# FUSION OF $q$-TENSOR OPERATORS: QUASI-HOPF-ALGEBRAIC POINT OF VIEW 

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#### Abstract

We discuss the fusion of tensor operators for $U_{q}(\mathcal{J})$ by means of the $R$-matrix approach. The problem is reduced to construction of the twisting element $\mathcal{F}$ which appears in Drinfeld's description of quasi-Hopf algebras. The discussion is illustrated by explicit calculations for the case of $U_{q}(s l(2))$. Bibliography: 20 titles.


## 1. Introduction

## §1.1. Motivations and notation

Originally the theory of tensor operators arose as a result of a group-theoretical approach to quantum mechanics [1]. In turn, the further development of representation theory was inspired by the physical interpretation of its mathematical content. The relatively recent appearance of the theory of quantum groups [2] has led to the development of the theory of $q$-deformed tensor operators [3]. The latter turned out to be not a pure mathematical construction; it is employed, in particular, in the description of the quantum WZW model given in $[4,5]$.

In the present paper, we discuss some aspects of the fusion procedure for (deformed) tensor operators in the $R$-matrix formulation [6]. We consider a special case of the fusion scheme. Namely, given basic tensor operators for two irreducible representations $\rho^{I}$ and $\rho^{J}$, we construct a set of basic tensor operators for the irreducible representation $\rho^{K}$ appearing in the decomposition of $\rho^{I} \otimes \rho^{J}$. This problem is closely related to Drinfeld's construction of quasi-Hopf algebras [7]. Our aim is to obtain exact prescriptions applicable in practice. However, to present a precise statement of the problem, we first need to give a rather detailed introduction to the subject.

We suppose that the reader is familiar with the notion of Hopf algebra. The latter is an associative algebra $\mathcal{G}$ equipped with a unit $\epsilon \in \mathcal{G}$, a homomorphism $\Delta: \mathcal{G} \mapsto \mathcal{G} \otimes \mathcal{G}$ (the co-product), an anti-automorphism $S: \mathcal{G} \mapsto \mathcal{G}$ (the antipode), and a one-dimensional representation $\epsilon: \mathcal{G} \mapsto \mathbb{C}$ (the co-unit) which satisfy a certain set of axioms [8]. A quasitriangular Hopf algebra [2] possesses in addition an invertible element $R \in \mathcal{G} \otimes \mathcal{G}$ (the universal $R$-matrix) satisfying certain relations which, in particular, imply the Yang-Baxter equation. Throughout the paper, we use the so-called $R$-matrix formalism [9, 10]. Recall that its main ingredients are the operator-valued matrices ( $L$-operators)

$$
\begin{equation*}
L_{+}^{I}=\left(\rho^{I} \otimes i d\right) R_{+}, \quad L_{-}^{I}=\left(\rho^{I} \otimes i d\right) R_{-} \tag{1}
\end{equation*}
$$

and the numerical matrices ( $R$-matrices)

$$
\begin{equation*}
R_{+}^{I J}=\left(\rho^{I} \otimes \rho^{J}\right) R_{+}, \quad R_{-}^{I J}\left(\rho^{I} \otimes \rho^{J}\right) R_{-} \tag{2}
\end{equation*}
$$

where $\rho^{I}$ is an irreducible representation of $\mathcal{G}$, and $R_{+}=R, R_{-}=\left(R_{+}^{\prime}\right)^{-1}$. In what follows, ' stands for the permutation in $\mathcal{G} \otimes \mathcal{G}$.

We consider only the case where $\mathcal{G}=U_{q}(\mathcal{J})$ with $|q|=1$, and $\mathcal{J}$ is a semi-simple Lie algebra (the way of generalizing to an arbitrary semisimple modular Hopf algebra $\mathcal{J}$ is described in [11]). For simplicity, we also assume that $q$ is not a root of unity.

We perform all explicit computations only in the case of $U_{q}(s l(2))$, but they can certainly be repeated at least for $U_{q}(s l(n))$. Let us also emphasize that, although we deal with deformed tensor operators and preserve the index $q$ in some formulas, the classical (i.e., nondeformed) theory is recovered in the limit $q=1$ and, therefore, it does not need special comments.

