HYPER VERSION

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Singularities of the Seiberg-Witten map

Andrei G. Bytsko

Steklov Mathematics Institute, Fontanka 27
St.Petersburg, 191011, Russia
E-mail: bytsko@pdmi.ras.ru

Abstract: We construct an explicit solution of the Seiberg-Witten map for a linear gauge field on the non-commutative plane. We observe that this solution as well as the solution for a constant curvature diverge when the non-commutativity parameter θ reaches certain event horizon in the θ-space. This implies that an ordinary Yang-Mills theory can be continuously deformed by the Seiberg-Witten map into a non-commutative theory only within one connected component of the θ-space.

Keywords: D-branes, Gauge Symmetry, Non-Commutative Geometry.

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Contents

1. Introduction

Gauge theories on non-commutative spaces arise as low energy effective theories on D-brane world volumes in the presence of the background Neveu-Schwarz B-field in the theory of open strings [1, 2]. The simplest but important example of a non-commutative space is the space $\mathbb{R}^d_\theta$ which corresponds to a flat background. The coordinates x^i of $\mathbb{R}^d_\theta$ obey the Heisenberg commutation relations,

$$[x^i, x^j] = i\,\theta^{ij}\,, \tag{1.1}$$

where $\theta^{ij} = -\theta^{ji}$ is a constant real-valued anti-symmetric matrix. Functions on the space $\mathbb{R}^d_\theta$ can be identified with ordinary functions on $\mathbb{R}^d$ with the non-commutative product given by the Moyal formula (here and below summation over repeated indices is implied),

$$(u * v)(x) = \left(e^{\frac{i}{2}\theta^{ij}\partial^x_i\partial^y_j}\, u(x)v(y)\right)_{y=x}. \tag{1.2}$$

As was argued in [2], non-commutative Yang-Mills theory on $\mathbb{R}^d_\theta$ is equivalent to ordinary Yang-Mills theory on $\mathbb{R}^d$. This means that there exists a transformation relating gauge fields on $\mathbb{R}^d_\theta$ with different values of the deformation parameter θ. More precisely, let θ and $\theta + \delta\theta$ be two infinitesimally close values of the deformation parameter. Then, there exists the Seiberg-Witten (SW) map $A \to \hat{A}(A)$ and $\lambda \to \hat{\lambda}(\lambda, A)$ of the gauge fields and infinitesimal gauge parameters on $\mathbb{R}^d_\theta$ to those on

$\mathbb{R}^d_{\theta+\delta\theta}$ such that $\hat{A}(A + \delta_\lambda A) = \hat{A}(A) + \delta_{\hat{\lambda}(\lambda,A)}\hat{A}(A)$. Explicitly, the SW map is given by [2]:

$$\delta_\theta A_i = -\frac{1}{4}\delta\theta^{kl}\{A_k, \partial_l A_i + F_{li}\}_\theta \,, \tag{1.3}$$

$$\delta_\theta F_{ij} = \frac{1}{4}\delta\theta^{kl}\Big(2\,\{F_{ik}, F_{jl}\}_\theta - \{A_k, D_l F_{ij} + \partial_l F_{ij}\}_\theta\Big)\,, \tag{1.4}$$

where F_{ij} is the curvature of the gauge field defined as follows:

$$F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]_\theta \,. \tag{1.5}$$

Here and below $[..]_\theta$ and $\{..\}_\theta$ stand, respectively, for commutator and anti-commutator with respect to the $*$-product.

For slowly varying fields on a D-brane, the effective action is described by the Dirac-Born-Infeld (DBI) lagrangian [3] which depends on B, F and the closed string metric g. Using the above equations for $\delta_\theta A_i$ and $\delta_\theta F_{ij}$, Seiberg and Witten established that this action is equivalent (up to a total derivative and $\mathcal{O}(\partial F)$ terms) to a non-commutative version of the DBI action. The latter has no explicit B-dependence, but it involves the non-commutative curvature $F(\theta)$ with

$$\theta(B) = -(g_\alpha + B)^{-1}B\,(g_\alpha - B)^{-1}\,, \tag{1.6}$$

where $g_\alpha = g/(2\pi\alpha')$, α' is the inverse to the string tension.

