A NEW APPROACH TO THE REPRESENTATION THEORY OF THE SYMMETRIC GROUPS. IV. $\mathbb{Z}_2$-GRADED GROUPS AND ALGEBRAS

A. M. VERSHIK AND A. N. SERGEEV

Contents

1. Introduction

In this paper we formulate the main notions of the theory of locally semisimple $\mathbb{Z}_2$-graded finite-dimensional algebras and their representations. The main result is a translation of the inductive method of constructing the representation theory (the method of Gelfand–Tsetlin algebras) developed in [OV, VO, V] to the case of $\mathbb{Z}_2$-graded algebras. In particular, it allows us to use this method for constructing projective representations of the symmetric groups.

Although the general theory of $\mathbb{Z}_2$-graded semisimple algebras and their representations was partially described in [KL, J], nevertheless we present a systematic treatment of the main notions of this theory, keeping in mind their application to inductive chains of semisimple $\mathbb{Z}_2$-graded algebras. In this case, we have a number of interesting new phenomena, which are related to the fact that there are two types of simple $\mathbb{Z}_2$-graded finite-dimensional algebras (algebras of type $M(n, m)$ and $Q(n)$) and, consequently, two types of simple modules, and the branching graph of an inductive family has an additional structure (involution). The central question concerns the relation between representations of a graded algebra (group) and representations of its even part, for example, representations of symmetric and alternating groups. The answer to this question is comparatively simple: there is a bijection between these two classes of modules (see Proposition 3.7), and this relation can be used in both directions. The classical example is the description of representations of the alternating groups via representations of the symmetric groups. The converse problem, which is much more difficult, is to describe projective representations of the symmetric groups. We investigate this problem below. Surprisingly enough, the knowledge of ordinary representations of the symmetric group does not help to solve this problem and in fact is not used in the related papers. In other interesting examples, both representations of a $\mathbb{Z}_2$-graded algebra and representations of its even part are unknown, and we look for them simultaneously. Moreover, sometimes, having an explicit description of a graded algebra, we cannot explicitly describe its even part.

We suggest to construct representations of inductive chains of $\mathbb{Z}_2$-graded groups and algebras using the same techniques as were used in [OV, VO, V] for describing representations of the symmetric groups; this is the main goal of the paper. One example that we consider in more detail is the problem of describing projective representations of the symmetric group. This problem is reduced to that of describing simple modules of a certain $\mathbb{Z}_2$-graded algebra which was considered by I. Schur.

Projective representations of the symmetric groups were studied by many authors (e.g., [Mo, Se3, N1, N2]). We use this example to illustrate the new approach to the problem concerning representations of $\mathbb{Z}_2$-graded chains of algebras. First of all, we find conditions under which the branching of representations of a chain of semisimple $\mathbb{Z}_2$-graded algebras is simple. In what follows, the most important part is played by a generalization of the notion of Gelfand–Tsetlin algebra for a chain $\mathfrak{A} =< A(1) \subseteq A(2) \subseteq \cdots \subseteq A(n) >$ of semisimple $\mathbb{Z}_2$-graded algebras. First of all, one should generalize the notion of center and define the so-called supercenter of a $\mathbb{Z}_2$-graded algebra. For a chain of $\mathbb{Z}_2$-graded algebras, the notion of the Gelfand–Tsetlin algebra splits into several notions, because, in contrast to the nongraded case, the algebra $SGZ(\mathfrak{Y}))$ generated by the supercentralizers of the successive subalgebras (which in what follows will be called the *Gelfand–Tsetlin superalgebra*) does not coincide with the algebra $SZ(\mathfrak{Y}))$ generated by the supercenters of the algebras $A(k)$. In general, the algebras $SGZ(\mathfrak{Y}))$ are not commutative, even in the case of a simple branching, but their structure turns out to be standard: it is the tensor product of a commutative algebra and a Clifford algebra. Between the algebra $SGZ(\mathfrak{Y}))$ and its supercenter $SZ(\mathfrak{Y}))$ there is a place for the ordinary Gelfand–Tsetlin algebra $GZ(\mathfrak{Y}))$ of the even part of the chain $\mathfrak{Y}$. The analysis of representations and characters of these algebras is the essence of the method. Similarly to [OV, VO], we find the spectrum (the list of irreducible representations) of the algebra $SGZ(\mathfrak{Y}))$. This is done by using essentially the same technique: we reduce the problem to the description of an analog of the Hecke algebra, which in turn allows us to describe the irreducible representations of the algebra $A(n)$. This method is used for describing the projective representations of the symmetric groups, constructing an analog of Young's forms, bases, etc. As in [OV, VO], the so-called strict

Young tableaux, which parameterize a distinguished basis in representations, turn out to be the points of the spectrum of an appropriate Gelfand–Tsetlin algebra, and the irreducible projective representations of S_n are indexed by the strict diagrams, i.e., the orbits of admissible substitutions of points of the spectrum. This description of projective representations is one of the goals of the paper.

In the representation theory of $\mathbb{Z}_2$-graded groups and algebras and their branching diagrams, there arise many combinatorial problems, which apparently have not yet been studied. Even the quite well-known question about representations of the alternating group S_n^+ regarded as the even part of the $\mathbb{Z}_2$-graded group S_n is not sufficiently studied from this point of view. Note that, for example, a description of the alternating group similar to the description of Coxeter groups (more exactly, its presentation as a "local group" in the sense of [V1]) was obtained only quite recently in [VV]. This presentation should be used for obtaining a direct construction of representations of S_n^+ independently of representations of S_n. To this end, one should develop and apply the whole machinery of Gelfand–Tsetlin algebras. We hope to return to this problem in another paper. By analogy with the symmetric groups, one should consider projective representations of other classical Weyl groups, construct a normal form, describe the subalgebra generated by the supercentralizers, etc. One may also hope that the ideology of an inductive construction of representation theory will be applicable also in the theory of superalgebras, in particular, Lie superalgebras.

Let us briefly describe the contents of the paper. The second section contains the main definitions of the theory of $\mathbb{Z}_2$-graded associative finite-dimensional semisimple algebras. In particular, we describe the structure of these algebras and give a description of simple algebras. Like in the classical case, an important role is played by the notions of center and centralizer, which in the graded case have several versions. The third section contains a brief treatment of the theory of modules over $\mathbb{Z}_2$-graded algebras. Here the main theorem reduces the description of graded modules over a graded algebra to the description of nongraded modules over some other algebra. We also present theorems describing the relations between graded modules and nongraded modules; graded modules over an algebra and nongraded modules over its even part. Note that a brief and clear introduction to the theory of associative superalgebras and modules over them can be found in [KL]. In the fourth section we introduce one of the main objects of the paper: inductive families of $\mathbb{Z}_2$-graded semisimple finite-dimensional algebras. We define the branching graph of such a family and prove a theorem characterizing these graphs. Besides, we show that the branching graphs of simple modules and the branching graphs of inductive families of algebras, which *a priori* are defined in different ways, coincide. We also give a criterion for the simplicity of branching, which is based on the notion of graded centralizer.

The fifth section is devoted to Gelfand–Tsetlin algebras. Here, in contrast to the nongraded case, there arise several natural analogs of this algebra.

The main result of this section is a theorem describing the relation between representations of the Gelfand–Tsetlin superalgebra and representations of the original algebra. In the case of a simple branching, these results are interpreted in terms of the corresponding branching graph. In particular, this allows us to define a natural equivalence relation on the path space of a graded graph.

In the most important last section, we present an application of the theory developed in the previous sections to the study of projective representations of the symmetric groups. We explicitly describe the Gelfand–Tsetlin superalgebra in terms of odd analogs of the Young–Jucys–Murphy elements, which allows us to describe the spectrum of this superalgebra following the method of [OV].

2. Main definitions

In what follows, we assume that the ground field is $\mathbb{C}$.

Recall the definition of a $\mathbb{Z}_2$-graded algebra.

Definition 2.1. *A $\mathbb{Z}_2$-graded algebra is an algebra A that has a direct sum decomposition $A = A_0 \oplus A_1$ such that if $a \in A_i$, $b \in A_j$, then $ab \in A_{i+j}$, for $i, j \in \mathbb{Z}_2$. We will also write the condition $a \in A_i$ in the form $p(a) = i$ and call p the parity function. The even part A_0 is a subalgebra of A, and the odd part A_1 is an A_0-module.*

An equivalent definition is as follows: a $\mathbb{Z}_2$-graded algebra is an algebra A with an automorphism θ such that $\theta^2 = 1$. Here A_0 is the eigensubspace corresponding to the eigenvalue 1, and A_1 is the eigensubspace corresponding to the eigenvalue -1. We will refer to θ as the parity automorphism.

A homomorphism of $\mathbb{Z}_2$-graded algebras is a homomorphism of ordinary algebras that is also grading-preserving.

An algebra A regarded without the grading will be denoted by $|A|$.

A subalgebra of a $\mathbb{Z}_2$-graded algebra A is a subalgebra $B \subset A$ in the ordinary sense that inherits the grading: $B = B_0 + B_1$, $B_0 = B \cap A_0$, $B_1 = B \cap A_1$.

In a similar way, a two-sided ideal in a $\mathbb{Z}_2$-graded algebra A is a two-sided ideal I that inherits the grading: $I = I \cap A_0 + I \cap A_1$. Analogously, we define the notions of left and right ideals.

As we will see below, the following algebra is useful when considering $\mathbb{Z}_2$-graded algebras:

$$A[\varepsilon] = \{a + b\varepsilon \mid a, b \in A,\ \varepsilon^2 = 1,\ \varepsilon a = \theta(a)\varepsilon\}.$$

Definition 2.2. *A $\mathbb{Z}_2$-graded algebra is called simple if it contains no two-sided $\mathbb{Z}_2$-graded ideals other than 0 and the algebra itself.*

Definition 2.3. *A $\mathbb{Z}_2$-graded algebra A is called semisimple if every two-sided graded ideal of A has a graded complement, i.e., for every such ideal I there exists a two-sided graded ideal I' such that $A = I \oplus I'$.*

Example 2.4. [1] For $n, m \geq 0$, denote by $M(n,m)$ the set of matrices of the following form:

$$M(n,m)_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad M(n,m)_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\},$$

where A is a square matrix of order n, D is a square matrix of order m, and B, C are rectangular matrices of orders $n \times m$ and $m \times n$, respectively. Here the parity automorphism θ is an inner automorphism: there is a unique (up to sign) element $J \in M(n,m)_0$ satisfying $J^2 = 1$ such that $\theta(M) = JMJ^{-1}$; namely,

$$J = \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix},$$

where 1_k is the identity matrix of order n.

It is not difficult to check that the algebra $M(n,m)$ is simple both as a $\mathbb{Z}_2$-graded algebra and as a nongraded algebra.

Example 2.5. Denote by $Q(n)$ the set of matrices of the following form:

$$Q(n)_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \right\}, \quad Q(n)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right\},$$

where A, B are square matrices of order n and the parity automorphism is of the same form as in the previous example; note that $J \notin Q$.

It is not difficult to check that $Q(n)$ is simple as a graded algebra and is not simple as a nongraded algebra.

The following well-known theorem describes the structure of simple $\mathbb{Z}_2$-graded algebras; its proof can be found, e.g., in [J, KL].

Theorem 2.6. *Every finite-dimensional simple $\mathbb{Z}_2$-graded algebra over $\mathbb{C}$ is graded isomorphic either to $M(n,m)$ or to $Q(n)$, where n, m are arbitrary positive integers.*

The previous theorem shows that there are more $\mathbb{Z}_2$-graded simple finite-dimensional algebras than nongraded ones. Nevertheless, it turns out that the class of semisimple finite-dimensional algebras does not depend on the grading [J, KL].