[^0]
## §1.2. (Deformed) tensor operators. Generating matrices

Let a given quasitriangular Hopf algebra $\mathcal{G}$ be the symmetry algebra of some physical model. This means that the operators corresponding to the physical variables in this model are classified with respect to their transformation properties with respect to the adjoint action of $\mathcal{G}$. Recall that if $\mathcal{H}$ is a Hilbert space such that $\mathcal{G} \subset$ End $\mathcal{H}$, then the ( $q$-deformed) adjoint action of an element $\xi \in \mathcal{G}$ on any element $\eta \in \operatorname{End} \mathcal{H}$ is defined as follows [3]:

$$
\begin{equation*}
\left(\operatorname{ad}_{q} \xi\right) \eta=\sum_{k} \xi_{k}^{1} \eta S\left(\xi_{k}^{2}\right) \tag{3}
\end{equation*}
$$

where $\xi_{k}^{a}$ are the components of the co-product $\Delta \xi=\sum_{k} \xi_{k}^{1} \otimes \xi_{k}^{2} \in \mathcal{G} \otimes \mathcal{G}$, and $S(\xi) \in \mathcal{G}$ stands for the antipode of $\xi$.

From the physical point of view, the space $\mathcal{H}$ in (3) is the Hilbert space of the model in question. Since $\mathcal{G}$ is a (quantum) Lie algebra, one often chooses $\mathcal{H}$ as the corresponding model space. The latter is defined as the sum of all irreducible representations taken with multiplicity one, $\mathcal{M}=\oplus_{I} \mathcal{H}_{I}$ ( $I$ runs over all highest weights).

Let $\rho^{J}: \mathcal{G} \mapsto \operatorname{End} V^{J}$ be an irreducible representation of $\mathcal{G}$ with highest weight $J$ and representation space $V^{J}$. A set of operators $\left\{T_{m}^{J}\right\}_{m=1}^{\operatorname{dim} \rho_{\rho}^{J}}$ acting on the Hilbert space $\mathcal{H}$ is called a tensor operator (of highest weight $J$ ) if

$$
\begin{equation*}
\left(\operatorname{ad}_{q} \xi\right) T_{m}^{J}=\sum_{n} T_{n}^{J}\left(\rho^{J}(\xi)\right)_{n m} \quad \text { for all } \quad \xi \in \mathcal{G} . \tag{4}
\end{equation*}
$$

If $\left\{T_{n}^{I}\right\}_{m=1}^{\operatorname{dim}_{o}^{I}}$ and $\left\{T_{m^{\prime}}^{J}\right\}_{m^{\prime}=1}^{\operatorname{dim}_{\rho}^{J}}$ are tensor operators acting on the same Hilbert space, then, using the corresponding (deformed) Clebsch-Gordan coefficients, we can construct the following tensor operator of weight $K^{\prime}[3]:$

$$
T_{n^{\prime \prime}}^{K}=\sum_{m, m^{\prime}}\left\{\begin{array}{ccc}
I & J & K  \tag{5}\\
m & m^{\prime} & m^{\prime \prime}
\end{array}\right\}_{q} T_{m}^{I} T_{m^{\prime}}^{J} .
$$

This formula describes the fusion of tensor operators.
In the case $\mathcal{G}=U_{q}(s l(2))$, the tensor operators are labeled by spin $J$, and definition (4) takes the form

$$
\begin{gather*}
X^{ \pm} T_{m}^{J} q^{H}-q^{H \mp 1} T_{m}^{J} X^{ \pm}=\sqrt{[J \mp m][J \pm m+1]} T_{m \pm 1}^{J},  \tag{6}\\
q^{H} T_{m}^{J} q^{-H}=q^{m} T_{m}^{J},
\end{gather*}
$$

where $[x]=\left(q^{x}-q^{-x}\right) /\left(q-q^{-1}\right)$ denotes the $q$-number, and $X_{ \pm}$and $H$ are the generators satisfying the relations

$$
\left[H, X_{ \pm}\right]= \pm X_{ \pm}, \quad\left[X_{+}, X_{-}\right]=[2 H] .
$$

An example of a (deformed) tensor operator (of spin 1) is provided by the following set of combinations of generators:

$$
\begin{equation*}
T_{1}^{1}=q^{-H} X_{+}, \quad T_{0}^{1}=\left(q^{-1} X_{-} X_{+}-q X_{+} X_{-}\right) / \sqrt{[2]}, \quad T_{-1}^{1}=-q^{-H} X_{-} . \tag{7}
\end{equation*}
$$

However, this is a rather special case because, generally speaking, the components of tensor operators act on the model space as shifts between different subspaces $\mathcal{H}^{I}$, whereas $\mathcal{H}$ are invariant subspaces for the components in (7).