The above mentioned equivalence of two DBI actions is based on the idea that one can use the SW map to *continuously* deform the initial (ordinary) Yang-Mills theory with $\theta = 0$ into a non-commutative Yang-Mills theory with $\theta = \theta(B)$. That is, it is implicitly assumed that in the θ-space there exists a continuous path γ connecting the origin with the point $\theta = \theta(B)$ such that A_i and F_{ij} converge *everywhere* on γ. However, convergence of solutions for the SW map is a rather difficult question which, to our knowledge, has not been addressed in the literature. In the present paper we will investigate the situation for the U(1) constant curvature case. Already in this simplest case solutions of the SW map will demonstrate a non-trivial feature — they cannot be continued beyond certain "event horizon".

2. SW map for constant curvature

2.1 Existence of event horizon

For $\mathbb{R}^d_\theta$, the non-commutativity parameter θ has $\frac{1}{2}d(d-1)$ independent entries and thus can be viewed as a point in $\frac{1}{2}d(d-1)$-dimensional euclidean space, which we will refer to as the θ-space. Given an initial value of the gauge field A_i for $\theta = 0$, we can solve, at least in principle, the differential equation (1.3) as a series in θ. As we explained above, it is important to determine the region of the θ-space where this solution converges.

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Notice that, in general, solutions of the SW map depend on a path in the θ-space along which we carry out the deformation. That is, the result of action of two infinitesimal shifts $\delta_1\theta$ and $\delta_2\theta$ on A_i or F_{ij} depends on their order [4]. Analogous statement holds for a gauge group element $g(x)$ even in the zero curvature case [5]. However, if the curvature is constant (independent of the coordinates), its SW map (1.4) considerably simplifies and the corresponding solution does not depend on the deformation path. Therefore, we discuss first properties of the solution of eq. (1.4) in the U(1) case for constant curvature.

If the curvature $(F_0)_{ij}$ in the initial (ordinary) theory is independent of the coordinates, then so is $F_{ij}(\theta)$ constructed according to the SW map, which in this case reads $\delta F_{ij} = -\delta\theta^{kl} F_{ik} F_{lj}$. Here we have the ordinary product on the right hand side. Therefore, as was noted in [2], this equation can be rewritten as a matrix differential equation (the Lorentz indices are regarded as matrix indices)

$$\delta F = -F\,\delta\theta\,F\,. \tag{2.1}$$

Then the corresponding solution is given in the matrix form as follows:

$$F = (1 + F_0\theta)^{-1} F_0\,. \tag{2.2}$$

It was remarked in [2], that since (2.2) has a pole at $\theta = -F_0^{-1}$, there is no noncommutative gauge theory for this value of θ equivalent to the initial commutative theory. However, this analysis of eq. (2.2) is not exhaustive. Indeed, consider $\mathbb{R}^d_\theta$ and let F_0 be non degenerate, i.e. $\operatorname{rank} F_0 = d$ (so, d is even). Then F diverges for all values of θ such that

$$\det(1 + F_0\theta) = 0\,. \tag{2.3}$$

This equation defines a hypersurface Γ in the θ-space. To describe Γ more explicitly, we note that (2.3) is solved by the substitution

$$\theta = -F_0^{-1} + \theta'\,, \tag{2.4}$$

where θ' is an arbitrary real-valued $d \times d$ anti-symmetric matrix such that

$$\det\theta' = 0\,. \tag{2.5}$$

The latter equation describes a cone (as it is invariant with respect to the rescaling $\theta' \to \mathrm{const}\,\theta'$).

An important fact that follows from the above description of Γ is that the complement of Γ in the θ-space *is not* connected. More precisely, this complement has two connected components. Indeed, the complement of (2.5) consists of all $d \times d$ real-valued anti-symmetric matrices M such that $\det M > 0$. By an orthogonal transformation, $M' = oMo^t$, $o \in \mathrm{O}(d)$, any such matrix can be brought to the canonical block-diagonal form,

$$M' = \mathrm{diag}\left(m_1\sigma,\, m_2\sigma, \ldots, m_{d/2}\sigma\right), \qquad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$m_1 \geq m_2 \geq \cdots \geq m_{d/2} > 0\,. \tag{2.6}$$

The group $\mathrm{O}(d)$ has two connected components, therefore so does the set of all $d \times d$ real-valued anti-symmetric matrices M with positive determinants. The two connected components correspond to two possible signs of the Pfaffian $\mathrm{Pf}\, M$.

