measure-preserving automorphisms; for automorphisms preserving a finite measure, this cannot happen. It is essential that the orbit partition of the Pascal automorphism in the natural representation is a subpartition of the orbit partition of the odometer into *finitely many* parts. In the other paper [8], the Pascal automorphism and the formula $P(0^m 1^k 10*) = 1^k 0^{m+1} 1*$ mentioned above were used in the study of a statistical problem (the so-called Kakutani problem). We will return to this link elsewhere and relate this problem to random walks on the group $\mathbb{Z}$.

[2] I am grateful to Professor Hajian for informing me about this paper after my talk about the Pascal automorphism at the Northeastern University (Boston) in April 2011.

4 The stationary model and encoding of the Pascal automorphism; the Pascal measure

Now we describe the action of the Pascal automorphism in more traditional terms, namely, as the shift in the space of two-sided sequences of zeros and ones (i.e., again in the two-sided infinite-dimensional cube) equipped with some shift-invariant measure. It is this representation that will be used in what follows.

Consider a stationary model of the Pascal automorphism.

Definition 4. Given $x \in \mathbf{Z}_2$, define a new two-sided sequence $y_n(x)$ of zeros and ones as follows:

$$y_n = (P^n x)_1, \quad n \in \mathbb{Z};$$

here $(\cdot)_1$ is the first digit (0 or 1) of the dyadic number in the parentheses. Thus we have a map

$$S : \prod_{1}^{\infty}\{0,1\} \equiv \mathbf{Z}_2 \to Y = \mathbf{2}^{\mathbb{Z}} = \prod_{-\infty}^{\infty}\{0,1\},$$

which is given by the formula

$$\mathbf{Z}_2 \ni x \mapsto Sx = y \equiv \{y_n\}_{n\in\mathbb{Z}} : \quad y_n(x) = (P^n x)_1,\ n \in \mathbb{Z}.$$

The map S sends the Bernoulli measure ν on the infinite-dimensional cube to some measure $S_*\nu \equiv \pi$ on another (two-sided infinite-dimensional) cube $Y = \prod_{-\infty}^{\infty}\{0,1\}$, which we will call the *Pascal measure*.

In terms of paths in the Pascal graph, the map S can be described as follows. Given such a path, regarded as a sequence of vertices in the Pascal graph, this map associates with it the sequence of changes of the first edge in the course of the adic evolution of the path.

Theorem 1. *The partition of the space $\mathbf{Z}_2$ into two sets according to the value of the first coordinate is a (one-sided) generator of the Pascal automorphism. In other words, almost every point is uniquely determined by the sequence of the first coordinates of its images under the action of the positive powers of the Pascal automorphism:*

$$x \leftrightarrow \{(P^n x)_1\}_{n\in\mathbb{N}}$$

is a bijection for almost all $x \in \mathbf{Z}_2$. Thus S is an isomorphism of measure spaces which sends the Pascal automorphism P of the space $\mathbf{Z}_2$ with the Bernoulli measure ν to the two-sided (right) shift in the space $Y = 2^{\mathbb{Z}}$ with the stationary measure $\pi = S_*\nu$.

Proof. It is easy to see that two noncoinciding elements of the group $\mathbf{Z}_2$ whose first distinct digits have index n generate sequences $(P^k x)_1$ that have at least one noncoinciding digit with index less than 2^n. □

Remarks. 1. The same partition according to the first coordinate is obviously not a generator for the odometer.
2. The support of the Pascal measure is of great interest, and we study it below. In [16] (see also [10]) it is proved that the number of cylinders in the image (the "complexity of the Pascal automorphism"), i.e., the number of words of length n, is asymptotically equal to $n^3/6$.

Definition 5. The S-image of a point x will be called its *Pascal image*. It is defined for all $x \in \mathbf{Z}_2$ with infinitely many zeros and ones.

In order to study the Pascal automorphism, it is convenient to parameterize dyadic integers (i.e., elements of $\mathbf{Z}_2$) with infinitely many zeros and ones by sequences of pairs of positive integers $(m_i(x), k_i(x))$ in the following way:

$$x = (0^{m_1(x)}1^{k_1(x)}\mathbf{10}0^{m_2(x)}1^{k_2(x)}\mathbf{10}0^{m_3(x)}1^{k_3(x)}\dots).$$

In other words, the numbers $m_i(x) \ge 0$ (respectively, $k_i(x) \ge 0$) are the lengths of the words consisting of zeros (respectively, ones) between two (the $(i-1)$th and the ith) occurrences of the word "**10**" in the binary expansion of the number x. The sequence of pairs $(m_i(x), k_i(x))$ will be called the *pair coordinates* of x. Obviously, the ordinary coordinates can be recovered from them in a trivial way.

