On the scientific work of M. Sh. Birman in 1998–2007

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January 17, 2008 is the 80-th birthday of the outstanding mathematician Mikhail Shlëmovich Birman. He is the author of many most important results in the spectral theory of abstract and differential operators and in the scattering theory. He contributed much to other areas of Real Analysis: to approximation theory, function theory, theory of integral operators, etc.

Papers [**1***, **3***, **4***] are devoted to the scientific achievements of M. Birman. In the present paper, we give a survey of his results obtained during the last ten years (in 1998 2007). The list of publications of M. Birman for the period before 1998 can be found in [**2***]. The next item of this collection is the continuation of this list. Herewith, the numbering of publications continues numbering from [**2***]. References [**130, 131**], which will be discussed below in detail, are repeated. We also repeat references [**138, 139, 140, 141**] from [**2***], since they contained non-complete data.

The main subject which M. Birman was working on during the last ten years (and is currently working on) is the spectral theory of periodic differential operators (DO's). Precisely, the following three themes can be named: 1) the problem of absolute continuity of the spectrum of periodic operators of mathematical physics; 2) threshold properties and homogenization of periodic DO's; 3) discrete spectrum in the gaps of a periodic elliptic operator perturbed by potential decaying at infinity.

Besides, M. Birman returned to his "old" subject, namely, to the study of asymptotics of discrete spectrum for the Maxwell operator in a resonator in non-smooth situation.

§1. Absolute continuity of the spectrum of periodic operators of mathematical physics

We start with the series of papers [**131, 138, 140, 141, 142, 143, 148**] devoted to the problem of absolute continuity of the spectrum of periodic differential operators. All papers of this series are joint with T. Suslina. Below in Subsections 1.1 1.3 we give necessary preliminary information.

1.1. Direct integral. Let P be a self-adjoint elliptic (in an appropriate sense) lower semi-bounded operator in $L_2(\mathbb{R}^d)$ generated by a second order linear differential expression $\mathcal{P}(\mathbf{x},\mathbf{D})$ ($\mathbf{x}\in\mathbb{R}^d$, $\mathbf{D}=-i\nabla$), which is periodic in $\mathbf{x}$ with respect

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to some lattice $\Gamma \subset \mathbb{R}^d$. By Ω we denote the elementary cell of the lattice Γ. Let $\widetilde{\Gamma}$ be the dual lattice, and let $\widetilde{\Omega}$ be the central Brillouin zone of the lattice $\widetilde{\Gamma}$.

The operator P is decomposed in the direct integral of the operators $P(\mathbf{k})$ acting in $L_2(\Omega)$ and depending on the parameter $\mathbf{k} \in \mathbb{R}^d$ (the quasi-momentum). The operator $P(\mathbf{k})$ is given by the differential expression $\mathcal{P}(\mathbf{x}, \mathbf{D}+\mathbf{k})$ with periodic boundary conditions. The operator P is unitarily equivalent to the direct integral of operators $P(\mathbf{k})$:

$$\mathcal{U}P\mathcal{U}^{-1} = \int_{\widetilde{\Omega}} \oplus P(\mathbf{k})\, d\mathbf{k}. \tag{1.1}$$

Here $\mathcal{U}$ is the Gelfand transformation.

The spectrum of each operator $P(\mathbf{k})$ is discrete. Let $E_j(\mathbf{k})$, $j \in \mathbb{N}$, be the eigenvalues of the operator $P(\mathbf{k})$ numbered in non-decreasing order counting multiplicities. The *band functions* $E_j(\cdot)$ are continuous and $\widetilde{\Gamma}$-periodic. The spectrum of the operator P has a *band structure*: it consists of the closed intervals $\operatorname{Ran} E_j$, $j \in \mathbb{N}$. Spectral bands can overlap. The intervals free of the spectrum are called *gaps*.

1.2. The problem of the absence of degenerate spectral bands. If some band function is constant: $E_j(\mathbf{k}) = \lambda = \text{Const}$, then the corresponding band degenerates into the point λ, which is an eigenvalue of infinite multiplicity for the operator P. For instance, such case can be realized for elliptic fourth order operator (an example of such an operator is given in the book by P. Kuchment), and also for the operator $-\operatorname{div} g(\mathbf{x})\nabla$ with $d \geq 3$ and a non-Lipschitz matrix g (such example was found in 2001 by N. Filonov).

If a periodic elliptic operator has no degenerate bands, then its spectrum *is absolutely continuous*.

1.3. The Thomas approach. For the first time, the absence of degenerate spectral bands for the Schrödinger operator $H = -\Delta + V(\mathbf{x})$ in $L_2(\mathbb{R}^3)$ with a real-valued periodic potential V was proved in the famous paper by L. Thomas in 1973. (For arbitrary dimension $d \geq 2$, the Thomas proof is presented in the book by M. Reed and B. Simon.) Later the general Thomas approach was used in almost all papers on this subject. The operator family $H(\mathbf{k}) = (\mathbf{D}+\mathbf{k})^2 + V(\mathbf{x})$ (with periodic boundary conditions) acting in $L_2(\Omega)$ is analytically extended to complex values of the parameter $\mathbf{k} \in \mathbb{C}^d$. We fix a unit vector $\mathbf{e} \in \mathbb{R}^d$ and a vector $\mathbf{k}' \in \mathbb{R}^d$, $\mathbf{k}' \perp \mathbf{e}$, and put $\mathbf{k} = k_1\mathbf{e} + \mathbf{k}'$, where $k_1 \in \mathbb{C}$. With respect to the parameter $k_1 \in \mathbb{C}$, the family $H(\mathbf{k}) = H(k_1\mathbf{e} + \mathbf{k}')$ is an analytic operator family with compact resolvent. Let $k_1 = \mu + iy$, where $\mu \in \mathbb{R}$ is fixed, and $y \in \mathbb{R}$ is the main parameter. We denote $H(\mathbf{k}) = H(y)$. Thomas showed that, for appropriate $\mathbf{e}$ and μ, the operator $H(y)$ is invertible for large values of $|y|$, and $\|H(y)^{-1}\| \to 0$ as $|y| \to \infty$. Precisely, the following *Thomas estimate* is true:

$$\|H(y)^{-1}\| \leq C|y|^{-1}, \quad |y| \geq y_0 > 0. \tag{1.2}$$

By the analytic Fredholm alternative, this easily implies the absence of degenerate bands.

First, estimate (1.2) is proved (by the Fourier series) for the "free" operator $H_0(y)$ corresponding to the case $V = 0$. Next, the potential V can be taken into account as an additive perturbation, and estimate (1.2) is carried over to the case of the "perturbed" operator $H(y) = H_0(y) + V$. Herewith, the smaller assumptions on

V are, the bigger efforts are needed for adding V. In the initial paper by Thomas, it was assumed that $V \in L_2(\Omega)$ (for $d = 3$).