If $0 < \mathrm{rank}\, F_0 = k < d$, we can apply an orthogonal transformation, $F_0' = oF_0o^t$, such that $(F_0')_{ij} = 0$ for all $i, j > k$. Then eqs. (2.2) and (2.3) impose restrictions only on the upper-left $k \times k$-minor of θ. The remaining $n = \frac{1}{2}(d-k)(d+k-1)$ components of θ are arbitrary. Thus, in the case of degenerate F_0 we have $\Gamma = \Gamma_k \times \mathbb{R}^n$, where Γ_k corresponds to eq. (2.3) for the non-degenerate minor of F_0' (and (2.3) is considered in $\frac{1}{2}k(k-1)$-dimensional space). Consequently, the complements of Γ and Γ_k have the same number of connected components, that is two. For instance, Γ is just a single point for $d = 2$, hence for $d = 3$ we obtain (as $n = 2$ here) that $\Gamma = \mathbb{R}^2$ is a 2-plane in $\mathbb{R}^3$. Below we will assume that $\mathrm{rank}\, F_0 = d$ and d is even.

Summarizing, we see that eq. (2.3) defines the hypersurface Γ whose complement consists of two connected components. Γ can be viewed as an *event horizon* for the constant curvature version of the SW map since the corresponding solution (2.2) diverges everywhere on Γ. As a consequence, the initial (ordinary) Yang-Mills theory can be continuously deformed into a non-commutative theory only within that connected component, call it Ω_+, which contains the origin of the θ-space.

2.2 Estimates on good values of θ

The above discussion shows that θ is "good", i.e. it belongs to Ω_+ if the sign of $\mathrm{Pf}(\theta + F_0^{-1})$ coincides with that of $\mathrm{Pf}(F_0^{-1})$. Indeed, $\mathrm{Pf}(\theta + F_0^{-1})$ vanishes only on Γ, hence its sign is constant within a connected component. Since $\mathrm{Pf}(M^{-1}) = (-1)^{d/2}(\mathrm{Pf}\, M)^{-1}$, the condition that θ belongs to Ω_+ is equivalent to the requirement

$$(-1)^{d/2}\ \mathrm{Pf}\, F_0\ \mathrm{Pf}\left(\theta + F_0^{-1}\right) > 0\,. \tag{2.7}$$

For instance, in the $d = 2$ case we have $\theta = \vartheta\,\sigma$, $F_0 = f\,\sigma$ (with σ as in (2.6)), and (2.7) is fulfilled if $\vartheta f < 1$. In the case of $d = 4$ any anti-symmetric matrix M can be represented by a pair of 3-vectors, $\vec{e}_M = (M_{12}, M_{13}, M_{14})$, $\vec{h}_M = (M_{34}, -M_{24}, M_{23})$. In this parameterization $\mathrm{Pf}\ M = \vec{e}_M \cdot \vec{h}_M$, so we can rewrite (2.7) as follows:

$$1 + \mathrm{Pf}\ \theta\ \ \mathrm{Pf}\, F_0 > \vec{e}_\theta \cdot \vec{e}_{F_0} + \vec{h}_\theta \cdot \vec{h}_{F_0}\,. \tag{2.8}$$

Let us remark that Γ for $d = 4$ is a cone with the base $S^2 \times S^2$. Indeed, Γ is defined by the equation $\vec{e}_M \cdot \vec{h}_M = 0$ (with $M = \theta + F_0^{-1}$). Its intersection with the unit 5-sphere, $(\vec{e}_M)^2 + (\vec{h}_M)^2 = 1$, can be parameterized as $\vec{e}_M = \vec{a} + \vec{b}$, $\vec{h}_M = \vec{a} - \vec{b}$, where $\vec{a}$ and $\vec{b}$ are 3-vectors of length 1/2.

For higher dimensions condition (2.7) becomes rather involved. But it is certainly fulfilled for θ with sufficiently small euclidean norm, $\|\theta\| = \sqrt{-\operatorname{tr}\theta^2}$. More precisely, if f_1 is the maximal eigenvalue for the canonical form (2.6) of F_0, then the distance r between the origin of the θ-space and the hypersurface Γ is given by (see appendix A)

$$r = \sqrt{2}\, f_1^{-1} \,. \tag{2.9}$$

That is any θ such that $\|\theta\| < r$ is guaranteed to belong to Ω_+. Note that this estimate does not assume any specific form of θ, whereas in the context of the DBI action on a D-brane we are interested in $\theta(B)$ given by (1.6). Here we should take into account that the values of $\theta(B)$ belong to some *compact* domain in the θ-space. To show this, it is convenient to introduce new variables,