It is clear that the vectors $(m_i(x), k_i(x))$, regarded as functions of x, form a sequence (in i) of independent identically distributed random two-dimensional vectors, with the distribution

$$\Pr\{m_i = m, k_i = k\} = 2^{-(m+k-2)}, \quad m, k = 0, 1, 2\dots,$$

for every i.

The map $S : \mathbf{Z}_2 \to Y = \prod_{-\infty}^{\infty}\{0,1\}$ defined above can be written in a more specific form. This leads to the notion of *supporting words* introduced above.

Assume that the first pair coordinate of an element $x \in \mathbf{Z}_2$ is $m_1(x) = m \geq 0$, $k_1(x) = k \geq 0$; consider the elements $x, Px, P^2x, \dots P^sx$, with $s = C^m_{m+k+1}$, and write down the first coordinates of these elements. We will obtain a word of length s, which, by definition, is the beginning of the S-image of x.

Lemma 3. *The first coordinates of the elements* $x, Px, P^2x, \dots, P^sx$, $s = C^m_{m+k+1}$, *form the supporting word* $O(m+1,k)$ *in the sense of the definition from § 1.*

The rules describing the transformation of the pair coordinates under the Pascal automorphism, i.e., the expressions for $m_i(Px)$, $k_i(Px)$ in terms of $m_i(x)$, $k_i(x)$, are easy to formulate; however, for our purposes, only the transformation rule for the first coordinate is of importance.

The following recurrence rule for the transformation of the pair coordinates can be checked straightforwardly:

if $m_1(x) = 0$, then $m_1(Px) = \delta_{k_1(x)}$, $k_1(Px) = k_1(x) + 1$;

if $m_1(x) > 0$, then $m_1(Px) = (m_1(x) - 1)\delta_{k_1(x)}$, $k_1(Px) = k_1(x) - 1$.

However, it is most important to study the structure of supporting words, whose concatenations form almost all orbits with respect to the Pascal measure (i.e., the Pascal ensemble).

Example. The exotic sequence. Consider the S-image of a simplest $(1/2, 1/2)$-sequence, namely, of the point $x = (10)^\infty \in \mathbf{Z}_2$ (regarded as a real number, x is equal to $2/3$). The corresponding path in the Pascal graph is the central path passing through the vertices with coordinates $(n, [(n+1)/2])$, $n = 0, 1, \dots$:

$$00 \to 11 \to 21 \to 32 \to \cdots \to 43 \to 54.$$

Here is the beginning of the Pascal image of the point $x = (10)^\infty$ (or the corresponding path in the Pascal graph):

$$x = 1010101010* \to Px = 01101010* \to \dots$$
$$\to P^5x = 11000110* \to \cdots \to P^{14}x = 00011110 * .$$

The corresponding concatenation of supporting words constituting the sequence Sx in the Pascal ensemble is

$$Sx = O(0,0)O(1,0)O(2,1)O(3,2)O(4,3)\dots O(n, n+1)\dots .$$

An explicit form of the supporting word $O(4,3)$ is given below.

In other words, the Pascal (S-)image of this sequence is the sequence of the first coordinates of the adically ordered vertices of the middle hyperplanes of the cubes whose dimensions successively increase by one, $n = 1, 2, \dots$.

It is easier to describe this sequence as follows. Consider the set F_n of all positive integers whose binary expansion contains exactly $2n+1$ digits (the highest digit is equal to 1), i.e., of all integers in the interval $(2^{2n}, 2^{2n+1}-1)$, with the number of zeros equal to $n+1$, i.e., exceeding the number of ones (equal to n) by one; arrange these numbers in each set F_n according to the adic order, and then join them into a single sequence $F_1, F_2, \dots$. We obtain a sequence of positive integers, which starts as follows:

$$F_0 = \{0\}, \quad F_1 = \{1,2,4\}, \quad F_2 = \{3,5,6,9,10,12,17,18,20,24\}, \quad F_3 = \dots,$$

i.e.,

$$0,1,2,4,3,5,6,9,10,12,17,18,20,24,\dots .$$

Then the S-image of the point $x = (10)^\infty$ is the sequence of the *parities* of these numbers; in our case,

$$Sx = (0,1,0,0,1,1,1,0,1,0,0,1,0,0,0,\dots).$$

Definition 6. The Pascal image Sx of the point $x = (10)^\infty$ will be called the *exotic sequence of zeros and ones.*