1.4. Two-dimensional magnetic Schrödinger operator. The periodic *magnetic Schrödinger operator* $M = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^2 + V(\mathbf{x})$ (the magnetic Hamiltonian) is a much more difficult case. Here $\mathbf{A}(\mathbf{x})$ is the vector-valued (magnetic) potential. Due to the gauge transformation, the potential $\mathbf{A}$ can be subject to the gauge conditions $\operatorname{div} \mathbf{A} = 0$, $\int_\Omega \mathbf{A}(\mathbf{x})\, d\mathbf{x} = 0$. If M is considered as a perturbed operator with respect to H_0, then the perturbation $M - H_0$ is a first order differential operator. For the corresponding operators in $L_2(\Omega)$ depending on the parameter y, the perturbation $M(y) - H_0(y)$ contains term of order $|y|$. That is why it is impossible to prove estimate of the form (1.2) for the operator $M(y)$ considering $M(y)$ as an additive perturbation of the "free" operator $H_0(y)$. This difficulty was not overcome for almost 25 years.

In 1997 in the break-through paper [**131**], absolute continuity of the spectrum of the periodic magnetic Hamiltonian was proved in dimension $d = 2$. The magnetic operator M was interpreted as a "multiplicative perturbation" of the operator H_0. Let us explain this in detail. For simplicity, suppose that $\mathbf{A} \in C^1$. Approach of [**131**] is based on the study of the operator $P = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^2 + \partial_1 A_2(\mathbf{x}) - \partial_2 A_1(\mathbf{x})$ (one of two blocks of the corresponding Pauli operator). The operator P admits a convenient factorization. Indeed, there exists a real-valued periodic function $\varphi(\mathbf{x})$ such that $\nabla\varphi = \{A_2, -A_1\}$. In terms of φ, the operator P is factorized in five factors: $P = e^{-\varphi}(D_1 + iD_2)e^{2\varphi}(D_1 - iD_2)e^{-\varphi}$. Each of them is either a differential operator with constant coefficients, or multiplication by a positive function. This allows one to consider the operator $P(y)$ (acting in $L_2(\Omega)$) as a multiplicative perturbation of the operator $H_0(y)$ and to prove an analogue of estimate (1.2) for the operator $P(y)$. After that, it is easy to take the scalar potential $W = V - \partial_1 A_2 + \partial_2 A_1$ into account as an additive perturbation and to prove the required estimate for the operator $M(y) = P(y) + W$.

In [**131**], it was assumed that the vector-valued potential $\mathbf{A}$ is continuous, and that $V \in L_2(\Omega)$. Later in [**138**] the scheme was modified and adapted to the operators defined in terms of quadratic forms. This allowed the authors to relax conditions on coefficients: in [**138**] it was assumed that $\mathbf{A} \in L_\rho(\Omega)$ with $\rho > 2$ and $V \in L_r(\Omega)$ with $r > 1$. These conditions are optimal in the L_p-scale. Thus, the result on absolute continuity of the spectrum of the two-dimensional periodic magnetic Hamiltonian was extended to the case of discontinuous vector-valued potentials.

1.5. Further development of the subject. After almost 25 years of stagnation, the paper [**131**] gave rise to a series of papers by many authors on absolute continuity of the spectrum of periodic operators.

For $d \geq 3$, the problem of absolute continuity of the spectrum of periodic magnetic Hamiltonian turned out to be much more difficult than for $d = 2$. (If $d \geq 3$, the Pauli operator does not admit a convenient factorization.) This problem has been solved in 1997 in the remarkable paper by A. Sobolev. He applied the technique of pseudodifferential operators on torus and showed that, for an appropriate choice of the direction of complex quasi-momentum (i. e., the direction of the vector $\mathbf{e}$) depending on the potential $\mathbf{A}$, an analogue of estimate (1.2) for the operator $M(y)$ is still true for $d \geq 3$. It turned out that a certain analogue of factorization (up to lower terms) still takes place, but on the "pseudodifferential level".

In [**141**], a survey of the results known at that time on absolute continuity of the spectrum of periodic operators was given. Besides, the result on absolute continuity of the spectrum of the magnetic Schrödinger operator for $d \geq 3$ was extended to a wider class of potentials. In particular, for $d = 3, 4$ the condition $V \in L^0_{d/2,\infty}(\Omega)$ from [**141**] is optimal in the Lorentz scale.

Later conditions on coefficients were relaxed in the papers by I. Lapin, R. Shterenberg and in a series of papers by Z. Shen.

The most difficult case is an operator with variable leading coefficients. However, the case of a scalar metric analyzed in [**141**] is relatively simple. Then the operator has the form

$$H(g, \mathbf{A}, V) = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^* g(\mathbf{x})(\mathbf{D} - \mathbf{A}(\mathbf{x})) + V(\mathbf{x}), \tag{1.3}$$

where $g(\mathbf{x}) = \omega^2(\mathbf{x})a$, a is a constant positive matrix, and $\omega(\mathbf{x})$ is a bounded and positive definite (scalar) function. In [**141**], it was shown that the result on absolute continuity can be easily carried over from the case of the magnetic Schrödinger operator to this case (under some smoothness assumptions on ω), by using the identity $H(\omega^2 a, \mathbf{A}, V) = \omega^{-1}H(a, \mathbf{A}, \omega^{-2}V + V_\omega)\omega^{-1}$, where $V_\omega = \omega^{-1}\mathrm{div}\, a\nabla\omega$.

The case of metric $g(\mathbf{x})$ of general type turned out to be much more difficult. Up to now, the problem of absolute continuity for the operator with variable metric is solved only in the two-dimensional case (here the first result belongs to A. Morame, 1998), and for $d \geq 3$ in the case where the operator has a special symmetry: it must be invariant under reflection with respect to some axis (the result of L. Friedlander, 2001). For $d \geq 3$, the problem of absolute continuity of the spectrum of the operator (1.3) with metric g of general type still remains open.

1.6. Singular potentials. In [**142**], the absolute continuity of the spectrum was established for the two-dimensional Schrödinger operator of the form

$$H_\Sigma = (\mathbf{D} - \mathbf{A})^2 + V(\mathbf{x}) + \sigma(\mathbf{x})\delta_\Sigma(\mathbf{x}). \tag{1.4}$$

Potential in (1.4) includes the delta-like term $\sigma(\mathbf{x})\delta_\Sigma(\mathbf{x})$ supported on a periodic system Σ of piecewise-smooth curves. Such operators are of interest in the theory of photonic crystals. Using the version of the Thomas approach adapted for the operators defined in terms of quadratic forms (this version was suggested in [**138**]), it is possible to take a delta-like potential into account as an additive perturbation.

Later, absolute continuity of the spectrum for the operator (1.4) was obtained in dimension $d \geq 3$ by T. Suslina and R. Shterenberg in 2001 (under some restrictions on a periodic system Σ of $(d-1)$-dimensional surfaces). For $d = 2$, the case of more general singular potentials (given in terms of measures) was studied by R. Shterenberg (2000, 2001).