$$\begin{aligned} \tilde{\theta} &= \sqrt{g_\alpha}\,\theta\,\sqrt{g_\alpha}\,, \\ \tilde{B} &= (\sqrt{g_\alpha}\,)^{-1}\,B\,(\sqrt{g_\alpha}\,)^{-1}\,, \\ \tilde{F}_0 &= (\sqrt{g_\alpha}\,)^{-1}F_0(\sqrt{g_\alpha}\,)^{-1}\,, \end{aligned} \tag{2.10}$$

where $\sqrt{g_\alpha}$ is defined as a symmetric matrix. Then (1.6) acquires the form

$$\tilde{\theta}(\tilde{B}) = -\tilde{B}\,(1 - \tilde{B}^2)^{-1} \,. \tag{2.11}$$

Both sides here are $d \times d$ anti-symmetric matrices, so this equation defines a map from $\mathbb{R}^{d(d-1)/2}$ into itself. This map is continuous because $\det(1 - M^2) \geq 1$ for any anti-symmetric M (as follows from (2.6)). Moreover, this map is bounded. Indeed, if b_i are the eigenvalues corresponding to the canonical form (2.6) of $\tilde{B}$, then the eigenvalues of $\tilde{\theta}$ are given by $\vartheta_i = b_i/(1 + b_i^2)$. Therefore, the image of the map $\tilde{\theta}(\tilde{B})$ (and hence of $\theta(B)$) is a compact set in the θ-space. This set belongs to a ball of the radius $\frac{1}{2}\sqrt{d}$ (for odd d the radius is $\frac{1}{2}\sqrt{d-1}$ since $\operatorname{rank}\tilde{\theta} \leq d-1$). We remark, however, that for $d \geq 4$ the boundary of the set is not a sphere since the boundary consists of matrices which have canonical forms with eigenvalues $\vartheta_1 = 1/2$, $0 \leq \vartheta_i \leq 1/2$, $i = 2, \ldots, d/2$.

For our purposes, the most important feature of the map (2.11) is that each eigenvalue of $\tilde{\theta}(\tilde{B})$ has a uniform bound, $\vartheta_i \leq 1/2$. It allows us to state that if

$$\tilde{f}_1 < 2 \,, \tag{2.12}$$

where $\tilde{f}_1$ is the maximal eigenvalue of $\tilde{F}_0$, then $\theta(B)$ belongs to Ω_+ for any value of the B-field. To prove this we apply transformation (2.10) to eq. (2.4). Then, provided that (2.12) holds, the canonical form of the r.h.s. of (2.4) has at least one eigenvalue that is greater than 1/2 (addition of a lower rank matrix θ' can change at most $d/2-1$ eigenvalues). Hence the l.h.s. of (2.4) cannot be identified as $\tilde{\theta}(\tilde{B})$, i.e. the image of the map $\theta(B)$ does not intersect Γ. This implies that the image, since it is compact, lies entirely in Ω_+.

If eq. (2.12) is not fulfilled, then the image of the map $\theta(B)$ intersects the event horizon Γ (one can construct B which solves (2.4) along the lines of constructing θ_0 in the appendix A). In this case a more detailed analysis (employing, e.g. (2.7)) is needed to decide whether for given B the value of $\theta(B)$ belongs to Ω_+. For instance, in the $d=4$ case eq. (2.8) rewritten in terms of the variables (2.10) (so that (2.11) holds) looks as follows:

$$1+\vec{e}_{\tilde{B}}\cdot(\vec{e}_{\tilde{B}}+\vec{e}_{\tilde{F}_0})+\vec{h}_{\tilde{B}}\cdot(\vec{h}_{\tilde{B}}+\vec{h}_{\tilde{F}_0})+(\vec{e}_{\tilde{B}}+\vec{e}_{\tilde{F}_0})\cdot(\vec{h}_{\tilde{B}}+\vec{h}_{\tilde{F}_0})\ \mathrm{Pf}\,\tilde{\mathrm{B}}>0\,. \tag{2.13}$$

3. SW map for linear gauge fields

Discussing equivalence of commutative and non-commutative Yang-Mills theories, it is instructive to study also solutions of the SW map (1.3) for the gauge field A_i and investigate, in particular, what singularities they have. However, this task is complicated even if the corresponding curvature is constant. First, the gauge transformations in the non-commutative theory are given by [2]

$$\delta_\lambda A_i=\partial_i\lambda+i[\lambda,A_i]_\theta\,,\qquad \delta_\lambda F_{ij}=i[\lambda,F_{ij}]_\theta\,, \tag{3.1}$$

where λ is the gauge parameter. In the U(1) case a constant curvature F_{ij} is gauge-invariant whereas A_i is not. Furthermore, a solution of the SW map for the gauge field A_i depends on the choice of a deformation path in the θ-space even in the zero curvature case [4, 5].