Theorem 2.7. 1) *A finite-dimensional graded algebra is semisimple as a graded algebra if and only if it is semisimple as a nongraded algebra.*

2) *Every finite-dimensional semisimple graded algebra is the sum of finitely many simple algebras, i.e., $M(n,m)$ and $Q(k)$.*

Corollary 2.8. *The even part of a graded semisimple finite-dimensional algebra is semisimple.*

[1] Regarding the algebras below as Lie superalgebras, we obtain two principal series of Lie superalgebras: $\mathfrak{gl}(n,m)$ and $\mathfrak{q}(n)$.

Corollary 2.9. *The number of simple components of a semisimple finite-dimensional graded algebra is equal to the dimension of the even part of its ordinary center, and the number of simple components of type Q is equal to the dimension of the odd part of its ordinary center.*

Remark 2.10. It follows from the previous theorem that every finite-dimensional semisimple graded algebra A admits a unique decomposition of the form $A = A_M \oplus A_Q$, where A_M is the sum of all simple components of type M and A_Q is the sum of all simple components of type Q. Further, as follows from Example 2.4, there exists an element $J_A \in A_M$ such that $\theta(a) = J_A a J_A^{-1}$, $a \in A_M$.

One of the most important notions of the classical theory of algebras is that of the center of an algebra. In the graded case, a natural analog of the center is the following algebra, which arises from considering the center of the algebra $A[\varepsilon]$.

Definition 2.11. *The center of a graded algebra A is the algebra that is the sum of the even part of the center of the nongraded algebra and the even part of the twisted center:*

$$Z(A) = Z(|A|)_0 + Z^\theta(|A|)_0,$$

where $Z(|A|)_0 = \{a \in A_0 \mid ab = ba \text{ for every } b \in A\}$ is the even part of the ordinary center and $Z^\theta(|A|)_0 = \{a \in A_0 \mid ab = \theta(b)a \text{ for every } b \in A\}$ is the even part of the twisted center.

Definition 2.12. *The graded centralizer of a subalgebra B in a graded algebra A is the algebra that is the sum of the even part of the nongraded centralizer and the even part of the twisted centralizer:*

$$Z(A, B) = Z_0(|A|, |B|) + Z_0^\theta(|A|, |B|),$$

where $Z_0(|A|, |B|) = \{a \in A_0 \mid ab = ba \text{ for every } b \in B\}$ is the even part of the ordinary centralizer and $Z_0^\theta(|A|, |B|) = \{a \in A_0 \mid ab = \theta(b)a \text{ for every } b \in B\}$ is the even part of the twisted centralizer.

Note that if the algebra A is finite-dimensional and semisimple, then its graded center coincides with the center of its even part: $Z(A) = Z(A_0)$. This can easily be checked by reducing to the case of a simple algebra. A similar assertion is also true for the centralizer, see Lemma 3.3.

The following notion, which is used in the theory of Lie superalgebras, turns out to be useful.

Definition 2.13. *The supercentralizer of a subalgebra $B \subset A$ is the algebra*

$$SZ(A, B) = \{a \in A \mid ab = (-1)^{p(a)p(b)} ba \text{ for every } b \in B\}.$$

In particular, the supercentralizer $SZ(A, A)$ is called the supercenter of A and is denoted by $SZ(A)$.

Note that for a semisimple finite-dimensional algebra, the supercenter coincides with the even part of the ordinary center: $SZ(A) = Z(|A|)_0$.

We will also need the notion of a $\mathbb{Z}_2$-graded group.

Definition 2.14. *A finite group G is called $\mathbb{Z}_2$-graded if it contains a distinguished normal subgroup G_0 of index 2, and the decomposition into the cosets of G_0 is the decomposition into the even and odd parts: $G = G_0 \bigcup G_1$.*

The group algebra of a $\mathbb{Z}_2$-graded group has a natural $\mathbb{Z}_2$-grading. It is not difficult to prove the following fact.

Theorem 2.15. *The group algebra of a finite graded group with the natural $\mathbb{Z}_2$-grading is a semisimple $\mathbb{Z}_2$-graded algebra.*

This theorem is a special case of a more general result proved in [NS].

For brevity, in what follows we often use the term "a graded algebra (group, module)" instead of "a $\mathbb{Z}_2$-graded algebra (group, module)." Here are the main examples of graded groups and algebras.

1) The symmetric group S_n in which the parity of an element $g \in S_n$ is defined as the parity of the permutation g, and the group algebra $\mathbb{C}[S_n]$ endowed with the corresponding grading.

2) The $\mathbb{C}$-algebra $\mathfrak{A}_n$ corresponding to projective representations of the symmetric group S_n; it is generated by elements $\tau_1, \dots, \tau_n$ satisfying the relations

$$\tau_k^2 = 1, \quad (\tau_k \tau_{k+1})^3 = 1, \quad (\tau_k \tau_l)^2 = -1 \text{ if } \mid k - l \mid > 1.$$

The graded algebra $\mathfrak{A}_n$ is not the group algebra of any graded group.

3) The semidirect product $S_n \ltimes C_n$ of the Clifford algebra C_n (with the natural action of the symmetric group) and the symmetric group S_n; here all generators of the Clifford algebra are assumed to be odd, and all elements of the symmetric group are assumed to be even. Note that we have the relation

$$S_n \ltimes C_n = \mathfrak{A}_n \otimes C_n,$$

where the tensor product in the right-hand side is understood as the tensor product of graded algebras. The corresponding definition is as follows.

Definition 2.16. *The tensor product $A \otimes B$ of graded algebras A and B is the graded algebra that coincides as a vector space with the ordinary tensor product of the algebras A and B, with the parity automorphism and multiplication defined by the following rules:*

$$\theta_{A \otimes B}(a \otimes b) = \theta_A(a) \otimes \theta_B(b), \quad (a \otimes b)(a' \otimes b') = (-1)^{ji} aa' \otimes bb',$$

where $a' \in A_i$, $b \in B_j$.

Note that the tensor product of graded algebras, regarded as a nongraded algebra, is not, in general, isomorphic to their tensor product as ordinary algebras.

Lemma 2.17. *The tensor product of semisimple finite-dimensional associative graded algebras is semisimple.*

Proof. The proof reduces to the case of a simple graded algebra. In this case, it is easy to check the equalities

$$M(n,m) \otimes M(n',m') = M(nn' + mm', nm' + nm'),$$
$$M(n,m) \otimes Q(l) = Q((m+n)l), \quad Q(n) \otimes Q(m) = M(nm, nm).$$

□

3. $\mathbb{Z}_2$-graded modules

We will consider the category GMod of graded modules over a $\mathbb{Z}_2$-graded algebra A.

Definition 3.1. *A graded module over a graded algebra A is an A-module V that has a decomposition $V = V_0 \oplus V_1$ such that the corresponding grading is preserved by the action of A, i.e., $A_i V_j \subset V_{i+j}$, where $i, j \in \mathbb{Z}_2$.*

Recall also that the graded dimension $\operatorname{Gdim} V$ of a graded module V is the pair $(\dim V_0, \dim V_1)$. The set of morphisms in the category GMod is the set of homomorphisms $f : V \to U$ of ordinary modules that preserve the grading, i.e., satisfy $f(V_i) \subset U_i$, $i \in \mathbb{Z}_2$.

On the category GMod there exists a natural functor of changing the parity: $V \to P(V)$, where $P(V)_0 = V_{\bar{1}}$, $P(V)_1 = V_{\bar{0}}$, and the new action coincides with the old one.

Definition 3.2. *A graded module V is called simple or irreducible if it has no graded submodules other than the zero one and V itself.*

Examples.

1) The algebra $M(n,m)$ has two nonisomorphic graded simple modules V and $P(V)$. Note that the graded dimension of one of them equals (n,m), while the graded dimension of the other one equals (m,n). It is easy to check that both modules are simple and isomorphic as nongraded modules. Such modules will be called a pair of *antipodal modules* of type M. Note that the algebra $M(n,m)$ is the algebra of all graded endomorphisms of the $\mathbb{Z}_2$-graded vector spaces V and $P(V)$.

2) The algebra $Q(n)$ has only one simple graded module $V = P(V)$, where V is the standard representation and $\operatorname{Gdim} V = (n,n)$. It is easy to check that V is reducible as a nongraded module and that it is the direct sum of two nonisomorphic simple nongraded modules. Irreducible modules of this type will be called modules of type Q.

Lemma 3.3. *If A is a semisimple finite-dimensional algebra and B is a semisimple subalgebra of A, then*

$$Z(A,B) = Z(A_0, B_0).$$

Proof. Clearly, $Z(A,B) = Z_0(|A|,|B|) + Z_0^\theta(|A|,|B|) \subset Z(A_0,B_0)$. Thus it suffices to show that both algebras have the same dimension over $\mathbb{C}$. Further, it is not difficult to check that we may restrict ourselves to the case where A is a simple graded algebra, i.e., $A = M(n,m)$ or $A = Q(n)$. Consider the first case. Let V be one of the irreducible modules over A. We may restrict ourselves to the case where V is the sum of B-modules $U, P(U)$ for some irreducible B-module U. The following cases are possible: 1) $B = B_0$; 2) $B \neq B_0$.

In the first case, $Z_0(A,B) = Z_0^\theta(|A|,|B|) = Z(A_0,B_0)$. In the second case, we have two possibilities: $U \neq P(U)$ and $U = P(U)$.

In the first one, $V = U^k \oplus P(U)^l$ for some k, l, whence $\dim(Z_0(|A|,|B|) + Z_0^\theta(|A|,|B|)) = 2k^2 + 2l^2 = \dim Z(A_0,B_0)$. In the second one, $V = U^k$ for some k, whence $\dim(Z_0(|A|,|B|) + Z_0^\theta(|A|,|B|)) = 4k^2 = \dim Z(A_0,B_0)$. $\square$

Let θ be an automorphism of an algebra A. Then on the category of nongraded modules we have the natural functor $V \to V^\theta$, where $V^\theta = V$ and the new action is defined by the formula $a \star v = \theta(a)v$.

Corollary 3.4. *Let A be a graded algebra, θ be the parity automorphism of A, and*

$$\mathrm{Rep}(|A|) = (E_1, E_1^\theta, \dots, E_r, E_r^\theta, F_1 = F_1^\theta, \dots, F_s = F_s^\theta)$$

be the complete set of pairwise nonisomorphic nongraded modules. Then

$$\mathrm{Rep}(A) = (F_1, P(F_1), \dots, F_s, P(F_s), E_1 \oplus E_1^\theta, \dots, E_r \oplus E_r^\theta)$$

is the complete set of pairwise nonisomorphic graded modules.

It turns out that the category of graded modules is isomorphic to a certain category of nongraded modules over some other algebra. Consider the algebra

$$A[\varepsilon] = \{a + b\varepsilon \mid a, b \in A,\ \varepsilon^2 = 1,\ \varepsilon a = \theta(a)\varepsilon\}.$$

Note that $A[\varepsilon]$ has the canonical automorphism φ defined by the formulas $\varphi(a) = a$, $a \in A$, and $\varphi(\varepsilon) = -\varepsilon$.

Proposition 3.5. *The category of nongraded $A[\varepsilon]$-modules is isomorphic to the category* GMod *of graded A-modules. Under this isomorphism, the functor of changing the parity goes to the functor $V \to V^\varphi$. A graded irreducible A-module is of type M if and only if $V \neq V^\varphi$; it is of type Q if and only if $V = V^\varphi$.*

Proof. Let us construct functors

$$F : \mathrm{Mod}(A[\varepsilon]) \longrightarrow \mathrm{GMod}(A) \quad \text{and} \quad G : \mathrm{GMod}(A) \longrightarrow \mathrm{Mod}(A[\varepsilon]).$$

Let V be an object of $\mathrm{Mod}(A[\varepsilon])$; then $\varepsilon \in \mathrm{End}(V)$. Since $\varepsilon^2 = 1$, it follows that V can be regarded as an object of the category $\mathrm{GMod}(A)$ with the following grading:

$$V_0 = \{v \in V \mid \varepsilon v = v\} \quad \text{and} \quad V_1 = \{v \in V \mid \varepsilon v = -v\}.$$

The functor F acts identically on morphisms. Since every morphism commutes with ε, it preserves the above grading and hence is a morphism of $\mathrm{GMod}(A)$.