1.7. Vector periodic problems. A progress for the magnetic Schrödinger operator allowed one to prove absolute continuity of the spectrum also for the periodic Dirac operator containing both electric and magnetic potentials. This was done in [**140**] on the basis of a relation between the square of the Dirac operator and the magnetic Schrödinger operator. (For the Dirac operator with periodic electric potential only, absolute continuity of the spectrum was obtained in 1990 by L. Danilov.)

In [**148**], absolute continuity of the spectrum was proved for the periodic isotropic operator of elasticity theory in the case where the shear modulus is constant (the so called "Hill body"). In this case, it is possible to reduce the problem to the known results for a scalar elliptic operator. If the shear modulus is variable, the problem of absolute continuity of the spectrum for the isotropic periodic operator of elasticity theory remains open.

Note that, in spite of the considerable progress, there remain many unsolved problems on absolute continuity of the spectrum of periodic operators.

§2. Threshold properties and homogenization for periodic differential operators

Important series of papers [**139, 145, 150, 151, 153, 155, 156, 158, 159, 161**] by M. Birman and T. Suslina is devoted to the study of threshold properties and homogenization problems for periodic differential operators. The starting point was the paper [**139**], where the spectral characteristics of the two-dimensional periodic Pauli operator at the bottom of the spectrum (the *threshold characteristics*) were studied. It became clear that "good" properties of the threshold characteristics are related not only to specific features of the Pauli operator, but also to existence of factorization of the form $\mathcal{X}^*\mathcal{X}$, where $\mathcal{X}$ is a homogeneous first order DO. This led to singling out of a wide class of matrix periodic DO's admitting a factorization of the form $\mathcal{X}^*\mathcal{X}$ and to the study of their threshold characteristics (see Subsection 2.3). During this work the idea was born that threshold properties of periodic DO's must be related to the homogenization theory in the small period limit. It was realized that the homogenization procedure for a periodic operator can be studied as a threshold effect at the bottom of the spectrum. In this way, the results of new type in the homogenization theory were obtained.

2.1. Differential operators in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. In the papers [**145, 151, 156, 158**], the class of matrix periodic differential operators $\mathcal{A}$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and admitting a factorization of the form $\mathcal{A} = \mathcal{X}^*\mathcal{X}$ was introduced and studied in detail. Here $\mathcal{X}$ is a homogeneous first order DO. Suppose that $\mathcal{X} = h(\mathbf{x})b(\mathbf{D})f(\mathbf{x})$, where an $(n\times n)$-matrix-valued function $f(\mathbf{x})$ and an $(m\times m)$-matrix-valued function $h(\mathbf{x})$ are periodic with respect to some lattice Γ and bounded together with their inverses. It is assumed that $m \geq n$. The operator $b(\mathbf{D})$ is a homogeneous first order DO; its symbol $b(\boldsymbol{\xi})$ is a linear homogeneous $(m\times n)$-matrix-valued function of $\boldsymbol{\xi}\in\mathbb{R}^d$ such that $\operatorname{rank} b(\boldsymbol{\xi}) = n$ for $\boldsymbol{\xi}\neq 0$. Formally, the operator $\mathcal{A}$ is given by the differential expression

$$\mathcal{A} = \mathcal{A}(g,f) = f(\mathbf{x})^* b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) f(\mathbf{x}), \quad g(\mathbf{x}) = h(\mathbf{x})^* h(\mathbf{x}). \tag{2.1}$$

The precise definition of $\mathcal{A}$ is given in terms of the corresponding quadratic form. Below, in the case where $f = \mathbf{1}_n$, we use the notation

$$\widehat{\mathcal{A}} = \widehat{\mathcal{A}}(g) = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}). \tag{2.2}$$

Many operators of mathematical physics admit such factorization. The acoustics operator and the operator of elasticity theory have the form (2.2), while the periodic Schrödinger operator and the two-dimensional Pauli operator can be written in the form (2.1) (with non-trivial f).

2.2. Main results on homogenization. We use the notation $\phi^\varepsilon(\mathbf{x}) = \phi(\varepsilon^{-1}\mathbf{x})$ for any Γ-periodic function ϕ and put $\mathcal{A}_\varepsilon = \mathcal{A}(g^\varepsilon, f^\varepsilon)$, $\widehat{\mathcal{A}}_\varepsilon = \widehat{\mathcal{A}}(g^\varepsilon)$. Coefficients of these operators are rapidly oscillating as $\varepsilon\to 0$. The *homogenization problem* for the operator $\mathcal{A}_\varepsilon$ is to study the behavior of the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ as $\varepsilon \to 0$.

We start with the results for the simpler operator $\widehat{\mathcal{A}}_\varepsilon$. *There exists an operator* $\widehat{\mathcal{A}}^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ *with constant coefficients such that*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1}\|_{L_2 \to L_2} \le C\varepsilon. \tag{2.3}$$

The constant positive matrix g^0 is called the *effective matrix*, and the operator $\widehat{\mathcal{A}}^0$ is called the *effective operator*. The existence of the effective operator and the *strong* resolvent convergence in L_2 have been known in the traditional homogenization theory. However, the resolvent convergence *in the operator norm* was not known before. The estimate (2.3) established for the first time in [**145, 151**] is order-sharp; the constant C is controlled explicitly in terms of the problem data.

The effective matrix g^0 is defined by the following rule. Let $\Lambda(\mathbf{x})$ be the periodic $(n \times m)$-matrix-valued function satisfying the equation

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0. \tag{2.4}$$

We put $\widetilde{g}(\mathbf{x}) = g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m)$. Then $g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}$.

For the operator $\mathcal{A}_\varepsilon$ (with $f \ne \mathbf{1}_n$), it is impossible to find a DO with constant coefficients such that the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ converges to the resolvent of this operator. However, it is possible to approximate the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ by the generalized resolvent of the effective operator $\widehat{\mathcal{A}}^0$ sandwiched between rapidly oscillating factors. The following approximation was found in [**151**]:

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (f^\varepsilon)^{-1}(\widehat{\mathcal{A}}^0 + \overline{Q})^{-1}((f^\varepsilon)^*)^{-1}\|_{L_2 \to L_2} \le C\varepsilon. \tag{2.5}$$

Here $Q(\mathbf{x}) = (f(\mathbf{x})f(\mathbf{x})^*)^{-1}$ and $\overline{Q} = |\Omega|^{-1}\int_\Omega Q(\mathbf{x})\, d\mathbf{x}$. It is important that, in the approximation (2.5), the inverse is taken for the DO with constant coefficients, though this approximation contains rapidly oscillating factors from both sides. One can get rid of these factors only by passing to the weak operator limit.