These technical problems are minimized if we consider a linear gauge field on $\mathbb{R}^d_\theta$:

$$A_i(x)=\alpha_i+a_{ij}\,x^j\,, \tag{3.2}$$

where α_i and a_{ij} do not depend on the coordinates (but, in general, depend on θ). The distinguished feature of such a field is that it *remains linear* under the SW map. This will allow us to solve eq. (1.3) explicitly.

To commence, we note that, like in the ordinary theory, the curvature F_{ij} corresponding to the linear field (3.2) is constant. Therefore, $F(\theta)$ obeys eq. (2.2). On the other hand, computing it according to (1.5), we obtain

$$F=a^t-a+a\,\theta\,a^t\,, \tag{3.3}$$

where again the Lorentz indices are regarded as matrix indices. Then, since both F_{ij} and $\partial_i A_j$ are coordinate independent, the SW map (1.3) for the gauge field acquires the form:

$$\delta\vec{\alpha}=\frac{1}{2}\,(a-F)\,\delta\theta\,\vec{\alpha}\,,\qquad \delta a=\frac{1}{2}\,(a-F)\,\delta\theta\,a\,. \tag{3.4}$$

Our aim now is to solve this system for given initial data, $\vec{\alpha}_0 = \vec{\alpha}(\theta = 0)$ and $a_0 = a(\theta = 0)$. The latter quantity is not entirely arbitrary but, by eq. (3.3), is related to the initial curvature, $a_0^t - a_0 = F_0$.

The first equation in (3.4) does not pose a problem. Indeed, we infer from the system (3.4) that $\delta(a^{-1}\vec{\alpha}) = 0$. Hence we obtain $\vec{\alpha} = a\, a_0^{-1}\, \vec{\alpha}_0$. Further, if $F_0 = 0$, then the second equation in (3.4) is also readily solved,

$$a = \left(1 - \frac{1}{2}\, a_0 \theta\right)^{-1} a_0 \,. \tag{3.5}$$

This formula is consistent with (3.3) (which should vanish) if we take into account that $F_0 = 0$ requires that $a_0 = a_0^t$. Formula (3.5) resembles that for the curvature (2.2). However, a_0 is now a symmetric matrix, so (3.5) has a different type of θ-dependence. For instance, in the $d = 2$ case we have $\theta = \vartheta\, \sigma$ (with σ as in (2.6)). Then (3.5) becomes

$$a(\theta) = \frac{a_0 + \frac{1}{2}\, \vartheta\, \sigma \det a_0}{1 + \frac{1}{4}\, \vartheta^2 \det a_0} \,. \tag{3.6}$$

Let now $F_0 \neq 0$. Introduce a new variable z such that

$$a^{-1} = z - F^{-1} \,. \tag{3.7}$$

Then, taking into account that $\delta\theta = \delta(F^{-1})$ (cf. eq. (2.1)), we rewrite the second equation in (3.4) as follows:

$$2\, \delta z\, F = -z\, \delta F \,. \tag{3.8}$$

It is easy to see from this equation that the result of action on $z(\theta)$ of two infinitesimal shifts $\delta_1\theta$ and $\delta_2\theta$ depends on their order. More precisely,

$$[\delta_2, \delta_1]\, z = \frac{1}{4}\, [F\, \delta_1\theta, F\, \delta_2\theta] \,. \tag{3.9}$$

Therefore, in general, to find $z(\theta)$ we have to choose a path in the θ-space along which we perform the SW transformation. In this context, it is rather an exception (which is in agreement with the computations in [4, 5]) that the solution (3.5) turns out to be path-independent.