The inverse functor also acts identically on objects and morphisms; the action of ε coincides with the action of the parity operator. It is not difficult to check that FG and GF are identical functors on the corresponding categories. $\square$

Lemma 3.6. *Assume that a graded algebra contains an odd element p such that $p^2 = 1$. Then one can introduce a grading on the algebra A_0 such that A will be isomorphic to $A_0[\varepsilon]$.*

Proof. Take the automorphism $a \to pap^{-1}$ as the parity automorphism on A_0. Then $A_0[\varepsilon] = A$. $\square$

Let us say that a $\mathbb{Z}_2$-grading of a semisimple algebra is *essential* if for all its simple components of type $M(n, m)$ both indices are positive: $n, m > 0$. For such an algebra, the odd and even parts of every simple module are nontrivial.

Proposition 3.7. *Let A be a semisimple finite-dimensional $\mathbb{Z}_2$-graded algebra with an essential grading. Then the functor*

$$I : V_0 \longrightarrow A \otimes_{A_0} V_0$$

from the category $\mathrm{Mod}(A_0)$ *to the category* $\mathrm{GMod}(A)$ *is an isomorphism of categories.*

Proof. It suffices to prove that the restriction functor

$$R : V \longrightarrow V_0$$

from the category $\mathrm{GMod}(A)$ to the category $\mathrm{Mod}(A_0)$ is the two-sided inverse to I. Indeed, $R \circ I(V_0) = A_0 \otimes_{A_0} V_0 = V_0$. Conversely, let us prove the equality $I \circ R(V) = V$. By additivity, it suffices to prove it for a simple module V. But the previous equality is equivalent to $A \otimes_{A_0} V_0 = V$. Since V is a simple A-module, this equality implies that V_0 is a simple A_0-module. By the assumptions of the lemma, the even part of every simple A-module is nontrivial. But the even part of the module $A \otimes_{A_0} V_0$ equals V_0. Thus this module is irreducible and coincides with V. The proposition is proved. $\square$

It follows from the proposition that the theory of graded modules of every semisimple $\mathbb{Z}_2$-graded algebra A reduces to the theory of ordinary representations of its even part A_0. However, this reduction does not always help to describe all representations of the algebra A, and, conversely, can be used in the opposite direction. This is the case for projective representations of the symmetric groups, where the even part has no simple realization, see Section 6 below.

4. Inductive families (chains) of $\mathbb{Z}_2$-graded algebras

Definition 4.1. *A $\mathbb{Z}_2$-graded graph is a graph that has an involutive automorphism.*

The main examples of $\mathbb{Z}_2$-graded graphs appear in the following situation. Consider a (finite) chain of $\mathbb{Z}_2$-graded algebras

$$\mathfrak{Y} =< \mathbb{C} = A(1) \subset A(2) \subset \cdots \subset A(n) >, \quad n = 1, 2, \ldots.$$

Denote by $GA_i^\wedge$ the set of isomorphism classes of irreducible objects of the category GMod.

Definition 4.2. *The branching graph of simple modules of the chain $\mathfrak{Y}$ is the directed graded*[2] *graph $Y(\mathfrak{Y})$ whose set of vertices is the disconnected union*

$$\bigcup_{i=1}^{n} GA(i)^\wedge$$

and the number of edges connecting a vertex $U \in GA_i^\wedge$ with a vertex $V \in GA_{i+1}^\wedge$ and directed from U to V is equal to the dimension of the vector space $\mathrm{GHom}_{A_i}(U, V)$ of A_i-homomorphisms between U and V.

The corresponding branching graph of nongraded modules will be denoted by $Y(|\mathfrak{Y}|)$. The branching graph of the even subalgebras will be denoted by $Y(\mathfrak{Y}_0)$.

On the category of $\mathbb{Z}_2$-graded modules we have the involutive functor $V \to P(V)$ of changing the parity. Hence every graph of the form $Y(\mathfrak{Y})$ has an automorphism ω such that $\omega^2 = 1$. Thus ω determines the structure of a $\mathbb{Z}_2$-graded graph on $Y(\mathfrak{Y})$. Besides, the zero level of every such graph contains two vertices, which are swapped by ω. It turns out that these properties are characteristic.

A characterization of the graded branching graphs of ordinary modules is as follows (see [VK]).

Theorem 4.3. *Let $Y = \cup_{i=0}^{n} Y_i$ be a finite graded graph with positive integer multiplicities of edges. Then it is the branching graph of a chain of nongraded modules if and only if the following conditions hold:*

1) *For every vertex y of Y that does not belong to Y_n there exists a vertex that immediately follows y.*

2) *For every vertex y of Y that does not belong to Y_0 there exists a vertex that immediately precedes y.*

3) *The set Y_0 consists of a single element.*

In the case of a chain of graded algebras, we have the following theorem.

Theorem 4.4. *Let $Y = \cup_{i=0}^{n} Y_i$ be a finite graded graph with positive integer multiplicities of edges. Then it is the branching graph of a chain of graded modules if and only if the following conditions hold.*

[2] One should not confuse the term "graded" in the context related to the branching graph with the $\mathbb{Z}_2$-grading of algebras, modules, etc.

1) *For every vertex y of Y that does not belong to Y_n there exists a vertex that immediately follows y.*

2) *For every vertex y of Y that does not belong to Y_0 there exists a vertex that immediately precedes y.*

3) *There exists an involutive automorphism ω of the graph Y that preserves the grading.*

4) *The set Y_0 consists of two elements, and the automorphism ω swaps these elements.*

The construction of an algebra from the corresponding graph follows the same scheme as in [VK]. Namely, consider the set $B(Y)$ of loops of the graph Y, i.e., the set of pairs of paths (t,s) that begin at the same vertex of level 0 and end at the same vertex of level n. Consider the vector space A of functions f on $B(Y)$ that are invariant under ω, i.e., satisfy $f(\omega s,\omega t) = f(s,t)$, and define the parity operator in this space by the following formula: $\theta(f)(t,s) = f(t,s)$ if the paths s,t begin at the same vertex, and $\theta(f)(t,s) = -f(t,s)$ if the paths s,t begin at different vertices. The multiplication is defined as in [VK]:

$$(f * g)(s,t) = \sum_r f(s,r)g(r,t),$$

where $(s,t) \in B(Y)$ and the sum is taken over all paths r such that $(s,r),(r,t) \in B(Y)$. We can also describe the decomposition of the algebra A into simple components in terms of the graph Y. Denote by $\tilde{Y}_n$ the set of orbits with respect to the action of ω in Y_n. Since ω is an involutive automorphism, orbits can be of two types: those consisting of a single vertex (type Q), and those consisting of two vertices (type M). Let $t \in \tilde{Y}$, and let n_t, m_t be the numbers of paths going from the two vertices of Y_0 to one of the vertices belonging to t. Note that in the case of a vertex of type Q, we have $n_t = m_t$. Then

$$A = \oplus_{t\in\tilde{Y}_n} A_t,$$

where $A_t = M(n_t, m_t)$ if the vertex is of type M, and $A_t = Q(n_t)$ if the vertex is of type Q.

Definition 4.5. *Let us say that two chains $\mathfrak{Y}, \mathfrak{Y}'$ of $\mathbb{Z}_2$-graded algebras are equivalent if for $i = 1, \ldots, n$ there exist isomorphisms $f_i : A(i) \to A(i)'$ such that the diagrams*

$$\begin{array}{ccc} A(i+1) & \xrightarrow{f_{i+1}} & A'(i+1) \\ \uparrow & & \uparrow \\ A(i) & \xrightarrow{f_i} & A'(i) \end{array}$$

are commutative for $i = 1, \ldots, n-1$.

Proposition 4.6. *Equivalence classes of chains are in a bijection with the branching graphs of graded simple modules (regarded up to isomorphism).*

Proof. It suffices to consider the case of a chain of length 2. First note that it suffices to restrict ourselves to chains of the form $B \subset A$, $B \subset A'$ and prove that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \uparrow & & \uparrow \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array} \tag{1}$$

is commutative if and only if the corresponding simple A, A'-modules V, V' are isomorphic as B-modules. Replacing the algebra B by the projection to the corresponding simple component, we may assume that A, A' are simple algebras. Besides, we may replace the algebra A' by A. Thus it suffices to prove the following assertion. Let A be a simple $\mathbb{Z}_2$-graded algebra, V be a standard A-module, B be a $\mathbb{Z}_2$-graded subalgebra of A, and φ be an automorphism of A. Then φ acts on B identically if and only if it is of the form $\varphi(a)v = faf^{-1}v$, where f is an isomorphism either of the B-module V, or of the B-modules V and $P(V)$. Let us prove this assertion. Assume that φ is of this form. Then for $b \in B$ we obtain $\varphi(b)v = fbf^{-1}v = bv$. Hence $\varphi(b) = b$. Conversely, assume that $\varphi(b) = b$ for every $b \in B$. The parity automorphism of the algebra A is of the form $a \to JaJ^{-1}$, where J is the parity automorphism of the module V. Since φ preserves the parity, $\varphi(J) = \pm J$. Further, there exists a linear map $f : V \to V$ such that $\varphi(a)v = faf^{-1}v$. Since $\varphi(J) = \pm J$, it follows that f is a graded homomorphism either $V \to V$ or $V \to P(V)$. The conditions $\varphi(b) = b$, $\varphi(a)v = faf^{-1}v$ imply that f is a homomorphism of B-modules. The proposition is proved. □

A branching graph is called simple if it has no multiple edges. In order to formulate a simplicity criterion, we use the notion of centralizer from Section 2. The following lemma gives a simplicity criterion for graded modules.

Lemma 4.7. *Let $B \subset A$ be a short chain of graded algebras. The branching of the corresponding graded modules is simple if and only if the algebra $Z(A, B)$ is commutative, and in this case it is generated by the graded centers $Z(A), Z(B)$. (Cf. the simplicity criterion in* [OV, VO]*).*

Proof. It is not difficult to check that $Z(A[\varepsilon], B[\varepsilon]) = Z_0(|A|, |B|) + \varepsilon Z_0^\theta(|A|, |B|)$.

Hence the algebra $Z(A, B)$ is the image of the algebra $Z(A[\varepsilon], B[\varepsilon])$ under the homomorphism that sends ε to 1. It easily follows that $Z(A, B)$ is commutative if and only if so is $Z(A[\varepsilon], B[\varepsilon])$. In this case, it is well known that $Z(A, B)$ is generated by the subalgebras $Z(A[\varepsilon])$, $Z(B[\varepsilon])$. □

Like in the nongraded case, two types of problems arise for the graphs under consideration. The analysis problem: given a graded algebra, construct the graph of simple modules. And the synthesis problem: given a graph with involution, describe the algebra for which it is the graph of simple modules.

Let G be a $\mathbb{Z}_2$-graded group; then G_0, being a subgroup of index 2, is a normal subgroup of G, so that G is a $\mathbb{Z}_2$-extension of G_0. Recall (see [FH]) the relation between irreducible representations of these groups. The group

$\mathbb{Z}_2$ acts on the simple modules of the group G_0 and divides them into two classes: those that are fixed under the action of $\mathbb{Z}_2$ and those that are not. The direct sum of a module of the second class and its $\mathbb{Z}_2$-image is an irreducible $\mathbb{Z}_2$-graded G-module. Note that for $\mathbb{Z}_2$-graded algebras, the situation is similar but more complicated. The existence of such relations implies the existence of relations between the corresponding branching graphs $Y(\mathfrak{Y})$, $Y(|\mathfrak{Y}|)$, and $Y(\mathfrak{Y}_0)$.