In the paper [**156**], a qualitatively new result in the homogenization theory has been obtained. Namely, a more precise approximation for the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the operator norm in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with error estimate of order ε^2 was found. Here we describe this result only for the simpler operator $\widehat{\mathcal{A}}_\varepsilon$. The term $\varepsilon K(\varepsilon)$ should be added to the resolvent of the effective operator, where $K(\varepsilon)$ is the so called *corrector. The following estimate holds:*

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2 \to L_2} \le C\varepsilon^2. \tag{2.6}$$

The operator $K(\varepsilon)$ is the sum of *three* terms: $K(\varepsilon) = K_1(\varepsilon) + K_1(\varepsilon)^* + K_3$. The first two terms of the corrector are mutually adjoint and contain a rapidly oscillating factor Λ^ε, where Λ is the periodic solution of the equation (2.4). The operator $K_1(\varepsilon)$ is given by $K_1(\varepsilon) = \Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}$. Here Π_ε is an auxiliary smoothing operator. (In some cases it is possible to get rid of the smoothing operator and to put $K_1(\varepsilon) = \Lambda^\varepsilon b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}$.) The third term of the corrector does not depend on ε and is given by $K_3 = -(\widehat{\mathcal{A}}^0 + I)^{-1} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D})(\widehat{\mathcal{A}}^0 + I)^{-1}$, where $L(\mathbf{D})$ is the first order DO with the symbol $L(\boldsymbol{\xi}) = |\Omega|^{-1}\int_\Omega (\Lambda(\mathbf{x})^* b(\boldsymbol{\xi})^* \widetilde{g}(\mathbf{x}) + \widetilde{g}(\mathbf{x})^* b(\boldsymbol{\xi}) \Lambda(\mathbf{x}))\, d\mathbf{x}$.

The estimate (2.6) is order-sharp. As $\varepsilon \to 0$, the weak limit of the corrector $K(\varepsilon)$ is equal to K_3. In this respect, the third term of the corrector is the most important. In some cases, it may happen that $K_3 = 0$. In particular, this is the case for the scalar operator $\widehat{\mathcal{A}} = -\operatorname{div} g(\mathbf{x})\nabla$, where the matrix g has real entries. However, in the general case the third term of the corrector is non-trivial. This is typical for matrix operators, and also for scalar operators with complex-valued coefficients. Note that the traditional corrector used in the homogenization theory corresponds to the term $K_1(\varepsilon)$. The estimate (2.6) shows that, in order to approximate the resolvent in the L_2-operator norm with error term of order $O(\varepsilon^2)$, it is not sufficient to take into account only the traditional corrector $K_1(\varepsilon)$, but it is necessary to include all the three terms in the corrector. This result was quite unexpected.

In [**156**], an analogue of estimate (2.6) for more general operator $\mathcal{A}_\varepsilon$ was also obtained. In this case, approximation contains rapidly oscillating factors from both sides.

In [**158**], the resolvent $(\widehat{\mathcal{A}}_\varepsilon + I)^{-1}$ was approximated in the norm of operators acting from L_2 to the Sobolev space H^1, with the error term of order ε. It turned out that, to this end, it suffices to take into account only the first term of the corrector. *The following estimate is true*:

$$\|(\widehat{\mathcal{A}}_\varepsilon + I)^{-1} - (\widehat{\mathcal{A}}^0 + I)^{-1} - \varepsilon K_1(\varepsilon)\|_{L_2 \to H^1} \leq C\varepsilon. \tag{2.7}$$

Comparing estimates (2.6) and (2.7), we see that the form of the corrector depends on the question considered. All the estimates under consideration are order-sharp, and constants in estimates are controlled explicitly in terms of L_∞-norms of the coefficients g, g^{-1}, f, f^{-1}, the lower and upper bounds for the symbol $b(\boldsymbol{\xi})^* b(\boldsymbol{\xi})$ on the sphere $|\boldsymbol{\xi}| = 1$, and parameters of the lattice Γ.

2.3. Method of investigation. Let T_ε be the *unitary* scaling transformation in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ given by $(T_\varepsilon \mathbf{u})(\mathbf{x}) = \varepsilon^{d/2}\mathbf{u}(\varepsilon\mathbf{x})$. The identity

$$(\mathcal{A}_\varepsilon + I)^{-1} = \varepsilon^2 T_\varepsilon^* (\mathcal{A} + \varepsilon^2 I)^{-1} T_\varepsilon$$

shows that, in order to approximate the resolvent $(\mathcal{A}_\varepsilon + I)^{-1}$ in the L_2-operator norm with error of order $O(\varepsilon)$, it suffices to approximate the operator $(\mathcal{A} + \varepsilon^2 I)^{-1}$ with error term of order $O(\varepsilon^{-1})$. The presence of the unitary factors T_ε is not an obstacle for proving estimates *in the operator norm*. At the same time, in the study of weak or strong convergence, the influence of these operators can not be controlled.

The bottom of the spectrum of the operator $\mathcal{A}$ is the point $\lambda = 0$. It is natural that the behavior of the resolvent of $\mathcal{A}$ in the point $\lambda = -\varepsilon^2$ (close to the bottom of the spectrum) can be described in terms of the threshold characteristics of $\mathcal{A}$. Thus, the *homogenization procedure can be studied as a threshold effect at the bottom of the spectrum.*

The operator $\mathcal{A}$ expands in the direct integral of operators $\mathcal{A}(\mathbf{k})$ acting in $L_2(\Omega; \mathbb{C}^n)$ (cf. Subsection 1.1). Let $E_j(\mathbf{k})$, $j \in \mathbb{N}$, be the consecutive eigenvalues of the operator $\mathcal{A}(\mathbf{k})$ (band functions). It turns out that for the operator (2.1) the first n spectral bands overlap and have the common bottom $\lambda = 0$. Herewith, minimum of band functions $E_j(\mathbf{k})$ is reached only at the point $\mathbf{k} = 0$:

$$\min_{\mathbf{k} \in \widetilde{\Omega}} E_j(\mathbf{k}) = E_j(0) = 0, \quad j = 1, \dots, n.$$

The edge of the $(n+1)$-th band is separated from zero: $\min E_{n+1}(\mathbf{k}) > 0$. The minimum point $\mathbf{k} = 0$ for each function E_j is non-degenerate: $E_j(\mathbf{k}) \geq c_* |\mathbf{k}|^2$, $j = 1, \dots, n$, $c_* > 0$. By the *threshold characteristics* of the operator $\mathcal{A}$ we mean the asymptotic behavior of the eigenvalues $E_j(\mathbf{k})$, $j = 1, \dots, n$, and the corresponding eigenfunctions of the operator $\mathcal{A}(\mathbf{k})$ near $\mathbf{k} = 0$.