Nevertheless, we can extract certain information about $z(\theta)$ from eq. (3.3) which now acquires the following form:

$$z^t\, F_0\, z = -F^{-1} \,. \tag{3.10}$$

In particular, we infer that $(\det z)^2 = (\det F_0 \det F)^{-1}$ vanishes if θ belongs to the event horizon Γ. Since $a^{-1} = (1 + z^t\, F_0)\, z$, we conclude that a is singular everywhere on Γ. This singularity is gauge-invariant as it does not depend on a_0. In addition to this singularity a may have another one if $\det(1 + z^t\, F_0)$ vanishes for some values of θ. This singularity depends on a_0 and is not gauge-invariant.

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To illustrate this discussion on the behaviour of $z(\theta)$, let us present a particular type of solutions to (3.8). Namely, consider only such paths in the θ-space that $[\delta\theta, F_0] = 0$. Assume for simplicity that all eigenvalues of F_0 are different. Then our requirement ensures that $\delta\theta$ and F_0 are brought to the canonical form by the same orthogonal transformation. This in turn implies that $[\delta\theta, \theta] = [F_0, \theta] = 0$. The later equation defines a linear subspace of the θ-space. On this subspace (3.9) vanishes and (3.8) is solved by $z(\theta) = z_0\sqrt{1 + F_0\theta}$, where the square root is defined as a real-valued symmetric matrix. The integration constant z_0 is determined from (3.7). Thus, we obtain a solution of (3.4):

$$\begin{aligned} a^{-1} &= \left(a_0^{-1} + F_0^{-1}\right)\sqrt{1 + F_0\theta} - F_0^{-1} - \theta \\ &= \left(a_0^{-1} + F_0^{-1} - F_0^{-1}\sqrt{1 + F_0\theta}\right)\sqrt{1 + F_0\theta}\,. \end{aligned} \tag{3.11}$$

It is easy to verify (again taking into account that $a_0^t - a_0 = F_0$) that this solution is consistent with (3.3). Notice also that (3.11) turns into (3.5) when F_0 goes to zero.

A linear gauge field corresponding to the solution (3.11) becomes singular everywhere on the hypersurface Γ and besides it diverges if $\det(a_0^{-1} + F_0^{-1}(1 - \sqrt{1 + F_0\theta})) = 0$. Actually, since (3.11) contains the square root of $(1 + F_0\theta)$, this solution makes sense only in the connected component Ω_+. Indeed, when θ passes through the event horizon Γ, some of the eigenvalues of $(1 + F_0\theta)$ become negative, and $\sqrt{1 + F_0\theta}$ cannot be defined appropriately. It is plausible that this picture holds for all solutions of (3.4) since, as seen from (3.8), they unavoidably involve some kind of a square root of F_0F^{-1}. This demonstrates again that the initial (ordinary) Yang-Mills theory can be continuously deformed into a non-commutative theory only within the connected component Ω_+ of the θ-space.

In the $d = 2$ case the θ-space is one dimensional and $\theta = \vartheta\,\sigma$ commutes with $F_0 = f\,\sigma$ (here σ is as in (2.6)). Therefore, in this case (3.11) gives the general solution. In the explicit form, it looks as follows:

$$a(\theta) = \frac{f^2}{\gamma_\theta\beta_\theta}\,a_0 + \frac{f(1-\gamma_\theta)\det a_0}{\gamma_\theta\beta_\theta}\sigma\,, \tag{3.12}$$

where $\gamma_\theta = \sqrt{1 - \vartheta f}$ and $\beta_\theta = f^2\gamma_\theta + (1-\gamma_\theta)^2\det a_0$. For example, if the gauge field in the ordinary Yang-Mills theory has the form $A_1 = -\frac{1}{2}x^2$, $A_2 = \frac{1}{2}x^1$, then it will evolve under the SW map into

$$A_1(\theta) = \frac{-x^2}{(1+\sqrt{1-\vartheta})\sqrt{1-\vartheta}}\,, \qquad A_2(\theta) = \frac{x^1}{(1+\sqrt{1-\vartheta})\sqrt{1-\vartheta}}\,. \tag{3.13}$$

Finally, it is interesting to remark that if $\det a_0 = 0$, then (3.12) simplifies to $a(\theta) = (1 - f\vartheta)^{-1}a_0$, i.e. to the same dependence on θ which the curvature $F(\theta)$ has (cf. (2.2)). However, it is not clear what is an analogue of this observation for higher dimensions d.