Now we can easily find the Pascal image of a general element $x \in \mathbf{Z}_2$, i.e., a general path in the Pascal graph. To this end, for a given x (i.e., a given path in the Pascal graph), write down all indices $r_i(x)$, $i = 1, 2, \dots,$ for which $x_{r_i} x_{r_i+1} = 10$. Set

$$\sum_{t=1}^{r_i(x)} x_t = \bar{k}_i(x), \qquad \bar{m}_i(x) = r_i(x) - \bar{k}_i(x).$$

It is not difficult to express $\bar{m}_i$, $\bar{k}_i$ in terms of sums of the pair coordinates $m_i(x)$, $k_i(x)$, $i = 1, 2, \dots$. In the above example with $x = (10)^\infty$, we obviously have $r_i = 2i - 1$, $\bar{k}_i = i - 1$, $\bar{m}_i = i$.

Theorem 2. *For $x \in \mathbf{Z}_2$, the image $Sx \in \prod_{-\infty}^{\infty}\{0;1\}$ is a sequence of concatenations of supporting words with monotonically increasing indices. For positive indices, it looks as follows:*

$$O(\bar{m}_1(x), \bar{k}_1(x)) O(\bar{m}_2(x), \bar{k}_2(x)) \dots,$$

where the parameters $\bar{m}_i$, $\bar{k}_i$ are defined above.

Thus the Pascal image Sx of a point x is the corresponding sequence of the first coordinates (or the parities, in the numerical interpretation). In particular, for the point $x = (10)^\infty$ we obtain the exotic sequence.

The growth of the parameters of the supporting words $O(\bar{m}, \bar{k})$ is controlled by the following simple rule.

Lemma 4. *The parameters $(\bar{m}_i, \bar{k}_i)$ of the current supporting word $O(\bar{m}_i, \bar{k}_i)$ can be expressed in terms of the parameters of the preceding supporting word $(\bar{m}_{i-1}, k_{i-1})$ as follows:*

$$\bar{m}_i = \bar{m}_{i-1} + 1, \qquad \bar{k}_i = \bar{k}_{i-1} + \delta_{m_i} k_i,$$

where $m_i = m_i(x)$, $k_i = k_i(x)$ are the pair coordinates defined above ($\delta_t = 1$ if $t = 0$, and $\delta_t = 0$ if $t > 0$).

Recall that $m_i(x)$, $r_i(x)$ are independent (of each other and in i) random variables with geometric distribution. Curiously, the coordinate m (the first pair coordinate) grows deterministically, increasing by one at each step, $m \mapsto m + 1$, while the second coordinate k grows randomly, with the mean value of the increment equal to $+1$.

Thus almost every sequence with respect to the Pascal measure is the concatenation of supporting words $O(m+1, k)$ of growing length constructed from an element $x \in \mathbf{Z}_2$, i.e., a path in the Pascal graph; more exactly, from its pair coordinates $m_i(x)$, $k_i(x)$, $i \in \mathbb{Z}$. It suffices to study only the positive part of the sequence (with $i > 0$), since it allows us to make a conclusion about the discreteness or continuity of the spectrum of the Pascal automorphism. It is not difficult to prove that the coordinates with negative indices are uniquely determined by the coordinates with positive indices for almost all points with respect to the Pascal measure.

An example of the dynamics of the Pascal automorphism. Consider an example of a fragment of an orbit of the Pascal automorphism:

$$m_1 = 3,\ k_1 = 3,\ x = 000111\mathbf{10}.$$

The length of this fragment is equal to $C^{k_1}_{m_1+k_1+1}$, but it is more convenient to begin it with the last word of the previous fragment, so that the number of words we consider is equal to $C^3_7 = 36$: $x \to Px = 11100001 \to P^2x \to$

$\cdots \to P^{35}x = 00001111$ (recall that we should find the first occurrence of the word **10** and then apply the algorithm described above). We arrange the 36 successive images of the point x in a 6×6 table:

00011110	11100001	11010001	10110001	01110001	11001001
10101001	01101001	10011001	01011001	00111001	11000101
10100101	01100101	10011001	01010101	00110101	10001101
01001101	00101101	00011101	11000011	10100011	10001101
10010011	01010011	00011101	10001011	01001011	00101011
00011011	10000111	01000111	00100111	00010111	00001111

The sequence of the first digits of this fragment is the supporting word $O(4,3)$; see § 1.