In contrast to the tradition, we will denote the alternating group by S_n^+ rather than A_n.

Proposition 3.7 implies the following lemma.

Lemma 4.8. *Assume that the odd part of every algebra of a chain $\mathfrak{Y}$ of $\mathbb{Z}_2$-graded algebras is nontrivial. Then the branching graph $Y(\mathfrak{Y})$ of this chain coincides with the branching graph $Y(|\mathfrak{Y}_0|)$.*

Example 4.9. Consider the involutive automorphism θ of the group algebra $\mathbb{C}[S_n]$ of the symmetric group defined by the formula $\theta(\sigma) = \mathrm{sgn}(\sigma)\sigma$, where $\mathrm{sgn}(\sigma)$ is the sign of a permutation σ, and the corresponding structure of a graded algebra on $\mathbb{C}[S_n]$. Let $Y(\mathbb{S})$ be the graded branching graph of the chain

$$\mathbb{S} =< \mathbb{C} = \mathbb{C}[S_1] \subset \mathbb{C}[S_2] \subset \cdots \subset \mathbb{C}[S_n] >$$

of the group algebras of the symmetric groups with this $\mathbb{Z}_2$-grading, and let $Y(|\mathbb{S}^+|)$ be the nongraded branching graph of the chain

$$\mathbb{S}^+ =< \mathbb{C} = \mathbb{C}[S_1^+] \subset \mathbb{C}[S_2^+] \subset \cdots \subset \mathbb{C}[S_n^+] >$$

of the group algebras of the alternating groups. Then it follows from Lemma 4.8 that the graph $Y(\mathbb{S})$ coincides with the graph $Y(|\mathbb{S}^+|)$ at all levels except the first one.

Note that an irreducible graded representation of the symmetric group coincides with the ordinary representation corresponding to a diagram λ if $\lambda = \lambda'$, and is equal to the direct sum of the ordinary representations corresponding to the diagrams λ, λ' if $\lambda \neq \lambda'$.

Consider the involutive automorphism θ of the group algebra $\mathbb{C}[S_n^+]$ of the alternating group given by the formula $\theta(f) = s_{12} f s_{12}$, where s_{12} is the transposition of the symbols $1, 2$, and the corresponding structure of a superalgebra on $\mathbb{C}[S_n^+]$; then Lemma 3.6 implies the following theorem.

Theorem 4.10. 1) *An ordinary irreducible representation of the group S_n remains irreducible regarded as a graded representation of the group A_n. It is of type Q if the corresponding Young diagram λ is self-conjugate, i.e., $\lambda = \lambda'$; it is of type M if the corresponding Young diagram λ is not self-conjugate, i.e., $\lambda \neq \lambda'$.*

2) *Let $Y(\mathbb{S}^+)$ be the graded branching graph of the chain*

$$\mathbb{S}^+ =< \mathbb{C} = \mathbb{C}[S_1^+] \subset \mathbb{C}[S_2^+] \subset \cdots \subset \mathbb{C}[S_n^+] >$$

of the group algebras of the alternating groups with the $\mathbb{Z}_2$-grading introduced above, and let $Y(|\mathbb{S}|)$ be the nongraded branching graph of the chain

$$\mathbb{S} =< \mathbb{C} = \mathbb{C}[S_1] \subset \mathbb{C}[S_2] \subset \cdots \subset \mathbb{C}[S_n] >$$

of the group algebras of the symmetric groups (i.e., the Young graph). Then $Y(\mathbb{S}^+)$ coincides with $Y(|\mathbb{S}|)$ at all levels except the first one.

5. Gelfand–Tsetlin algebras

Consider a chain of $\mathbb{Z}_2$-graded algebras

$$\mathfrak{Y} =< \mathbb{C} = A(1) \subset A(2) \subset \cdots \subset A(n) > .$$

Definition 5.1. *Assume that the branching graph of the chain $\mathfrak{Y}$ is simple. The Gelfand–Tsetlin algebra $GZ(\mathfrak{Y})$ is the algebra generated by the graded centers $Z(A_i)$, $i = 1, \dots, n$.*

Theorem 5.2. *Consider a chain of $\mathbb{Z}_2$-graded algebras*

$$\mathfrak{Y} =< \mathbb{C} = A(1) \subset A(2) \subset \cdots \subset A(n) >$$

with simple branching. Then the Gelfand–Tsetlin algebra $GZ(\mathfrak{Y})$ coincides with the ordinary Gelfand–Tsetlin algebra $GZ(\mathfrak{Y}_0)$ of the chain of even subalgebras. It is a maximal commutative subalgebra among the even commutative subalgebras in $A(n)$. Every irreducible graded representation of the algebra $A(n)$ has a homogeneous basis that consists of eigenvectors of this subalgebra.

Proof. Consider the chain of algebras

$$\mathbb{C} \subset \mathbb{C}[\varepsilon] = A(1)[\varepsilon] \subset A(2)[\varepsilon] \subset \cdots \subset A(n)[\varepsilon].$$

Since the branching is simple, the algebra generated by $Z(A(i+1)[\varepsilon], A(i)[\varepsilon])$, $i = 1, \dots, n-1$, is a maximal commutative subalgebra in $A(n)[\varepsilon]$. Further, consider the homomorphism $A(n)_0[\varepsilon] \to A(n)_0$ that sends ε to the identity. The image of every maximal commutative subalgebra in $A(n)_0[\varepsilon]$ is a maximal commutative subalgebra in $A(n)_0$. □

Thus, to a certain extent, the representation theory of a chain of $\mathbb{Z}_2$-graded algebras reduces to studying representations of the chain of the even parts of these algebras. However, it is not always possible to describe the even part of a graded algebra in a transparent way, so that it is useful to introduce another version of the notion of Gelfand–Tsetlin algebra, which is related to the notion of supercentralizer (see Section 2) used in the theory of Lie superalgebras.

Definition 5.3. *Consider a chain of $\mathbb{Z}_2$-graded algebras*

$$\mathfrak{Y} =< \mathbb{C} = A(1) \subset A(2) \subset \cdots \subset A(n) > .$$

1) *The algebra*

$$SGZ(\mathfrak{Y}) =< SZ(A(2), A(1)), \dots, SZ(A(n), A(n-1)) >$$

generated by the successive supercentralizers is called the Gelfand–Tsetlin superalgebra.

2) *Denote by $SZ(\mathfrak{Y})$ the commutative algebra generated by the supercenters $SZ(A(1)), \dots, SZ(A(n))$.*

As will be shown below, the algebra $SZ(\mathfrak{Y})$ is the supercenter of the algebra $SGZ(\mathfrak{Y})$.

In the nongraded case, for chains with simple branching the algebra generated by the successive centralizers coincides with the algebra generated by the centers: $SZ(\mathfrak{Y}) = SGZ(\mathfrak{Y})$. If the spectrum is not simple, $SGZ(\mathfrak{Y})$ is the so-called "big Gelfand–Tsetlin algebra." In the graded case, we have the following (in general, strict) inclusions

$$SZ(\mathfrak{Y}) \subset GZ(\mathfrak{Y}) \subset SGZ(\mathfrak{Y}),$$

the first two algebras being commutative. The role of the Gelfand–Tsetlin superalgebra is that the decompositions of irreducible $A(n)$-modules into irreducible $SGZ(\mathfrak{Y})$-modules are disjoint. In the case of trivial grading and simple spectrum, the algebra $SGZ(\mathfrak{Y})$ coincides with $SZ(\mathfrak{Y})$. The main result of this section is the following theorem.

Theorem 5.4. *Consider a chain of $\mathbb{Z}_2$-graded algebras*

$$\mathfrak{Y} =< \mathbb{C} = A(1) \subset A(2) \subset \cdots \subset A(n) > .$$

Then

1) *The restriction of an irreducible $A(n)$-module to the Gelfand–Tsetlin superalgebra has a simple spectrum.*

2) *The decompositions of different irreducible $A(n)$-modules into irreducible $SGZ(\mathfrak{Y})$-modules have no common components. The irreducibility type of a nonzero $SGZ(\mathfrak{Y})$-module coincides with the irreducibility type of the $A(n)$-module that contains it.*

Proof. The proof is based on a series of lemmas.

Lemma 5.5. *Let V be a finite-dimensional graded vector space, $A \subset \mathrm{End}(V)$ be a semisimple graded subalgebra of $\mathrm{End}(V)$, and A' be its supercentralizer. Then $(A')' = A$.*

The proof of the lemma is contained in [NS]. It is similar to von Neumann's theorem on the bicommutant of subalgebras in simple algebras, and can be proved analogously.

Lemma 5.6. *Let A be a $\mathbb{Z}_2$-graded algebra and B be a $\mathbb{Z}_2$-graded subalgebra of A. Then the equality $SZ(A, B) = SZ(B)$ holds if and only if every irreducible A-module decomposes into a multiplicity-free sum of simple B-modules and different irreducible A-modules have no common irreducible B-components. Moreover, every irreducible B-component has the same irreducibility type as the irreducible A-module that contains it.*

Proof. Assume that every irreducible A-module decomposes into a multiplicity-free sum of simple B-modules and different irreducible A-modules have no common irreducible B-components. Let $V_1, \dots, V_q$ be the complete set of pairwise nonisomorphic irreducible A-modules, $U_1, \dots, U_p$ be all different B-modules, and $b_1, \dots, b_p$ be the corresponding central idempotents. Let $a \in SZ(A, B)$. By the assumptions of the lemma, a is an even element. Since every irreducible A-module decomposes into a multiplicity-free sum of simple B-modules, the element a preserves each module U_j and acts in it as a scalar c_j. Since different irreducible A-modules have no common irreducible B-components, the difference $a - c_1b_1 - \dots - c_pb_p$ vanishes in every irreducible A-module. This proves that $SZ(A, B) = SZ(B)$.

Conversely, assume that $SZ(A, B) = SZ(B)$. Therefore $SZ(A) \subset SZ(B)$. Let e_1, e_2 be two different central idempotents in A corresponding to simple A-modules V_1, V_2; then

$$e_1 = c_1b_1 \cdots + c_pb_p, \quad e_2 = d_1b_1 + \dots + d_pb_p.$$

The condition $e_1e_2 = 0$ implies that $c_id_i = 0$ for $i = 1, \dots, p$. It follows that the modules V_1, V_2 have no common irreducible B-components. Moreover, the equalities $e_1^2 = e_1$, $e_2^2 = e_2$ imply that $c_i, d_i = 0, 1$. This proves that every irreducible A-module decomposes into a multiplicity-free sum of simple B-modules. The claim concerning the irreducibility type follows from the fact that $SZ(A, B)$ contains only even elements. $\square$

Lemma 5.7. *Let $f : A \longrightarrow B$ be a homomorphism of semisimple superalgebras and C be a simple subsuperalgebra in A. Then*

$$f(SZ(A, C)) = SZ(f(A), f(C)).$$

Proof. Clearly, $f(SZ(A, C)) \subset SZ(f(A), f(C))$. Let us prove the converse inclusion. Let $b = f(a) \in SZ(f(A), f(C))$. Since A is a semisimple superalgebra, $\ker f = eA$, where e is a central idempotent. We have $ac - (-1)^{p(a)p(c)}ca \in eA$ for every $c \in C$. Hence $a(1-e)c = (-1)^{p(a)p(b)}c(1-e)a$, also for every $c \in C$. Therefore $a(1-e) \in SZ(A, C)$. Hence $f(a) = f(a(1-e))$. $\square$

Lemma 5.8. *Let A be a semisimple graded algebra, B be a semisimple graded subalgebra of A, and C be a graded subalgebra in A generated by $SZ(A, B)$ and B. Given an irreducible A-module V and an irreducible B-module U, denote by $I_U(V)$ the sum of all B-submodules in V isomorphic to U or $P(U)$. Then $I_U(V)$ is an irreducible C-module, and every irreducible C-module is of this form. Besides, two nonzero modules of this form are isomorphic if and only if $U = U'$, $V = V'$.*

Proof. By Lemma 5.7, we can replace the algebras A and B by their images in $\mathrm{End}(V)$. Clearly, $I_U(V)$ is a C-module. Besides, the algebras $SZ(A, B)$ and B are semisimple. Hence their tensor product is also a semisimple algebra, and C, being the homomorphic image of this tensor product, is also semisimple. Further, we have a canonical decomposition $V = W \oplus I_U(V)$, where W is the sum of all B-submodules in V that are not isomorphic to U or $P(U)$. Thus we have a canonical homomorphism $\mathrm{End}(V) \to \mathrm{End}(I_U(V))$. Hence, in order to prove the irreducibility of $I_U(V)$, it suffices to compute the supercentralizer of C in $I_U(V)$. Further, V is an irreducible A-module, hence there are two possibilities: $\mathrm{End}(V) = A$ and $\mathrm{End}(V) = A[\pi]$, where π is an odd element from $\mathrm{End}(V)$ that supercommutes with A. In the first case, by Lemma 5.5, the supercentralizer of C in $\mathrm{End}(V)$ coincides with the center of B. Hence, by Lemma 5.7, the centralizer of C in $I_U(V)$ coincides with $\mathbb{C}$. In the second case, the same arguments show that the supercentralizer of C in $I_U(V)$ coincides with $\mathbb{C}[\pi]$.