The study of the resolvent $(\mathcal{A} + \varepsilon^2 I)^{-1}$ is reduced to the study of the resolvent $(\mathcal{A}(\mathbf{k}) + \varepsilon^2 I)^{-1}$ of the operator family $\mathcal{A}(\mathbf{k})$. This family depends on the parameter $\mathbf{k} \in \mathbb{R}^d$ analytically. For $\mathbf{k} = 0$, the kernel $\mathfrak{N}$ of the operator $\mathcal{A}(0)$ is n-dimensional. The natural desire is to use methods of the analytic perturbation theory. However, for $d > 1$ and $n > 1$ (multidimensional parameter and multiple eigenvalue) the analytic perturbation theory is not applicable. The solution suggested in [**145, 151**] is to introduce the one-dimensional parameter $t = |\mathbf{k}|$, putting $\mathbf{k} = t\boldsymbol{\theta}$, $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$. The operator family $\mathcal{A}(\mathbf{k}) = \mathcal{A}(t\boldsymbol{\theta}) = A(t, \boldsymbol{\theta})$ is studied by methods of the analytic perturbation theory with respect to the one-dimensional parameter t.

Herewith, a great deal of constructions can be carried over in *abstract* operator-theoretic terms. A crucial notion of the abstract scheme is the notion of the *spectral germ* $S(\boldsymbol{\theta})$ of the operator family $A(t, \boldsymbol{\theta})$. The germ is a self-adjoint operator acting in the n-dimensional space $\mathfrak{N} = \operatorname{Ker} \mathcal{A}(0)$. Let us give the spectral definition of the germ. By the analytic perturbation theory, for $t \leq t_0$ there exist real-analytic branches of eigenvalues $\lambda_l(t, \boldsymbol{\theta})$ and real-analytic branches of eigenvectors $\varphi_l(t, \boldsymbol{\theta})$ (orthonormal in $L_2(\Omega; \mathbb{C}^n)$), $l = 1, \dots, n$, of the operator $A(t, \boldsymbol{\theta})$. For sufficiently small t_*, the following convergent power series expansions hold:

$$
\begin{aligned}
&\lambda_l(t, \boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta}) t^2 + \mu_l(\boldsymbol{\theta}) t^3 + \dots, \\
&\varphi_l(t, \boldsymbol{\theta}) = \omega_l(\boldsymbol{\theta}) + \varphi_l^{(1)}(\boldsymbol{\theta})\, t + \dots, \quad l = 1, \dots, n, \quad t \leq t_*.
\end{aligned} \tag{2.8}
$$

The coefficients $\gamma_l(\boldsymbol{\theta})$ are positive: $\gamma_l(\boldsymbol{\theta}) \geq c_* > 0$. The vectors $\{\omega_1(\boldsymbol{\theta}), \dots, \omega_n(\boldsymbol{\theta})\}$ form an orthonormal basis in $\mathfrak{N}$. The coefficients $\gamma_l(\boldsymbol{\theta})$ and the vectors $\omega_l(\boldsymbol{\theta})$ are *threshold characteristics* of the operator $\mathcal{A}$ near the bottom of the spectrum.

According to [**145, 151**], the self-adjoint operator $S(\boldsymbol{\theta}) : \mathfrak{N} \to \mathfrak{N}$ such that

$$S(\boldsymbol{\theta}) \omega_l(\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta}) \omega_l(\boldsymbol{\theta}), \quad l = 1, \dots, n,$$

is called the *spectral germ* of the operator family $A(t, \boldsymbol{\theta})$. Thus, the spectral germ contains information about threshold characteristics of the operator $\mathcal{A}$. Therefore, the germ is responsible for threshold effects.

The key result from [**145, 151**] is the following approximation for the resolvent of the operator $A(t, \boldsymbol{\theta})$ by finite rank operators given in terms of the spectral germ

$$\|(A(t, \boldsymbol{\theta}) + \varepsilon^2 I)^{-1} - (t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P\|_{L_2(\Omega) \to L_2(\Omega)} \leq C \varepsilon^{-1}, \quad t \leq t_0. \tag{2.9}$$

Here P is the orthogonal projection of the space $L_2(\Omega; \mathbb{C}^n)$ onto $\mathfrak{N}$. Constants C and t_0 are controlled explicitly.

It is possible to calculate the spectral germ. Here we formulate the result for the simpler operator $\widehat{\mathcal{A}}$. In this case, the germ is represented as $S(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* g^0 b(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, where g^0 is the effective matrix. It turns out that the effective operator $\widehat{\mathcal{A}}^0$ has the same spectral germ as $\widehat{\mathcal{A}}$. Then, using estimate (2.9), it is possible to approximate the resolvent of the operator $\widehat{\mathcal{A}}$ by the resolvent of $\widehat{\mathcal{A}}^0$:

$$\|(\widehat{\mathcal{A}} + \varepsilon^2 I)^{-1} - (\widehat{\mathcal{A}}^0 + \varepsilon^2 I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C \varepsilon^{-1}.$$

This implies estimate (2.3) by the simple scale transformation.

In order to obtain more precise approximation (2.6), one has to refine (2.9) and to find approximation of the resolvent $(A(t, \boldsymbol{\theta}) + \varepsilon^2 I)^{-1}$ by some finite rank operators with an error term of order $O(1)$. In abstract terms, such result was obtained in [**155**]. For this, it is necessary to take into account terms up to order

t^3 in the expansions for eigenvalues and terms up to order t in the expansions for eigenvectors in (2.8). In [**156**], these abstract results were applied for the proof of estimate (2.6).

In order to approximate the resolvent in the norm of operators acting from L_2 to H^1, it is necessary to approximate the operator $A(t,\boldsymbol{\theta})^{1/2}(A(t,\boldsymbol{\theta})+\varepsilon^2 I)^{-1}$ in the operator norm in $L_2(\Omega;\mathbb{C}^n)$. In this way, estimate (2.7) was obtained in [**158**] .

Finally, in order to study the operator $\mathcal{A}_\varepsilon$ (with $f\neq \mathbf{1}$), the identity

$$(\mathcal{A}_\varepsilon + I)^{-1} = (f^\varepsilon)^{-1}(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}((f^\varepsilon)^*)^{-1}$$

was used, and the generalized resolvent $(\widehat{\mathcal{A}}_\varepsilon + Q^\varepsilon)^{-1}$ of $\widehat{\mathcal{A}}_\varepsilon$ was studied. For the generalized resolvent the analogues of estimates (2.3), (2.6), (2.7) were obtained.

2.4. In [**145, 151, 156, 158**], general results were applied to specific periodic operators of mathematical physics: to the acoustics operator, the operator of elasticity theory, the Schrödinger operator, the two-dimensional Pauli operator, and also to the stationary Maxwell system.

Homogenization for the stationary periodic Maxwell system is the most difficult problem. In the case where one of the coefficients (dielectric permittivity or magnetic permeability) is constant, the problem is (partially) reduced to the study of the second order operator $\widehat{\mathcal{A}} = \operatorname{rot}\eta(\mathbf{x})^{-1}\operatorname{rot} - \nabla\operatorname{div}$, which admits factorization of the form (2.2). Then general results for this class of operators are applicable. Such case was studied in [**151**, Chapter 7], but only in [**159**] approximations in the $L_2(\mathbb{R}^3)$-norm for all physical fields were obtained.