The data in Table 1 shed some light on the distribution of probabilities of cylinders of lengths 6 to 10 with respect to the Pascal measure.[3]

One can observe that the cylinders are divided into several groups such that inside each group the probabilities are equal. This simplifies obtaining lower bounds on the growth of the scaling entropy needed for proving the continuity of the spectrum of the Pascal automorphism within the entropy approach [6] described below.

5 Criteria for the continuity of the spectrum of an automorphism

We formulate several necessary and sufficient conditions for the spectrum of an automorphism to be purely continuous; more exactly, for the spectrum of the unitary operator U_P associated with the automorphism by the formula $U_P f(x) = f(Px)$ to be continuous in the orthogonal complement of the constants in the space $L^2(I^\infty, \mu)$. This problem for the Pascal automorphism, along with its definition, was suggested by the author [6] in 1980 and subsequently considered in a series of papers (e.g., [14, 16, 18, 17, 19]), where various useful properties of the Pascal automorphism were studied; however, the problem has not been solved up to now.

[3] These computations were performed on my request by the PhD students I. E. Manaev and A. R. Minabutdinov.

Table 1:

Length of the word	Number of groups	Cardinality of the groups	Measure of the groups
6	3	5	0,312484741
		12	0,374969482
		20	0,312213898
7	4	2	0,124998093
		14	0,437492371
		16	0,249990463
		24	0,187408447
8	5	1	0,062498093
		12	0,374994278
		19	0,296865463
		20	0,156238556
		28	0,109274864
9	5	10	0,312494278
		22	0,343740463
		24	0,187488556
		24	0,093736649
		32	0,062393188
10	6	8	0,249994278
		21	0,328117371
		28	0,218738556
		29	0,113267899
		28	0,054672241
		36	0,03504467

5.1 Entropy approach

The original plan for solving this problem was related to the scaling entropy, average metrics, etc. (see [6, 10]). The method suggested by the author for proving the continuity of the spectrum (see [6]) relied on the following fact: the spectrum is purely continuous if and only if the result of averaging an arbitrary semimetric along an orbit is a trivial (constant) metric, i.e.,

$$\lim_{n\to\infty} n^{-1}\sum_{s=0}^{n-1}\rho(P^s x, P^s x') = \text{const}$$

for almost all pairs x, x' with respect to the measure $\mu \times \mu$.

In particular, in [6] the following theorem is proved. Let T be an automorphism of a Lebesgue space (X, μ). The spectrum of the corresponding unitary operator U_T in $L^2(X, \mu)$ is purely continuous in the orthogonal complement of the constants if and only if for every admissible semimetric for which the limiting average metric is also admissible, the scaling sequence for the entropy is bounded.

The assumption about the admissibility of the limiting average metric is superfluous, because, as shown in [6], it holds for every admissible initial metric. Originally, the admissibility of the limiting average metric was proved for the class of compact and bounded admissible semimetrics. On the other hand, in [6] it is proved that the average semimetric is constant if and only if the scaling sequence for the entropy is unbounded, i.e., the ε-entropy of the spaces obtained by successive averagings tends to infinity. For the Hamming metric, a close result was earlier proved in [15]. Thus one might prove the continuity of the spectrum by bounding the growth of the entropy of the prelimit average metrics from below by some growing sequence. Moreover, it would suffice to do this only for cut semimetrics, which have always been regarded in ergodic theory not as metrics, but rather as generating partitions. Recall that a *cut semimetric* is a semimetric determined by a finite partition of a measure space into measurable subsets $\{A_i\}$, $i = 1, \dots, k$, by the following formula: $\rho(x, y) = \delta_i(x)i(y)$, where $i(x)$ is the index of the set A_i that contains x.

Summarizing, we can formulate the following criterion for the continuity of the spectrum of an automorphism.

Theorem 3. *Given an automorphism T, the spectrum of the operator U_T in the orthogonal complement of the constants is continuous if and only if any of the following equivalent conditions is satisfied:*

1. *For an arbitrary cut semimetric ρ, the limit of the average metrics is a constant metric:*

$$\lim_n n^{-1} \sum_{k=0}^{n-1} \rho(T^k x, T^k y) = \text{const} \quad a.e.$$

2. *For an arbitrary initial semimetric, the ε-entropy of its averages grows unboundedly.*

We emphasize that the partition determining a cut metric in this theorem is not at all assumed to be a generator.