Conversely, let us prove that every irreducible module is of this form. Let W be an irreducible C-module and U be an irreducible B-submodule of W. Consider the induced module $A \otimes_C W$ and the irreducible component V of this module that contains W. Then, by the above, the module $I_U(V)$ is irreducible and hence coincides with W.

It remains to prove only that the condition $I_U(V) = I_{U'}(V')$ implies $(U, V) = (U', V')$. This assertion is equivalent to the fact that different irreducible A-modules have no common irreducible C-components. By Lemma 5.6, this is equivalent to the condition $SZ(A, C) = SZ(C)$. But it is easy to see from the definition that $SZ(A, C) = SZ(B')$, where $B' = SZ(A, B)$. Hence it suffices to show that $SZ(B') = SZ(C)$. But the algebra C is semisimple, and it is a homomorphic image of the algebra $B' \otimes B$, which is also semisimple with supercenter equal to $SZ(B)' \otimes SZ(B)$. Hence the supercenter of C is equal to the product of the supercenters of B' and B. But it is easy to see that $SZ(B) \subset SZ(B')$, whence $SZ(C) = SZ(B')$. □

Corollary 5.9. *Consider a chain of $\mathbb{Z}_2$-graded semisimple finite-dimensional algebras*

$$\mathfrak{Y} =< A(1) \subset \cdots \subset A(n-1) \subset A(n) >$$

(we do not assume that $A(1) = \mathbb{C}$). Also let $V_1, \dots, V_{n-1}, V_n$ be irreducible modules over the algebras $A(1), \dots, A(n-1), A(n)$, respectively, and $SGZ(\mathfrak{Y})$ be the algebra generated by the supercentralizers $SZ(A(i+1), A(i))$, $i = 1, \dots, n-1$, and the algebra $A(1)$. Then the algebra $SGZ(\mathfrak{Y})$ is semisimple.

Denote by $I(V_1, V_2, \dots, V_n)$ the subspace $I_{V_1}(\dots(I_{V_{n-1}}(V_n)\dots))$. Then the nonzero subspaces

$$I(V_1, V_2, \dots, V_n)$$

are irreducible $SGZ(\mathfrak{Y})$-modules, and every irreducible $SGZ(\mathfrak{Y})$-module is of this form. Moreover, if

$$I(V_1, \dots, V_{n-1}, V_n) = I(V_1', \dots, V_{n-1}', V_n'),$$

then $V_1 = V_1', \dots, V_n = V_n'$.

Proof. Induction on n. If $n = 2$, then this is the assertion of the previous lemma. Let $n > 2$. Denote by C the subalgebra in $A(n)$ generated by $SZ(A(i+1), A(i))$, $i = 2, \dots, n-1$, and by B the algebra generated by $A(2)$ and C. Consider the chain $A(1) \subset B$. By the induction hypothesis, the algebra C is semisimple, hence the tensor product $A(2) \otimes C$ is also semisimple. Further, it is easy to check that the supercentralizer of the algebra $A(1)$ in the algebra $A(2) \otimes C$ equals $SZ(A(2), A(1)) \otimes C$. Since the algebra B is a homomorphic image of the algebra $A(2) \otimes C$, it follows from Lemma 5.7 that $SZ(B, A(1))$ equals $SGZ(\mathfrak{Y})$. Now we can use Lemma 5.8. $\square$

Now let us prove the theorem. To this end, we first prove that the supercenter of the algebra $SGZ(\mathfrak{Y})$ is generated by the supercenters of the algebras $A(1), \dots, A(n)$. Let W be an irreducible $SGZ(\mathfrak{Y})$-module and z_W be the element of the supercenter that acts as 1 in the modules W, $P(W)$ and acts as 0 in all the other irreducible modules. By the previous lemma, $W = I_{V_1}(\dots(I_{V_{n-1}}(V_n)\dots))$. Let z_{V_i}, $i = 1, \dots, n$, be the elements similar to z_W. Then the difference $z_W - z_{V_1} \dots z_{V_n}$ acts as 0 in every irreducible $SGZ(\mathfrak{Y})$-module and hence vanishes. This proves that the supercenter of $SGZ(\mathfrak{Y})$ is generated by the supercenters of $A(1), \dots, A(n)$. Since irreducible representations of $SGZ(\mathfrak{Y})$, regarded up to the functor P, are in a bijection with homomorphisms of the supercenter of $SGZ(\mathfrak{Y})$, the theorem follows. $\square$

In the case of a simple branching, a description of subspaces of irreducible modules over the Gelfand–Tsetlin superalgebra can be given in terms of the corresponding branching graph.

Definition 5.10. *Assume that the branching graph of a chain $\mathfrak{A}$ is simple. In this case, for every path*

$$T = \lambda_1 \nearrow \dots \nearrow \lambda_i \nearrow \lambda_{i+1} \nearrow \dots \nearrow \lambda_n$$

in the branching graph there is a vector v_T, unique up to a nonzero constant, in the corresponding irreducible representation. This vector is called the Gelfand–Tsetlin vector. The set of Gelfand–Tsetlin vectors forms a basis, which is called the Gelfand–Tsetlin basis.

Definition 5.11. *Two vectors v, w of the Gelfand–Tsetlin basis are called equivalent if for the corresponding paths*

$$v \subset A(1)v \subset A(2)v \subset \dots \subset A(n)v, \quad w \subset A(1)w \subset A(2)w \subset \dots \subset A(n)w$$

in the branching graph, the following equations hold: $A(n)w = A(n)v$, and for every $1 \le i \le n-1$, either $A(i)w = A(i)v$ or $P(A(i)w) = A(i)v$.

Lemma 5.12. *Equivalent vectors of the Gelfand–Tsetlin basis determine the same homomorphism $\chi : SZ(\mathfrak{A}) \to \mathbb{C}$ of the supercenter of the algebra $SGZ(\mathfrak{A})$. The linear span of these equivalent vectors coincides with V_χ.*

Proof. Let v, w be equivalent Gelfand–Tsetlin vectors and $z \in Z(A(i))$ for $1 \le i \le n$. Since the irreducible modules $A(i)v$ and $A(i)w$ differ only by the parity, z acts on v and w by the same scalar. Therefore the vectors v, w determine the same homomorphism $\chi : SZ(\mathfrak{Y}) \to \mathbb{C}$. Now let us prove the converse. It is enough to prove the following claim: if a acts as 1 on every vector of the Gelfand–Tsetlin basis that is equivalent to v and acts as 0 on every nonequivalent vector, then $a \in SZ(\mathfrak{Y})$. Let

$$v \subset A(1)v \subset A(2)v \subset \cdots \subset A(n)v$$

be the path corresponding to the vector v. Since the branching is simple, we can find $z_1, \ldots, z_i, \ldots, z_n$ such that z_i acts as 1 in $A(i)v$, $P(A(i)v)$ and acts as 0 in every irreducible module nonisomorphic to $A(i)v$, $P(A(i)v)$. Then the product $z_1 \ldots z_n$ acts in the same way as a in every irreducible $A(n)$ module. Therefore $a = z_1 \ldots z_n$. □

Corollary 5.13. *If the branching of a chain of graded algebras is simple, then the corresponding Gelfand–Tsetlin algebra is the direct sum of Clifford algebras.*

6. Projective representations of the symmetric groups

I. Schur [Sch] proved that the symmetric group S_n has only one nontrivial central extension and suggested a method for finding projective representations of S_n. This central extension, which we will denote by $\tilde{S_n}$, is a $\mathbb{Z}_2$-graded group in the sense defined above. However, the corresponding grading of the group algebra of $\tilde{S_n}$ is more complicated than in the case of $\mathbb{C}[S_n]$ considered above. Proper projective representations of the symmetric group coincide with representations of some $\mathbb{Z}_2$-graded algebra $\mathfrak{A}_n$, which is described below and which is the "half" of the group algebra of $\tilde{S_n}$, more exactly, the quotient of $\mathbb{C}[\tilde{S_n}]$ modulo the "half" ideal. The second ideal, complementary to the first one, is the image of the group algebra $\mathbb{C}[S_n]$ under the (non-identity-preserving) embedding, and the quotient of $\mathbb{C}[\tilde{S_n}]$ modulo this ideal coincides with $\mathbb{C}[S_n]$. Here both quotient algebras, $\mathbb{C}[S_n]$ and $\mathfrak{A}_n$, inherit the $\mathbb{Z}_2$-grading of the algebra $\mathbb{C}[\tilde{S_n}]$. It follows from the above considerations that representations of every $\mathbb{Z}_2$-graded algebra with an essential grading are in a bijective correspondence with representation of its even part. However, the even part of $\mathfrak{A}_n$ still has no convenient model. Hence, in order to describe its representations, we use the notions of graded representations of the Gelfand–Tsetlin superalgebra $SGZ(\mathfrak{Y})$ introduced above. We reproduce the method of constructing the representation theory of the symmetric groups suggested in [OV, VO]. Namely, we construct analogs of the YJM-elements, spectra, etc., and finally obtain the known list of projective representations of the symmetric groups.

We consider two algebras related to projective representations of the symmetric group. The first one is the algebra $\mathfrak{A}_n$ described below, and the second one is the (graded) tensor product of the Clifford algebra $\mathcal{C}_n$ with n generators and the algebra $\mathfrak{A}_n$. For $\mathfrak{A}_n$, Young's orthogonal form was constructed in [N1]; and for $\mathcal{C}_n \otimes \mathfrak{A}_n$, Young's orthogonal and seminormal forms were constructed in [N2]. We construct seminormal forms for both algebras. It turns out that for $\mathcal{C}_n \otimes \mathfrak{A}_n$ Young's formulas look simpler and coincide with the formulas obtained by M. Nazarov in [N2] by other methods.

Thus graded representations of $\mathfrak{A}_n$ are in a one-to-one correspondence with nongraded proper projective representations of the alternating group, or, which is the same, with representations of a central extension of the alternating group in which the central element acts in a nontrivial way.