The problem is even more difficult, if both coefficients are not constant. The corresponding "model" second order operator does not belong to the class of operators of the form (2.2) or (2.1). Nevertheless, it is possible to apply the results of the abstract scheme from [**151, 158**]. The homogenization problem for the Maxwell system in the case where both coefficients are variable periodic matrix-valued functions was studied in the papers by T. Suslina (2004, 2007), where approximations in the $L_2(\mathbb{R}^3)$-norm for all fields were obtained.

We also mention the papers [**150, 153**], where an analogue of homogenization procedure for the scalar elliptic operator $\mathcal{A} = -\operatorname{div} g(\mathbf{x})\nabla + p(\mathbf{x})$ (with real-valued periodic coefficients) near the edge of an *internal gap* was considered. Let $\mathcal{A}_\varepsilon = -\operatorname{div} g^\varepsilon\nabla + \varepsilon^{-2}p^\varepsilon$. Let λ be the right edge of a gap in the spectrum of the operator $\mathcal{A}$. For the operator $\mathcal{A}_\varepsilon$, the edge of the corresponding gap turns into the point $\varepsilon^{-2}\lambda$ (lying in the high-energy area). The problem is to approximate the operator $(\mathcal{A}_\varepsilon - (\varepsilon^{-2}\lambda - \varkappa^2)I)^{-1}$ for small ε in the operator norm in $L_2(\mathbb{R}^d)$. (Here the number $\varkappa$ is such that the point $\lambda-\varkappa^2$ lies in the initial gap in the spectrum of $\mathcal{A}$.) Such approximation with sharp-order error estimate was obtained in [**153**] under some assumptions about the spectral characteristics of $\mathcal{A}$ near the edge of the gap. (The one-dimensional case was studied before in [**150**].) In the problems of this type interaction between the threshold effects and the high-energetic effects is observed.

2.5. The papers [**145, 151**] stimulated interest in the homogenization theory to approximations of the resolvent with error estimates in the operator norm. V. Zhikov suggested a different method of obtaining operator estimates in the homogenization theory. By this method estimates of the form (2.3) and (2.7) were obtained for the scalar operator $\widehat{\mathcal{A}} = -\operatorname{div} g(\mathbf{x})\nabla$ (V. Zhikov, 2005) and for the operator of elasticity theory (V. Zhikov and S. Pastukhova, 2005). In the recent paper by D. Borisov (2008), estimates of the form (2.3) and (2.7) were obtained for an operator with coefficients depending both on rapid and slow variables.

§3. Discrete spectrum of a perturbed periodic elliptic operator

After 1998, M. Birman continued to study the discrete spectrum in the gaps of a periodic elliptic operator perturbed by a potential decaying at infinity. Two papers by M. Birman were devoted to this subject. One of them is the paper [**144**] joint with A. Laptev and T. Suslina and the second one is the paper [**146**] joint with M. Solomyak.

3.1. Setting of the problem. Consider the operator $A(\alpha) = A - \alpha V$ acting in $L_2(\mathbb{R}^d)$, where A is an "unperturbed" operator, V is a "perturbation", and $\alpha \geq 0$ is a *coupling constant.*

The unperturbed operator A is an elliptic second order operator of the form $A = -\operatorname{div} g(\mathbf{x})\nabla + p(\mathbf{x})$ with real-valued coefficients that are periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$. The matrix-valued function $g(\mathbf{x})$ is bounded and uniformly positive definite, and the potential $p(\mathbf{x})$ is bounded. The precise definition of the operator is given in terms of the quadratic form. One may assume that the bottom of the spectrum of A is the point $\lambda = 0$. The spectrum of the operator A has a band structure (see Subsection 1.1). There exists a semi-bounded gap $(-\infty, 0)$; there may be also internal gaps in the spectrum of A.

The perturbation V is the operator of multiplication by a real-valued function $V(\mathbf{x})$. It is assumed that $V(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, in an appropriate sense. In general, for the perturbed operator $A(\alpha)$ eigenvalues appear in the semi-bounded gap $(-\infty, 0)$ and also in the internal gaps. Let us discuss the negative discrete spectrum. If the potential V_+ decays sufficiently fast, then the number of negative eigenvalues is finite. Here $2V_+(\mathbf{x}) = |V(\mathbf{x})| + V(\mathbf{x})$. Let $N(\alpha, \lambda)$ be the number of eigenvalues (counting multiplicities) of the operator $A(\alpha)$ lying below a point $\lambda \leq 0$ called the observation point. The full number of negative eigenvalues $N(\alpha) = N(\alpha, 0)$ is of a special interest. Asymptotics of $N(\alpha, \lambda)$ as $\alpha \to +\infty$ is studied.

The corresponding asymptotic formulas are rather diverse, depending on dimension d, on the rate of decay of V and on the nature of operator A. They were studied in a series of papers of M. Birman in 1991 1997. See, e. g., [**110, 127, 130, 134**] and, especially, the survey [**128**] and references therein. (Here and in what follows, references correspond to the list [**2***]). The cases $d \geq 3$ and $d = 2$ turn out to be essentially different.

3.2. The case $d \geq 3$. For $d \geq 3$, a "regular" case is distinguished by the condition $V \in L_{d/2}(\mathbb{R}^d)$. If this condition is satisfied, then the following Rozenblum-Lieb-Cwikel estimate is true:

$$N(\alpha, \lambda) \leq c(d)\alpha^{d/2} \int_{\mathbb{R}^d} V_+(\mathbf{x})^{d/2}\, d\mathbf{x}, \quad \lambda \leq 0, \tag{3.1}$$

and the Weyl asymptotics holds:

$$\begin{aligned} &N(\alpha, \lambda) \sim \alpha^{d/2} J_d(V, g), \quad \alpha \to +\infty, \quad \lambda \leq 0, \\ &J_d(V, g) = \frac{\omega_d}{(2\pi)^d} \alpha^{d/2} \int_{\mathbb{R}^d} V_+(\mathbf{x})^{d/2} (\det g(\mathbf{x}))^{-1/2}\, d\mathbf{x}. \end{aligned} \tag{3.2}$$

Here ω_d is the volume of the unit ball in $\mathbb{R}^d$.

If $V(\mathbf{x}) \geq 0$, then the following properties are true: 1) the finiteness of the Weyl coefficient J_d (which is equivalent to the condition $V \in L_{d/2}$) implies the Weyl asymptotics; 2) the asymptotics (3.2) is uniform in λ up to the edge $\lambda = 0$; 3) if $N(\alpha, \lambda) \leq C\alpha^{d/2}$ for some λ, then $V \in L_{d/2}$, and hence, the asymptotics (3.2) holds (i. e., the Weyl order $\alpha^{d/2}$ ensures the Weyl asymptotics).