5.2 Besicovitch–Hamming almost periodicity and the NBH property

Consider the following semimetric in the spaces of (one- or two-sided) infinite sequences $\{x_n\}_{n\in\mathbb{N}}$ (or $\{x_n\}_{n\in\mathbb{Z}}$) of symbols $x_n \in A$ in a finite alphabet A:

$$\rho(x,y) = \liminf_{|n|\to\infty} \frac{1}{2n+1} \#\{k : |k| \leq n, \{x_k \neq y_k\}\}$$

for two-sided infinite sequences;

$$\rho(x,y) = \liminf_{n\to\infty} \frac{1}{n} \#\{k : k = 1,\dots,n, \{x_k \neq y_k\}\}$$

for one-sided infinite sequences. It should be called the Besicovitch–Hamming (BH) metric, since it is the limit of the Besicovitch metrics B_p as $p \to \infty$, as well as the limit (as $n \to \infty$) of the Hamming metrics on finite sequences.

For any stationary (i.e., invariant under the one- or two-sided shift) measure μ on the space $A^{\mathbb{Z}}$ (or $A^{\mathbb{N}}$), one can replace $\liminf$ in the definition of this metric with $\lim$ for almost all sequences (by the ergodic theorem).

Definition 7. A sequence $\{x_n\}$ is called *Besicovitch–Hamming (BH) almost periodic* if the set of its images under the (one- or two-sided) shift is relatively compact in the BH semimetric.

Our sketch of the proof that the spectrum is not discrete relies on the following well-known fact.

Lemma 5. *The shift S in the space $A^{\mathbb{Z}}$ of sequences in a finite alphabet $A = \{1, 2, \dots, l\}$ with stationary (shift-invariant) measure μ has a pure point spectrum if and only if almost all realizations $\{x_n\} \in A^{\mathbb{Z}}$ (or $\{x_n\} \in A^{\mathbb{N}}$) are BH almost periodic A-valued functions on $\mathbb{Z}$ (respectively, $\mathbb{N}$).*

Indeed, it suffices to apply von Neumann's discrete spectrum theorem and observe that the restriction of any bounded measurable function on a compact Abelian group to a countable $\mathbb{Z}$-subgroup, regarded as a function on $\mathbb{Z}$, is BH almost periodic.[4]

Since the BH almost periodicity of an orbit is equivalent to the relative compactness of the set of translations of this orbit, in order to establish the existence of a continuous component in the spectrum, one should verify that the translations of almost every orbit are not compact. Thus the procedure is to prove that *almost every, with respect to a given stationary measure, orbit is bounded away from the periodic orbits by a nonzero BH distance not depending on the length of the period.* In fact, it suffices to prove that it is bounded away not from all periodic sequences, but only from sequences whose periods are arbitrary finite subwords of a given sequence.

Usually, it suffices to prove this property only for one typical orbit.

Example. In order to prove the existence of a continuous component in the spectrum of the Morse automorphism (see [11]), it suffices first to check that the distance between the famous Morse–Thue sequence $0110100110010110\dots$ (the fixed point of the Morse–Hedlund substitution) and any periodic sequence is at least $1/2$. Extending this fact to almost all, with respect to the Morse measure, orbits presents no difficulties, since the structure of almost every orbit is similar to that of the Morse–Thue sequence. Indeed, to obtain almost every sequence from the Morse–Thue sequence, one should make the change ("fault") $0 \leftrightarrow 1$ at independent moments of time of the form $2^{n(\omega)}$ with geometric distribution in n. Under this operation, the bound on the distance discussed above between the modified words and the periodic words remains the same. This fact does not depend even on the distribution of the moments of "faults."

[4] It is worth mentioning that there is a great amount of confusion in the terminology related to almost periodic functions. For instance, by almost periodic sequences one means (see [13]) minimal or recurrent sequences, i.e., sequences x such that every word occurring in x occurs infinitely many times. But, in general, such sequences are even not BH almost periodic, though the BH condition should be considered as the weakest almost periodicity condition. According to the tradition introduced by von Neumann, almost periodicity is always related to the relative compactness of the set of group translations.