Let us proceed to a more detailed description of the groups and algebras under consideration. The group $\tilde{S}_n$ is given by generators $\tilde{\sigma}_k$, $k = 0, 1, \ldots, n-1$, satisfying the relations

$$\tilde{\sigma_k}^2 = 1, \quad \tilde{\sigma}_k\tilde{\sigma}_0 = \tilde{\sigma}_0\tilde{\sigma}_k, \quad (\tilde{\sigma}_k\tilde{\sigma}_{k+1})^3 = 1, \quad k = 0, 1, \ldots, n-1,$$
$$\tilde{\sigma}_i\tilde{\sigma}_k = \tilde{\sigma}_0\tilde{\sigma}_k\tilde{\sigma}_i, \quad i, k = 0, 1, \ldots, n-1.$$

The grading is defined as follows: the identity element is assumed to be even, and all the elements $\tilde{\sigma}_k$, $k > 0$, are assumed to be odd. Note that $\tilde{\sigma}_0$ is even. The quotient of the group $\tilde{S}_n$ over its center $\mathbb{Z}_2 = \{\tilde{\sigma}_0\}$ is the symmetric group, so that $\tilde{S}_n$ is a $\mathbb{Z}_2$-extension of S_n, but this extension is not trivial. The group algebra $\mathbb{C}[\tilde{S}_n]$ decomposes into the sum of two ideals. The first one is generated by the relation $\tilde{\sigma}_0 - 1 = 0$, and the quotient modulo this ideal is the group algebra of the symmetric group with the ordinary parity grading. The second one is generated by the relation $\tilde{\sigma}_0 + 1 = 0$ and does not corresponds to any normal subgroup of $\tilde{S}_n$; the quotient modulo this ideal is, by definition, a graded algebra $\mathfrak{A}_n$, which is not a group algebra. Representations of this algebra are exactly proper projective representations of the symmetric group. Note also that the normal subgroup that is the even part of $\tilde{S}_n$ is the restriction of the central extension of S_n to the alternating group S_n^+, i.e., a $\mathbb{Z}_2$-extension of the alternating group; this extension is also nontrivial.

Thus the algebra $\mathfrak{A}_n$ is generated by elements $\tau_1, \ldots, \tau_n$ that are the images of the generators $\tilde{\sigma}_k$, $k > 0$, of $\tilde{S}_n$ and satisfy the relations

$$\tau_k^2 = 1, \quad (\tau_k\tau_{k+1})^3 = 1, \quad (\tau_k\tau_l)^2 = -1 \text{ if } \mid k - l \mid > 1. \qquad (2)$$

The algebra $\mathfrak{A}_n$ has a natural grading inherited from $\mathbb{C}[\tilde{S}_n]$: $p(\tau_i) = 1$, $i = 1, \ldots, n-1$. We can rewrite the defining relations in the form

$$\tau_k^2 = 1, \quad \tau_k\tau_{k+1}\tau_k = \tau_{k+1}\tau_k\tau_{k+1}, \quad \tau_k\tau_l = -\tau_l\tau_k \text{ if } \mid k - l \mid > 1. \qquad (3)$$

In fact, it is more convenient to describe the algebra $\mathfrak{A}_n$ in a slightly different way. Let us define elements τ_{ij} for $i < j$ by induction as follows:

$$\tau_{ii+1} = \tau_i, \quad \tau_{ij} = -\tau_{ij-1}\tau_j\tau_{ij-1}, \quad \tau_{ji} = -\tau_{ij},$$

where $i = 1, \ldots, n-1$. It is easy to check the following relations for $i, j \in \{1, \ldots, n\}$ and $i \neq j$:

$$\tau_{ij} = -\tau_{ji}, \quad \tau_{ij}^2 = 1, \quad \tau_{ij}\tau_{kl} = -\tau_{kl}\tau_{ij} \text{ if } \{i, j\} \cap \{k, l\} = \emptyset,$$
$$\tau_{ij}\tau_{jk}\tau_{ij} = \tau_{jk}\tau_{ij}\tau_{jk} = -\tau_{ij},$$

where i, j, k are pairwise distinct. We will regard $\mathfrak{A}_n$ as a superalgebra, setting $p(\tau_{ij}) = 1$. Note that we can obtain a similar description of the algebra $\mathfrak{A}_n^+$. To this end, recall the following result from [VV]. The group

S_n^+ is generated by elements $x_i = (i, i+1, i+2)$, $i = 1, \dots, n-2$, satisfying the relations

$$x_i^3 = 1, \quad i = 1, \dots, n-2,$$
$$(x_i x_{i+1})^2 = 1, \quad i = 1, \dots, n-3,$$
$$x_i x_j = x_j x_i, \quad |i-j| > 2,\ i, j = i = 1, \dots, n-2,$$
$$x_i x_{i+1}^{-1} x_{i+2} = x_{i+2} x_i, \quad i = 1, \dots, n-4.$$

This easily implies the following proposition.

Proposition 6.1. *The algebra $\mathfrak{A}_n^+$ is generated by elements y_i, $i = 1, \dots, n-2$, satisfying the relations*

$$y_i^3 = 1, \quad i = 1, \dots, n-2,$$
$$(y_i y_{i+1})^2 = -1, \quad i = 1, \dots, n-3,$$
$$y_i y_j = y_j y_i, \quad |i-j| > 2,\ i, j = i = 1, \dots, n-2,$$
$$y_i y_{i+1}^{-1} y_{i+2} = -y_{i+2} y_i, \quad i = 1, \dots, n-4.$$

Now let us describe the supercenter (see Definition 2.13) of the algebra $\mathfrak{A}_n$.

Theorem 6.2. *Let $\alpha \vdash n$ be a partition of n such that all parts of α are odd. The elements C_n^α with such α form a basis of the supercenter of the algebra $\mathfrak{A}_n$.*

Proof. Let $i_1, \dots, i_p$ be a sequence of pairwise distinct elements from $\{1, 2, \dots, n\}$. Set

$$\tau_{i_1 i_2 \dots i_p} = \tau_{i_1 i_2} \tau_{i_2 i_3} \dots \tau_{i_{p-1} i_p}.$$

Then it is easy to check that $\tau_{i_1 i_2 \dots i_p} = -\tau_{i_2 i_3 \dots i_p i_1}$ for an even p, and $\tau_{i_1 i_2 \dots i_p} = \tau_{i_2 i_3 \dots i_p i_1}$ for an odd p.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$ be a partition of m, where $m \le n$ and $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_s \ge 2$. Set

$$C_n^\alpha = \sum \tau_{i_1^{(1)} \dots i_{\alpha_1}^{(1)}} \tau_{i_1^{(2)} \dots i_{\alpha_2}^{(2)}} \dots \tau_{i_1^{(s)} \dots i_{\alpha_s}^{(s)}},$$

where the sum is taken over all pairwise distinct $i_j^{(r)}$ from $\{1, 2, \dots, n\}$.

Define the following action of the symmetric group S_n on $\mathfrak{A}_n$:

$$\sigma_{ij}(\tau_{kl}) = -\tau_{ij} \tau_{kl} \tau_{ij},$$

where $\sigma_{ij} \in S_n$ is a transposition. We see that in order to describe the supercenter, we must describe invariant elements with respect to this action. The supercenter is the linear span of the elements C_n^α with α an arbitrary partition of n. Let us show that if α_j is even for some j, then C_n^α vanishes. Indeed,

$$C_n^\alpha = \sum \tau_{i_1^{(1)} \dots i_{\alpha_1}^{(1)}} \dots \tau_{i_1^{(j)} \dots i_{\alpha_j}^{(j)}} \dots \tau_{i_1^{(s)} \dots i_{\alpha_s}^{(s)}}$$
$$= \sum \tau_{i_1^{(1)} \dots i_{\alpha_1}^{(1)}} \dots \tau_{i_2^{(j)} \dots i_{\alpha_j}^{(j)} i_1^{(j)}} \dots \tau_{i_1^{(s)} \dots i_{\alpha_s}^{(s)}} = -C_n^\alpha.$$

Therefore $C_n^\alpha = 0$. If all parts of α are odd, then $C_n^\alpha \neq 0$. Note that for different $\alpha^{(1)}, \ldots, \alpha^{(q)}$ the corresponding elements $C_n^{\alpha^{(1)}}, \ldots, C_n^{\alpha^{(q)}}$ have no common summands and hence are linearly independent. $\square$

Now let us describe the supercentralizer $SZ(\mathfrak{A}_n, \mathfrak{A}_{n-1})$. To this end, we introduce the following analogs of the YJM-elements.

Definition 6.3. [Se] *The Young–Jucys–Murphy (YJM) element is the following element of the algebra $\mathfrak{A}_n$:*

$$\pi_n = \tau_{1n} + \tau_{2n} + \cdots + \tau_{n-1n}.$$

Note that $\pi_1 = 0$ by definition. Note also that analogs of the YJM-elements for the algebra $\mathcal{C}_n \otimes \mathfrak{A}_n$ were introduced by M. Nazarov in [N2] and denoted by x_n; they coincide, up to an element of the Clifford algebra, with π_n, namely, $x_n = \frac{1}{\sqrt{2}} p_n \pi_n$. It is known that the elements x_i, $i = 1, \ldots, n$, pairwise commute. It follows that the elements π_i, $i = 1, \ldots, n$, pairwise anti-commute. Below we will give an independent proof of this fact.

Theorem 6.4. *The supercentralizer $SZ(\mathfrak{A}_n, \mathfrak{A}_{n-1})$ is generated by the supercenter $SZ(\mathfrak{A}_{n-1})$ and the element π_n. The Gelfand–Tsetlin superalgebra coincides with the algebra $\mathbb{C}[\pi_1, \pi_2, \ldots, \pi_n]$, and the algebra $SZ(\mathfrak{Y})$ generated by the supercenters coincides with $\mathbb{C}[\pi_1^2, \pi_2^2, \ldots, \pi_n^2]$.*

Proof. As in the previous theorem, we must average elements of $\mathfrak{A}_n$ with respect to the action of S_{n-1}. Set

$$C_n^{\alpha,p} = \sum \tau_{i_1 \ldots i_p n} \tau_{i_1^{(1)} \ldots i_{\alpha_1}^{(1)}} \cdots \tau_{i_1^{(j)} \ldots i_{\alpha_j}^{(j)}} \cdots \tau_{i_1^{(s)} \ldots i_{\alpha_s}^{(s)}},$$

where the sum is taken over all pairwise distinct i_l, i_r^j from $\{1, 2, \ldots, n-1\}$. It is not difficult to check that $SZ(\mathfrak{A}_n, \mathfrak{A}_{n-1})$ is the linear span of $C_n^{\alpha,p}$ with $\alpha_1 + \cdots + \alpha_s + p \leq n$. As in the proof of the previous theorem, we see that all parts of α are odd and p is arbitrary. Let us use the induction on $\alpha_1 + \cdots + \alpha_s + p$. If $p = 0$, then $C_n^{\alpha,0} \in SZ(\mathfrak{A}_{n-1})$. Hence we may assume that $p > 0$. Consider the product

$$\begin{aligned}
\pi_n C_n^{\alpha,p-1} &= \sum \tau_{in} \tau_{i_1 \ldots i_{p-1} n} \tau_{i_1^{(1)} \ldots i_{\alpha_1}^{(1)}} \cdots \tau_{i_1^{(s)} \ldots i_{\alpha_s}^{(s)}} \\
&= C_n^{\alpha,p} + \sum_{i = i_1, \ldots, i_{p-1}} \tau_{in} \tau_{i_1 \ldots i_{p-1} n} \tau_{i_1^{(1)} \ldots i_{\alpha_1}^{(1)}} \cdots \tau_{i_1^{(s)} \ldots i_{\alpha_s}^{(s)}} \\
&\quad + \sum_{j=1}^{s} \sum_{i = i_1^{(j)}, \ldots, i_{\alpha_j}^{(j)}} \tau_{in} \tau_{i_1 \ldots i_{p-1} n} \tau_{i_1^{(1)} \ldots i_{\alpha_1}^{(1)}} \cdots \tau_{i_1^{(s)} \ldots i_{\alpha_s}^{(s)}}.
\end{aligned}$$

It is easy to check that

$$\tau_{i_q n} \tau_{i_1 \ldots i_{p-1} n} = (-1)^p \tau_{i_1 \ldots i_{q-1} n} \tau_{i_q \ldots i_{p-1} n}$$

and

$$\tau_{i_1^{(j)} n} \tau_{i_1 \ldots i_{p-1} n} \tau_{i_1^{(j)} \ldots i_{\alpha_j}^{(j)}} = \tau_{i_1 \ldots i_{p-1} i_2^{(j)} \ldots i_{\alpha_j}^{(j)} i_1^{(j)} n}.$$

Hence all the summands except $C_n^{\alpha,p}$ are linear combinations of the elements $C_n^{\beta,q}$ with $|\beta| + q = |\alpha| + p - 1 < |\alpha| + p$. This immediately implies the equality $SGZ(\mathfrak{Y}) = \mathbb{C}[\pi_1, \pi_2, \dots, \pi_n]$. Hence it suffices to show that $SZ(\mathfrak{A})$ coincides with $\mathbb{C}[\pi_1^2, \pi_2^2, \dots, \pi_n^2]$. We have

$$\pi_i^2 = i - 1 - \sum_{k,l<i, k\neq l} \tau_{kli} = \sum_{1\leq i,j,k\leq i-1} \tau_{ijk} - \sum_{1\leq i,j,k\leq i} \tau_{ijk},$$

where the indices i, j, k in the last sums are pairwise distinct. Therefore $\pi_i^2 \in SZ(\mathfrak{Y})$. On the other hand, we have already proved that $SZ(A_i) \subset \mathbb{C}[SZ(A_{i-1}), \pi_i^2]$. $\square$

In a similar way we can prove the following theorem.