In the "non-regular" case $V \notin L_{d/2}(\mathbb{R}^d)$, the Weyl asymptotics is violated. Moreover, the Weyl asymptotic order $\alpha^{d/2}$ is impossible. However, for any $q > d/2$ there are examples of potentials V such that $N(\alpha) \sim C\alpha^q$.

In the regular case asymptotics has a "high-energetic" origin, and in the non-regular case it has a "threshold" origin. The regular asymptotics was studied in [**127**], and the non-regular asymptotics was studied in [**134**].

3.3. Two-dimensional case. First, let $V(\mathbf{x}) \geq 0$. If $d = 2$, the three properties mentioned above are violated: 1) the condition $V \in L_1(\mathbb{R}^2)$ (which is equivalent to the finiteness of the Weyl coefficient J_2) is not sufficient for the validity of the Weyl asymptotics; 2) asymptotics is not uniform in $\lambda \leq 0$ (for any $q > 1$ there are examples of potentials such that $N(\alpha, \lambda)$ has the Weyl asymptotics for $\lambda < 0$, but $N(\alpha, \lambda)$ is of order α^q for $\lambda = 0$); 3) the estimate $N(\alpha) \leq C\alpha$ does not ensure the Weyl asymptotics (there are examples of potentials such that $N(\alpha) \sim c\alpha$, but $c \neq J_2$).

All these effects were discovered in the paper [**130**] by M. Birman and A. Laptev in the case where $A = -\Delta$. It is assumed that the potential $V(\mathbf{x})$ satisfies the condition

$$\Big(\int\limits_{|\mathbf{x}|\leq 1} |V|^\sigma \, d\mathbf{x}\Big)^{1/\sigma} + \sum_{k=1}^{\infty}\Big(\int\limits_{e^{k-1}\leq|\mathbf{x}|\leq e^k} |V|^\sigma |\mathbf{x}|^{2(\sigma-1)} \, d\mathbf{x}\Big)^{1/\sigma} < \infty, \quad \sigma > 1, \tag{3.3}$$

with some $\sigma > 1$. Relation (3.3) implies that $V \in L_1(\mathbb{R}^2)$. Condition (3.3) ensures the Weyl asymptotics for the function $N(\alpha, \lambda)$ if $\lambda < 0$, but not for $N(\alpha)$. In [**130**], for each $q > 1$, the class of potentials V such that asymptotics of the function $N(\alpha)$ is of order α^q was distinguished. Then asymptotics has a threshold origin, and the answer is formulated in terms of an auxiliary problem on the semi-axis, which is obtained by restricting the operator $-\Delta - \alpha V$ on the subspace of functions depending only on $r = |\mathbf{x}|$. Simultaneously, the potential V is averaged over the polar angle. In [**130**], the class of potentials such that the asymptotics of $N(\alpha)$ is of order α, and the asymptotic coefficient is the sum of the Weyl term and the threshold term, was also distinguished.

For the case where $d = 2$ and A is a *periodic elliptic operator*, the non-regular asymptotics was studied in [**144**]. Here two different cases are observed: the case of the purely threshold asymptotics of order $q > 1$, and the case of a competition between the Weyl asymptotics and the threshold asymptotics. The threshold term is described in terms of the auxiliary problem on the semi-axis which is now determined not only by potential V, but also by the threshold characteristics of the operator A at the bottom of the spectrum. By the threshold characteristics, we mean the positive matrix g^0 and the periodic function $\varphi(\mathbf{x})$ defined as follows. The matrix g^0 determines asymptotics of the first band function $E_1(\mathbf{k})$ as $|\mathbf{k}| \to 0$ (asymptotics has the form $E_1(\mathbf{k}) \sim \langle g^0\mathbf{k}, \mathbf{k}\rangle$), and $\varphi(\mathbf{x})$ is a positive periodic solution of the equation $A\varphi = 0$.

For $d = 2$, the non-regular asymptotics for the spectrum in internal gaps was studied by T. Suslina (2003).

3.4. Periodic waveguide. The paper [**146**] is close to [**144**] both by the nature of the problem considered and by the character of results. The main difference is in geometry: in [**146**] the operators act in the space $L_2(X)$, where the domain $X \subset \mathbb{R}^d$ is periodic in only one direction. The unperturbed operator is again $A = -\text{div}\, g(\mathbf{x})\nabla + p(\mathbf{x})$, with the Dirichlet or the Neumann boundary condition on ∂X. The functions $g(\mathbf{x})$ and $p(\mathbf{x})$ are also assumed to be periodic, which gives one the possibility to apply the Floquet-Bloch theory. By adding a constant to $p(\mathbf{x})$, one can always assume that $\lambda = 0$ is the lower point of spectrum of A.

The perturbation is introduced by the potential $V(\mathbf{x})$ decaying, in appropriate sense, along the same direction. The assumptions about $V(\mathbf{x})$ guarantee that for the operator $A - \alpha V$ the function $N(\alpha, \lambda)$ (see its definition in Subsection 3.1) has the Weyl-type asymptotic behavior (as $\alpha \to +\infty$) for any $\lambda < 0$. The main problem consists in studying $N(\alpha, 0)$.

Unlike the case $X = \mathbb{R}^d$, here the threshold effect arises in any dimension $d > 1$. To describe its influence on the spectrum, one introduces an auxiliary Schrödinger operator $A'_{\alpha Q} = -\partial_t^2 - \alpha Q$ acting in $L_2(\mathbb{R})$. The "effective potential" $Q(t)$ is expressed in terms of the original potential $V(\mathbf{x})$ and the Floquet data for the unperturbed operator A.

The main result can be written in the form

$$N(\alpha, 0) \sim [\text{Weyl's term}] + N'(\alpha, 0), \qquad \alpha \to +\infty,$$

where the second term corresponds to the operator $A'_{\alpha Q}$. All the effects mentioned in Subsection 3.3 exhibit here. So, as in [**144**] (and earlier, in [**130**]), one encounters a sort of competition between the Weyl term in asymptotics and another term, of the threshold origin.

§4. Asymptotic behavior of the spectrum of the non-smooth Maxwell operator

For the empty electro-magnetic resonator in a bounded domain Ω with smooth, ideally conducting boundary, the behavior of the eigenfrequencies was found by H. Weyl as far ago as in 1912. Results for the non-smooth situation (filled resonator with non-smooth dielectric permittivity $\varepsilon(\mathbf{x})$ and magnetic permeability $\mu(\mathbf{x})$, in the case of non-smooth boundary) were obtained only recently by M. Birman and his colleagues and students. The reason of such a long delay is not purely technical, its roots are of a rather fundamental nature.