At the same time, the spectrum of the Morse automorphism has a discrete component, which agrees with the fact that after identifying every orbit with its "antipode," $\{x_n\} \leftrightarrow \{\bar{x}_n\}$, $\bar{0} = 1$, $\bar{1} = 0$, the factor automorphism of the Morse automorphism in the quotient space coincides with the odometer.

To prove the pure continuity of the spectrum in our situation, one needs to verify a more complicated property (which does not hold for the Morse automorphism), namely, the *uniform almost periodicity*.

Consider an arbitrary cylinder function $f(\cdot)$, say $\{0;1\}$-valued, depending on finitely many coordinates $x_1, \ldots, x_k$ and all its translations $f(T^k \cdot)$, $k \in \mathbb{Z}$. Divide the words of length k into m groups, denoting them $b_1, \ldots, b_m$, $b_i \in B$. A cylinder partition of the space $A^{\mathbb{Z}}$ is a finite partition of $A^{\mathbb{Z}}$ whose elements are unions of elementary cylinders (an elementary cylinder is defined as the set of all sequences whose coordinates with given indices, say $n = 1, 2, \ldots, k$, are words belonging to some b_i for a fixed i). A cylinder partition gives rise to a cylinder factorization, i.e., a natural map from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$ commuting with the shift. If $m > 1$ (i.e., B consists of at least two collections of words), the cylinder factorization is called nontrivial.

Definition 8. An *NBH-sequence* is a sequence $\{x_n\}$, $x_n \in A$, enjoying the following property: for every cylinder function $f = f(x_1, \ldots, x_k)$ with zero integral, the sequence $\{f(T^k \cdot)\}$, $k \in \mathbb{Z}$, is not BH almost periodic (in k); of course, the sequence $\{x_n\}$ itself is not almost periodic either.

Theorem 4. *The shift S in the space $A^{\mathbb{Z}}$ of sequences in a finite alphabet $A = \{1, \ldots, l\}$ with a stationary (shift-invariant) measure μ has a purely continuous spectrum (in the orthogonal complement of the constants) if and only if almost all realizations $\{x_n\} \in A^{\mathbb{Z}}$ are NBH-sequences.*

Proof. It follows from the definition and the previous theorem that for every cylinder factorization, the spectrum of the factor automorphism contains a continuous component. We need to verify this for an arbitrary factorization, i.e., for a factorization constructed from an arbitrary finite partition. We use the following argument. First of all, any finite partition can be approximated with arbitrary accuracy by cylinder partitions. In terms of the corresponding *cut semimetrics*, this means that in the topology of convergence in measure,

$$(\mu \times \mu)\{(x, y) : |\rho(x, y) - \rho'(x, y)| > \varepsilon\} < \varepsilon.$$

The average distance between points for all semimetrics ρ_ε may be assumed fixed and equal to the average distance D with respect to the metric ρ.

Averaging these inequalities and applying the ergodic theorem, we obtain

$$(\mu \times \mu)\left\{(x, y) : \lim_n \sum_{k=0}^{n-1} |\rho(T^k x, T^k y) - \rho'(T^k x, T^k y)| > \varepsilon\right\} < \varepsilon(1-\varepsilon) + 2D\varepsilon.$$

Since the limiting average of ρ_ε is a constant metric, equal to the average distance D for almost all pairs, and since ε is arbitrary, the limiting average of ρ is also a constant metric. By the result mentioned above, this means that the spectrum of the odometer T is continuous. □

5.3 The main lemma

The main combinatorial property of almost every (with respect to the Pascal measure) sequence, i.e., that of the parities of concatenations of supporting words corresponding to almost all points of $\mathbf{Z}_2$, is as follows.

Lemma 6. *The Pascal image of almost every, with respect to the Bernoulli measure μ_p, $0 < p < 1$, point $x \in \mathbf{Z}_2$ is a sequence of zeros and ones enjoying the NBH property.*

The proof will be published elsewhere; its structure is identical for almost all points of the type under consideration, so that it can be reduced to proving the desired assertion for only one sequence, e.g., for the sequence that we have called exotic. In brief, everything is based on the fact that every sufficiently long finite periodic sequence with period composed from any fragment of the exotic (or similar) sequence is bounded away in the BH metric from a sufficiently long fragment of this sequence itself by a universal constant. This fact is of purely combinatorial (or numerical) nature. In the simpler example considered above, that of the Morse automorphism, the proof of the non-almost periodicity was based on a similar property.

Corollary 1. *The Pascal automorphism (with respect to the image of any Bernoulli measure) has a continuous spectrum in the orthogonal complement of the constants.*

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