Theorem 6.5. *The supercentralizer $SZ(\mathfrak{A}_n, \mathfrak{A}_{n-2})$ is generated by the supercenter $SZ(\mathfrak{A}_{n-2})$ and the elements $\pi_n, \pi_{n-1}, \tau_{n-1}$.*

Remark. According to Theorem 5.4 and the previous corollary, every irreducible $\mathfrak{A}_n$-module is the direct sum of pairwise nonisomorphic irreducible $SGZ(\mathfrak{Y})$-modules, each being of the form $V(a_1, \dots, a_n)$, where $a_1, \dots, a_n$ are eigenvalues of the elements $\pi_1^2, \pi_2^2, \dots, \pi_n^2$.

Corollary 6.6. *Consider the chain of $\mathbb{Z}_2$-graded algebras*

$$\mathbb{C} \subset \mathcal{C}_1 \otimes \mathfrak{A}_1 \subset \mathcal{C}_2 \otimes \mathfrak{A}_2 \subset \cdots \subset \mathcal{C}_n \otimes \mathfrak{A}_n.$$

Then the algebra generated by the supercentralizers of this chain coincides with the algebra $\mathcal{C}_n \otimes \mathbb{C}[\pi_1, \pi_2, \dots, \pi_n]$, and the algebra $SZ(\mathfrak{Y})$ generated by the supercenters coincides with $\mathbb{C}[\pi_1^2, \pi_2^2, \dots, \pi_n^2]$.

The following lemma describes some relations in the algebra $SZ(\mathfrak{A}_n, \mathfrak{A}_{n-2})$. Consider the following elements of the algebra $\mathfrak{A}_n$:

$$F_i = \tau_i(\pi_i^2 - \pi_{i+1}^2) + (\pi_{i+1} - \pi_i), \quad i = 1, 2, \dots, n-1.$$

Lemma 6.7. *For $i = 1, 2, \dots, n-1$, the following relations hold:*

$$\tau_i\pi_i + \pi_{i+1}\tau_i = 1, \quad (\pi_i - \pi_{i+1})\tau_i = \tau_i(\pi_i - \pi_{i+1}),$$

$$(\pi_i^2 - \pi_{i+1}^2)\tau_i + \tau_i(\pi_i^2 - \pi_{i+1}^2) = 2(\pi_i - \pi_{i+1}),$$

$$F_i\pi_i + \pi_{i+1}F_i = 0, \quad F_i\pi_{i+1} + \pi_iF_i = 0, \quad F_i^2 = \pi_i^2 + \pi_{i+1}^2 - (\pi_i^2 - \pi_{i+1}^2)^2.$$

Proof. We have

$$\tau_i\pi_i + \pi_{i+1}\tau_i = \tau_{ii+1}\sum_{j<i}\tau_{ji} + \left(\tau_{ii+1} + \sum_{j<i}\tau_{ji+1}\right)\tau_{ii+1}$$

$$= 1 + \sum_{j<i}\tau_{i+1ij} - \sum_{j<i}\tau_{ji+1i} = 1,$$

and the first relation is proved. The remaining relations easily follow from the first one. $\square$

The following lemma is needed for obtaining analogs of Young's formulas; it is a simple exercise in representation theory. Consider the algebra H generated by a semisimple commutative algebra A and elements τ, p, q that commute with A and satisfy the relations $\tau^2 = 1$, $pq + qp = 0$, $\tau p + q\tau = 1$. It is easy to check that the element $\Delta = p^2 + q^2 - (p^2 - q^2)^2$ belongs to the center of H.

Lemma 6.8. *Let V be an irreducible module over H that is semisimple as an $A[p, q]$-module. Then*

1) *If $\Delta = 0$ in V, then V is irreducible as an $A[p, q]$-module and p^2, q^2 act in V by multiplications by a, b, where $a \neq b$, $a + b = (a - b)^2$, and τ acts as the operator $\frac{p-q}{a-b}$.*

2) *If $\Delta \neq 0$ in V, then V is the direct sum of two irreducible $A[p, q]$-modules, in one of which (say, U) the elements p^2, q^2 act by multiplications by a, b, where $a \neq b$, $a + b \neq (a - b)^2$, and*

$$V = H \otimes_{A[p,q]} U.$$

Corollary 6.9. *Let V be an irreducible module over $\mathcal{C}_k \otimes \mathfrak{A}_n$ and $V(a_1, \ldots, a_n)$ be the subspace of common eigenvectors for $\pi_1^2, \pi_2^2, \ldots, \pi_n^2$ with eigenvalues $a_1, \ldots, a_n$. Then*

1) *$a_i \neq a_{i+1}$ for $i = 1, \ldots, n - 1$.*

2) *If $a_i + a_{i+1} = (a_i - a_{i+1})^2$, then τ_i acts in the subspace $V(a_1, \ldots, a_n)$ as the operator $\frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}}$.*

3) *If $a_i + a_{i+1} \neq (a_i - a_{i+1})^2$, then*

$$V(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \neq 0.$$

Proof. Consider V as a module over the subalgebra $SZ(\mathfrak{A}_{i+1}, \mathfrak{A}_{i-1})$, which is semisimple by Lemma 6.5. Consider the corresponding action of the algebra $H(p, q, \tau)$, which is also semisimple. Decompose the module V into isotypic components with respect to $H(p, q, \tau)$; then the subspace $V(a_1, \ldots, a_n)$ is contained in the isotypic component in which the element $\Delta = p^2 + q^2 - (p^2 - q^2)^2$ acts as 0. Therefore, by Lemma 6.8, $a_i \neq a_{i+1}$ and τ_i acts in $V(a_1, \ldots, a_n)$ as the operator $\frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}}$. This proves 2). Let us prove 3). Similarly to the above, we consider the isotypic component with a fixed value $\Delta \neq 0$. By Lemma 6.8, $V(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \neq 0$. □

Denote by $\mathrm{SSpec}(n)$ the set of all possible sequences of eigenvalues of the elements $\pi_1^2, \pi_2^2, \ldots, \pi_n^2$, and by $\mathrm{Spec}(n)$ the set of all possible sequences of eigenvalues of the ordinary YJM-elements. Also denote by $\mathrm{Spec}^+(n)$ the subset in $\mathrm{Spec}(n)$ consisting of $(c_1, c_2, \ldots, c_n)$ such that $c_i \geq 0$ for $i = 1, \ldots, n$.

Theorem 6.10. *Let $(a_1, a_2, \ldots, a_n) \in \mathrm{SSpec}(n)$. Then*

(i) *$a_i \geq 0$ for every $i = 1, \ldots, n$, and there exists a unique nonnegative integer b_i such that $a_i = \frac{1}{2} b_i (b_i + 1)$;*

(ii) *the map*

$$(a_1, a_2, \ldots, a_n) \longrightarrow (b_1, b_2, \ldots, b_n)$$

is a bijection of $\mathrm{SSpec}(n)$ *onto* $\mathrm{Spec}^+(n)$.

Proof. Let us prove the first assertion by induction on i. The uniqueness is obvious. Let us prove the existence. If $i = 1$, then $\pi_1 = 0$ and $a_1 = 0$, whence $b_1 = 0$. If $i = 2$, then $a_2 = \pi_2^2 = 1$ and $b_2 = 1$. Assume that $i > 1$ and $a_i = \frac{1}{2} b_i(b_i + 1)$ where b_i is a nonnegative integer. Then two cases are possible:

(a) $a_i + a_{i+1} = (a_i - a_{i+1})^2$;

(b) $a_i + a_{i+1} \neq (a_i - a_{i+1})^2$.

In case (a) we have the following equation on a_{i+1}:

$$a_{i+1}^2 - (2a_i + 1)a_{i+1} + a_i^2 - a_i = 0.$$

By the induction hypothesis, this equation has two roots, $a_{i+1} = \frac{1}{2}(b_i + 1)(b_i + 2)$ and $a_{i+1} = \frac{1}{2}(b_i - 1)b_i$, and $b_i \geq 0$. Therefore $a_{i+1} \geq 0$. In case (b), $(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \in \mathrm{SSpec}(n)$, and the assertion follows from the induction hypothesis.

Now let us prove the second claim of the theorem. Recall the following characterization of the set $\mathrm{Spec}(n)$. According to [OV], $(c_1, c_2, \ldots, c_n) \in \mathrm{Spec}(n)$ if and only if the following conditions hold:

1) $c_i \in \mathbb{Z}$, $c_1 = 0$;

2) $c_i \neq c_{i+1}$;

3) for every $i = 1, \ldots, n-2$, we have $(c_i, c_{i+1}, c_{i+2}) \neq (d, d+1, d)$ for any $d \in \mathbb{Z}$;

4) if $c_{i+1} \neq c_i \pm 1$, then $(c_1, \ldots, c_{i+1}, c_i \ldots, c_n) \in \mathrm{Spec}(n)$.

Let us prove that b_i, for $i = 1, \ldots, n$, satisfy the same conditions.

Assertions 1), 2), and 4) have already been proved. Let us prove assertion 3). Assume that there exists i such that $(b_i, b_{i+1}, b_{i+2}) = (b, b+1, b)$. Then

$$(a_i - a_{i+1})^2 = (-1 - b)^2 = a_i + a_{i+1}, \quad (a_{i+1} - a_i)^2 = (1 + b)^2 = a_{i+1} + a_{i+2}.$$

Hence, according to Corollary 6.9,

$$\tau_i = \frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}}, \quad \tau_{i+1} = \frac{\pi_{i+1} - \pi_{i+2}}{a_{i+1} - a_{i+2}}.$$

But it is not difficult to check that these relations contradict the relation

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1},$$

and assertion 4) is proved. Hence the map from assertion (ii) of the theorem is an injection. The remaining part of the proof follows the same scheme as for the symmetric group. First we prove that there exists a sequence of admissible[3] transpositions that sends a given strict standard tableau to the standard tableau of the same shape obtained by placing the integers $1, 2, \ldots, n$ successively into the cells of the first row, then the cells of the second row, etc. This implies that if the weight of a standard tableau has the same shape as a weight of some irreducible module, then it is itself a

[3] A transposition of indices is called admissible if it does not cause a violation of conditions 1)–4), see [OV].

weight of this module. Computing the number of irreducible representations and the number of strict partitions completes the proof. □

Let us proceed to the description of analogs of Young's formulas. In contrast to the ordinary Young's formulas, we do not use the Gelfand–Tsetlin basis, but write the corresponding formulas in terms of the Gelfand–Tsetlin superalgebra. Note that here we obtain a complete description of the action of the algebra $\mathcal{C}_k \otimes \mathfrak{A}_n$ in irreducible modules.