Formally, the stationary Maxwell operator $M_{\varepsilon,\mu}$ acts as

$$M_{\varepsilon,\mu}\{\mathbf{u}, \mathbf{v}\} = \{i\varepsilon^{-1}\text{rot}\, \mathbf{v}, -i\mu^{-1}\text{rot}\, \mathbf{u}\},$$

under the divergence free conditions

$$\text{div}\,(\varepsilon\mathbf{u}) = 0, \qquad \text{div}\,(\mu\mathbf{v}) = 0,$$

and the boundary conditions

$$\mathbf{u}_\tau|_{\partial\Omega} = 0, \qquad (\mu\mathbf{v})_\nu|_{\partial\Omega} = 0.$$

Here $\mathbf{u}$ and $\mathbf{v}$ stand for the electric and the magnetic component of an electromagnetic field, and τ and ν indicate the tangent and the normal component of a vector field on $\partial\Omega$. However, all the operations involved have to be defined in an appropriate way, and this turns out to be non-trivial if the boundary $\partial\Omega$ or the positive definite matrix-valued functions $\varepsilon(\mathbf{x})$, $\mu(\mathbf{x})$ are non-smooth. Such non-smooth situations are important for many applications (stratified media, domains with edges, conical points, or screens, etc.)

Prior to calculating the spectrum, one has to define $M_{\varepsilon,\mu}$ as a self-adjoint operator acting in an appropriate Hilbert space. In the case of filled resonator, the latter is an L_2-space of vector-valued functions, with the matrix weight generated by $\varepsilon(\mathbf{x})$, $\mu(\mathbf{x})$. The difficulties arise when one attempts to describe the operator domain of $M_{\varepsilon,\mu}$. The standard approaches, which include using the Sobolev spaces and quadratic forms, do not work here, since the operator is neither elliptic, nor semi-bounded. The choice of the "correct" self-adjoint realization is determined by the condition of finiteness of the electromagnetic energy. In the non-smooth cases it cannot be described in terms of Sobolev spaces. Ignoring these important facts led to errors in several papers, whose authors claimed that they found the spectral asymptotics of the Maxwell operator in the non-smooth domains.

Due to the block structure of the Maxwell operator, its spectrum is symmetric with respect to zero. For its calculation it is sufficient to work with the electric component of the electro-magnetic field. The above mentioned condition of finiteness of the energy dictates that this component must belong to the space

$$\Phi(\tau,\varepsilon) = \{\mathbf{u} \in L_2(\Omega;\mathbb{C}^3) : \operatorname{rot}\mathbf{u} \in L_2(\Omega;\mathbb{C}^3),\ \operatorname{div}(\varepsilon\mathbf{u}) = 0,\ \mathbf{u}_\tau|_{\partial\Omega} = 0\}.$$

Here the operators rot and div are defined in the sense of distributions, and the equality $\mathbf{u}_\tau|_{\partial\Omega} = 0$ is understood in an appropriate generalized sense.

If the boundary $\partial\Omega$ and the matrix-valued function $\varepsilon(\mathbf{x})$ are smooth, the space $\Phi(\tau,\varepsilon)$ can be easily described in classical terms. Then the asymptotic behavior of the eigenfrequencies can be studied by various standard tools. The resulting asymptotic formula looks as follows. Let $r(\boldsymbol{\xi})$ be the symbol of rot, i.e.

$$r(\boldsymbol{\xi}) = i\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}.$$

Denote by $\Lambda_1(\mathbf{x},\boldsymbol{\xi}), \Lambda_2(\mathbf{x},\boldsymbol{\xi})$ the positive eigenvalues of the algebraic problem

$$r(\boldsymbol{\xi})\mu(\mathbf{x})^{-1}r(\boldsymbol{\xi})\mathbf{h} = \Lambda\varepsilon(\mathbf{x})\mathbf{h}, \qquad \mathbf{h}\in\mathbb{C}^3,\ \mathbf{x}\in\Omega,\ \boldsymbol{\xi}\in\mathbb{R}^3;$$

the third eigenvalue is equal to zero. We put

$$\Gamma(\varepsilon,\mu,\Omega) = \frac{1}{24\pi^3}\int_\Omega\int_{|\boldsymbol{\xi}|=1}\left(\Lambda_1(\mathbf{x},\boldsymbol{\xi})^{-3/2} + \Lambda_2(\mathbf{x},\boldsymbol{\xi})^{-3/2}\right)dS(\boldsymbol{\xi})d\mathbf{x},$$

where dS denotes the area element on the unit sphere. Then the eigenfrequencies m_k satisfy the asymptotic formula

$$\lim_{k\to\infty} k^{-1}m_k^3 = \Gamma(\varepsilon,\mu,\Omega). \tag{4.1}$$

Under some, rather general assumptions this formula remains valid also in the non-smooth case. Its justification needs a careful study of the structure of the space $\Phi(\tau,\varepsilon)$. First of all, A. Alekseev and M. Birman [**65, 66**] established a general geometric scheme (in the sense of geometry of Hilbert spaces), showing that the problem in a filled resonator can be always reduced to the case of the empty one. For resonators with smooth boundary this led to the formula (4.1) for any measurable and bounded matrices $\varepsilon(\mathbf{x}), \mu(\mathbf{x})$, having the bounded inverse. The case of non-smooth boundary turned out to be more difficult. The further progress was achieved on the basis of the results on the analytic structure of vector fields belonging to the space $\Phi(\tau, \varepsilon)$. This analysis was developed by M. Birman and M. Solomyak in the papers [**90, 93, 97, 117**]. Namely, it was shown that in domains with Lipschitz boundary, and also in domains with screens, any vector field $\mathbf{u} \in \Phi(\tau, \mathbf{1})$ can be decomposed into the sum of a term from the Sobolev space $H_0^1(\Omega; \mathbb{C}^3)$ and another term which is the gradient of a weak solution of the Poisson equation $-\Delta w = f$, $w|_{\partial\Omega} = 0$, with some $f \in L_2(\Omega)$. This fact allowed these authors to justify (in [**91**]) the formula (4.1) for an empty resonator with Lipschitz boundary.

Recently it was shown by M. Birman and N. Filonov [**160**] that the existence of the above decomposition alone, without any explicit requirements about $\partial\Omega$, already implies the formula (4.1). This is a crucial result, since it reduces the problem of the calculation of spectral asymptotics for Maxwell operator to the problem (of Real Analysis) of describing singularities of vector fields of a certain class. In particular, this led to the proof of (4.1) for a filled resonator with Lipschitz boundary, under the same assumptions about $\varepsilon(\mathbf{x}), \mu(\mathbf{x})$ as in [**65, 66**], that is measurability and boundedness of the matrices and their inverses. Some technical tools, which are necessary for the proof of this result, were developed by M. Birman, A. Alekseev, and N. Filonov in [**157**].

A programme of studying spectral properties of the "non-smooth" Maxwell operator was initiated by M. Birman in early 70-tees, though his first papers on the subject are [**65**] and [**66**], joint with his student A. Alekseev. The papers [**157**] and [**160**] can be considered as concluding the long series of results devoted to realization of this programme.

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