4. Conclusion

We have considered solutions to the SW map in the constant curvature case and observed that they diverge when θ reaches certain event horizon Γ which divides the θ-space into two connected components. This implies that an ordinary Yang-Mills theory can be continuously deformed into a non-commutative theory only within the connected component Ω_+ which contains the origin of the θ-space. We have found some sufficient conditions for θ to belong to Ω_+ which involve only the maximal eigenvalue of the initial curvature F_0. These results can be of interest in the context of the deformation quantization approach to gauge theories on non-commutative manifolds. In particular, one can conjecture that there exist conditions on θ ensuring convergence of the SW map in the general case in terms of the maximal eigenvalue of the corresponding curvature $F_0(x)$.

From the string theory viewpoint, our results indicate that equivalence of the ordinary and non-commutative DBI actions holds possibly not for all values of the B-field. Although we studied here only the constant curvature case, we expect that the event horizon for the SW map exists also in a non-constant case (at least, for slow varying fields). A supporting evidence for this is provided by the formula relating $F_0(x)$ and $F(x)$ recently suggested in [6].

In the present paper we have constructed an explicit solution of the SW map for a linear gauge field (apart from formula (2.2) this is, to our knowledge, the first exact solution to the SW map). It possibly can be useful for further development of the perturbative approach to solving the SW map [6, 7]. In particular, our solution for the gauge field and the solution for the curvature can be written as follows:

$$F^{-1} - F_0^{-1} = \theta\,, \qquad a^{-1} + F^{-1} = \left(a_0^{-1} + F_0^{-1}\right)\sqrt{F_0 F^{-1}}\,. \tag{4.1}$$

This form of the solutions exhibits apparent symmetry. Namely, (4.1) remains invariant if we reverse the sign of θ and, at the same time, exchange the ordinary and the non-commutative variables (recall that here F and F_0 commute because of the specific choice of a subspace in the θ-space),

$$\theta \longleftrightarrow -\theta\,, \qquad F \longleftrightarrow F_0\,, \qquad a \longleftrightarrow a_0\,. \tag{4.2}$$

It would be very interesting to see whether solutions of the SW map in more general cases possess similar symmetries.

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A. Distance between origin and horizon

Let us find the distance r between the origin of the θ-space and the hypersurface Γ. That is, we want to find the minimum of the euclidean norm, $\|\theta\| = \sqrt{-\operatorname{tr}\theta^2}$, for θ which solve eq. (2.3). Let an orthogonal transformation with some $o \in \mathrm{O}(d)$ brings F_0 to the canonical form (2.6) with eigenvalues $f_1 \geq f_2 \geq \cdots \geq f_{d/2} > 0$. Then, clearly, $o^t \theta_0\, o$ solves (2.3) if we take $\theta_0 = \operatorname{diag}(f_1^{-1}\sigma, 0, \ldots, 0)$. In fact, $\|\theta_0\| = \sqrt{2} f_1^{-1}$ is *the* minimum which we are seeking. To prove this assertion, we notice that, since $\operatorname{rank}\theta' \leq d-2$ in (2.4), we can apply to this equation such an orthogonal transformation with some $o \in \mathrm{O}(d)$ that $o\,\theta' o^t$ acquires a form where entries outside of the upper-left $(d-2)\times(d-2)$-minor vanish. Then, as we want to minimize $\|\theta\|$, we should put $o\,\theta' o^t$ equal to the upper-left $(d-2)\times(d-2)$-minor of $oF_0^{-1}o^t$. Since the norm $\|..\|$ is invariant with respect to orthogonal transformations, we conclude that $\|\theta\|^2 \geq \|F_0^{-1}\|^2 - \|\hat{F}_0^{-1}\|^2$, where $\hat{F}_0^{-1}$ is any $(d-2)\times(d-2)$-minor of F_0^{-1}. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{\frac{d}{2}-1}$ be the eigenvalues (each of the multiplicity two) of $-\hat{F}_0^{-2}$. Taking into account that $-F_0^{-2}$ is a positive definite matrix, we can apply a theorem from the linear algebra (see, e.g. [8, chapter 10]), stating that λ_i satisfy the relation $f_i^{-2} \leq \lambda_i \leq f_{i+1}^{-2}$. Therefore, $\|\hat{F}_0^{-1}\|^2 \leq 2\sum_{i=2}^{d/2} f_i^{-2} = \|F_0^{-1}\|^2 - 2f_1^{-2}$ and hence we proved for eq. (2.4) that $\|\theta\|^2 \geq 2f_1^{-2}$. Thus, $\|\theta_0\|$ found above is the minimal possible value, i.e. it is the distance r between the origin of the θ-space and Γ.

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