Fix a strict partition α and choose a subspace $V^\alpha_{T_0}$ in the irreducible module V^α corresponding to the standard tableau obtained by filling the diagram row-wise. Let T be an arbitrary tableau of shape α. Then there exists a unique $s \in S_n$ such that $T = sT_0$. Denote by P_T the map $V^\alpha_{T_0} \to V^\alpha_T$ that is the composition of s and the projection of V^α to V^α_T parallel to $\oplus_{T' \neq T} V^\alpha_{T'}$.

Theorem 6.11 (Young's seminormal form for the algebra $\mathcal{C}_k \otimes \mathfrak{A}_n$). *Let α be a strict partition and T be an arbitrary tableau of shape α. Then the commutation between $\tau_i \in \mathfrak{A}_n$ and P_T is given by the following formulas:*

(i) If $a_i + a_{i+1} = (a_i - a_{i+1})^2$, then

$$\tau_i P_T = \frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}} P_T.$$

(ii) If $a_i + a_{i+1} \neq (a_i - a_{i+1})^2$ and $l(s_i T) > l(T)$, then

$$\tau_i P_T = \frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}} P_T + \frac{1}{\sqrt{2}} (p_i - p_{i+1}) P_{s_i T}.$$

(iii) If $a_i + a_{i+1} \neq (a_i - a_{i+1})^2$ and $l(s_i T) < l(T)$, then

$$\tau_i P_T = \frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}} P_T + \frac{1}{\sqrt{2}} (p_i - p_{i+1}) \left(1 - \frac{a_i + a_{i+1}}{(a_i + a_{i+1})^2}\right) P_{s_i T}.$$

Proof. First let us write an explicit formula for the map P_T. Let $T = sT_0$, and let $s = s_{i_1} \dots s_{i_l}$ be the reduced decomposition of a permutation s, all transpositions in this decomposition being admissible. Then

$$P_{sT_0} = P_{s_{i_1}} P_{s_{i_2}} \dots P_{s_{i_l}}, \tag{4}$$

where

$$P_{s_i} = -\frac{p_i - p_{i+1}}{\sqrt{2}} \left(\tau_i - \frac{\pi_i - \pi_{i+1}}{\pi_i^2 - \pi_{i+1}^2}\right).$$

It suffices to consider the case $l = 1$ and check that if $v \in V^\alpha_T$ and s_i is an admissible transposition, then $s_i v - v' = P_{s_i} v \in V^\alpha_{s_i T}$, where $v' \in V^\alpha_T$. But it is easy to check that $\pi_i^2 P_{s_i} = P_{s_i} \pi_{i+1}^2$, $\pi_i^2 P_{s_{i+1}} = P_{s_i} \pi_i^2$. Hence $P_{s_i} v \in V^\alpha_{s_i T}$. Further, assertion (i) follows from Lemma 6.8. Assertion (ii) follows from the relation $P_{s_i T} = P_{s_i} P_T$ if $l(s_i T) > l(T)$. Assertion (iii) follows from (ii). The theorem is proved. □

Remark 6.12. It is easy to check the relations $P_{sT_0} \pi_i = \pi_{s(i)} P_{sT_0}$, $P_{sT_0} p_i = p_{s(i)} P_{sT_0}$. Together with the relations from the previous theorem, they give a complete description of the action of the algebra $\mathcal{C}_k \otimes \mathfrak{A}_n$ in irreducible modules and coincide (after simple transformations) with the formulas obtained in [N2] by other methods.

Let us proceed to the description of analogs of Young's formulas for the algebra $\mathfrak{A}_n$. For this we need to define analogs of the maps P_T. In contrast to the previous case, we cannot do this using the projection to the corresponding eigenspace. Instead, we will write an analog of formula (4), which gives an explicit decomposition of this operator.

Fix a strict partition α, choose subspaces V_T^α, $V_{s_iT}^\alpha$, where s_i is an admissible transposition, and set

$$Q_{s_i} = \left(\tau_i - \frac{\pi_i - \pi_{i+1}}{\pi_i^2 - \pi_{i+1}^2}\right)\left(\frac{\pi_i}{\sqrt{\pi_i^2}} - \frac{\pi_{i+1}}{\sqrt{\pi_{i+1}^2}}\right)$$

if $a_i a_{i+1} \neq 0$, and

$$Q_{s_i} = \left(\tau_i - \frac{\pi_i - \pi_{i+1}}{\pi_i^2 - \pi_{i+1}^2}\right)\frac{\pi_i + \pi_{i+1}}{\sqrt{\pi_i^2} + \sqrt{\pi_{i+1}^2}}$$

if $a_i a_{i+1} = 0$.

As above, let $V_{T_0}^\alpha$ be the subspace in the irreducible module V^α corresponding to the standard tableau filled by rows. Let T be an arbitrary tableau of shape α. Then there exists a unique $s \in S_n$ such that $T = sT_0$ and $s = s_{i_1} \dots s_{i_l}$ is the reduced decomposition of the permutation s, and all transpositions in this decomposition are admissible. Set

$$Q_T = Q_{s_1} Q_{s_2} \dots Q_{s_l}.$$

Theorem 6.13 (Young's seminormal form for the algebra $\mathfrak{A}_n$)**.** *Let α be a strict partition and T be an arbitrary tableau of shape α. Then Q_T does not depend on the choice of a reduced decomposition of s into the product of admissible transpositions, and the commutation between $\tau_i \in \mathfrak{A}_n$ and Q_T is given by the following formulas:*

(i) If $a_i + a_{i+1} = (a_i - a_{i+1})^2$, then

$$\tau_i Q_T = \frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}} Q_T.$$

(ii) If $a_i + a_{i+1} \neq (a_i - a_{i+1})^2$, $a_i a_{i+1} \neq 0$, and $l(s_iT) > l(T)$, then

$$\tau_i Q_T = \frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}} Q_T - \frac{1}{2}\left(\frac{\pi_i}{\sqrt{a_{i+1}}} - \frac{\pi_{i+1}}{\sqrt{a_i}}\right) Q_{s_iT}.$$

If $l(s_iT) < l(T)$, then

$$\tau_i Q_T = \frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}} Q_T - \frac{1}{2}\left(\frac{\pi_i}{\sqrt{a_{i+1}}} - \frac{\pi_{i+1}}{\sqrt{a_i}}\right)\left(1 - \frac{a_i + a_{i+1}}{(a_i + a_{i+1})^2}\right) Q_{s_iT}.$$

(iii) If $a_i + a_{i+1} \neq (a_i - a_{i+1})^2$, $a_i a_{i+1} = 0$, and $l(s_i T) > l(T)$, then

$$\tau_i Q_T = \frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}} Q_T - \frac{\pi_i + \pi_{i+1}}{\sqrt{a_i} + \sqrt{a_{i+1}}} Q_{s_i T}.$$

If $l(s_i T) < l(T)$, then

$$\tau_i Q_T = \frac{\pi_i - \pi_{i+1}}{a_i - a_{i+1}} Q_T + \frac{\pi_i + \pi_{i+1}}{\sqrt{a_i} + \sqrt{a_{i+1}}} \left(1 - \frac{a_i + a_{i+1}}{(a_i + a_{i+1})^2}\right) Q_{s_i T}.$$

Proof. First let us check that Q_T does not depend on the choice of a reduced decomposition. For this it suffices to check the equality

$$Q_{s_i} Q_{s_{i+1}} Q_{s_i} = Q_{s_{i+1}} Q_{s_i} Q_{s_{i+1}},$$

which can be done by directly enumerating all possible cases. Then the proof follows the scheme of the previous theorem. □

Partially supported by EPSRC (grant EP/E004008/1).

References

[C] Cherednik I. V., Special bases of irreducible representations of a degenerate affine Hecke algebra (Russian), Funktsional. Anal. i Prilozhen. 20 (1986), no. 1, 87–88.

[FH] Fulton W., Harris J., *Representation Theory. A First Course.* Graduate Texts in Mathematics 129, Springer-Verlag, New York, 1991.

[J] Jozefiak T., Semisimple superalgebras, Lecture Notes Math. 1352, 1988, 96–113.

[Ju1] Jucys A., Symmetric polynomials and the center of the symmetric group ring, Report Math. Phys. 5, 1974, 107–112.

[Ju2] Jucys A., Factorization of Young's projection operators for symmetric groups, Litovsk. Fiz. Sb. 5, 1971, 1–10.

[Mo] Morris A., Projective representations of finite reflection groups. III. Comm. Algebra 32, no. 7 (2004), 2679–2694.

[M] Murphy G., A new construction of the Young seminormal representations of the symmetric group, J. Algebra 69 (1981), 287–291.

[N1] Nazarov M., Young's orthogonal form of irreducible projective representations of the symmetric group, J. London Math. Soc. (2), 42 (1990), 437–451.

[N2] Nazarov M., Young's symmetrizers for projective representations of the symmetric group, Adv. Math. 127, no. 2 (1997), 190–257.

[NS] Nazarov M., Sergeev A., Centralizer construction of the Yangian of the queer Lie superalgebra. Studies in Lie theory, 417–441, Progr. Math. 243, Birkhauser Boston, Boston, MA, 2006.

[OV] Okounkov A., Vershik A., A new approach to representation theory of symmetric groups. Selecta Math. (N. S.) 2, no. 4 (1996), 581–605.

[Se] Sergeev A., The Howe duality and the projective representations of symmetric groups, Represent. Theory 3 (1999), 416–434.

[Se1] Sergeev A., The invariant polynomials on simple Lie superalgebras, Represent. Theory 3 (1999), 250–280.

[Se3] Sergeev A. N., Tensor algebra of the identity representation as a module over the Lie superalgebras Gl(n, m) and $Q(n)$ (Russian), Mat. Sb. (N.S.) 123(165), no. 3 (1984), 422–430.

[KL] Kleshchev A., Linear and projective representations of symmetric groups, Cambridge Tracts in Mathematics 163, Cambridge University Press, Cambridge, 2005. xiv+277 pp.

[Sch] Schur J., On the representation of the symmetric and alternating groups by fractional linear substitutions. Translated from the German [J. Reine Angew. Math. 139 (1911), 155–250] by Marc-Felix Otto. Internat. J. Theoret. Phys. 40 (2001), no. 1, 413–458.

[VO] Vershik A., Okounkov A., A new approach to representation theory of symmetric groups. II, J. Math. Sci. (N.Y.) 131, no. 2 (2005), 5471–5494.

[V] Vershik A., A new approach to representation theory of symmetric groups. III. Induced representation and Frobenius correspondence, Moscow Math. J. 6, no. 3 (2006), 567–585.

[V1] Vershik A., Local algebras and a new version of Young's orthogonal form, In "Topics in Algebra, part 2: Commutative Rings and Algebraic Groups" (Warsaw 1988), Banach Cent. Publ. 26, Part 2 (1990), 467–473.

[VK] Vershik A. M., Kerov S. V, Locally semisimple algebras. Combinatorial theory and the K_0-functor (Russian), Current problems in mathematics. Newest results, Vol. 26, 3–56, 260, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1985.

[KW] Khongsap T., Wang W., Hecke–Clifford algebras and spin Hecke algebras I, `arXiv:07040201`.

[VV] Vershik A. M., Vsemirnov M. A., The local stationary presentation of the alternating groups and normal form, `arXiv:math/0703278` (submitted to J. Algebra).

St. Petersburg Department of Steklov Institute of Mathematics