Let us note that, along with the tensor operators of covariant type introduced in (4), one can define a contravariant tensor operator as a set of operators $\left\{\bar{T}_{m}^{J}\right\}_{m=1}^{\operatorname{dim}_{o}^{J}}$ satisfying the relations

$$
\begin{equation*}
\left(\operatorname{ad}_{q} \xi\right) \bar{T}_{m}^{J}=\sum_{n}\left(\rho^{J}(S(\xi))\right)_{m n} \bar{T}_{n}^{J} \quad \text { for all } \quad \xi \in \mathcal{G} . \tag{8}
\end{equation*}
$$

Further we consider only the covariant case since the theory and computations for the contravariant case are quite similar.

In the case of a quasitriangular Hopf algebra, we can describe tensor operators by means of the $R$-matrix language. Let $\rho^{J}$ be an irreducible representation of $\mathcal{G}$ with highest weight $J$ and representation space $V^{J}$. Consider the matrix $U^{J} \in \operatorname{End} V^{J} \otimes$ End $\mathcal{H}$ satisfying the following $R$-matrix relations:

$$
\begin{equation*}
\stackrel{1}{L}_{ \pm}^{I} \stackrel{2}{U}^{J}=\stackrel{2}{U^{J}} R_{ \pm}^{I J} \stackrel{1}{L}_{ \pm}^{I} \tag{9}
\end{equation*}
$$

where $L_{ \pm}^{J}$ and $R_{ \pm}^{I J}$ are defined as in (1)-(2). Equations (9) are equivalent to the statement that each row of $U^{J}$ satisfies (4), i.e., all rows of $U^{J}$ are tensor operators of weight $J[6]$. We shall refer to $U^{J}$ as generating matrices because, according to the Wigner-Eckart theorem, matrix elements of entries of $U^{J}$ (evaluated, say, on the corresponding model space) give the ( $q$-deformed) Clebsch-Gordan coefficients.

Note that $U^{J}$ in (9) can have an arbitrary number of rows which are not necessarily linearly independent tensor operators. However, from now on we shall work with the case of square generating matrices. Moreover, for $\mathcal{G}=U_{q}(s l(n))$ (and very likely even for $U_{q}(\mathcal{J})$ with any semisimple $C \mathcal{J}$ ) one can construct square generating matrices whose rows are linearly independent tensor operators [4,5,12]. Notice also that if $U^{J}$ satisfies (9) and $M$ is a matrix with entries commuting with all elements of $\mathcal{G}$, then

$$
\begin{equation*}
\widetilde{U}^{J}=M U^{J} \tag{10}
\end{equation*}
$$

also satisfies (9), i.e., $\tilde{U}^{J}$ also is a generating matrix.
Now let $U^{I}$ and $U^{J}$ satisfy (9). The analog of the fusion formula (5) for generating matrices reads [6]

$$
\begin{equation*}
U_{K}^{I J}=P_{K}^{I J} F^{I J} \stackrel{2}{U}^{J} \tilde{U}^{I} P_{K}^{I J} \in V^{I} \otimes V^{J} \otimes \operatorname{End} \mathcal{H}, \tag{11}
\end{equation*}
$$

where the left-hand side is a new generating matrix of weight $K$ written in the basis of $V^{I} \otimes V^{J}$. Here $F^{I J} \in V^{I} \otimes V^{J}$ stands for an arbitrary matrix whose entries commute with all elements of $\mathcal{G}$ and $P_{K}^{I J}$ denotes a projector (i.e., $\left.\left(P_{K}^{I J}\right)^{2}=P_{K}^{I J}\right)$ onto the subspace in $V^{I} \otimes V^{J}$ corresponding to the representation $\rho^{K}$.

One can rewrite (11) in the standard basis of the space $V^{K}$ :

$$
\begin{equation*}
U_{m n}^{K}=\epsilon_{m}^{t} U_{K}^{I J} \epsilon_{n}, \quad m, n=1, \ldots, \operatorname{dim} \rho^{K} . \tag{12}
\end{equation*}
$$

Here $\left\{\epsilon_{n}\right\}$ is an orthonormal set of the eigenvectors of the projector $P_{K}^{I J}$, i.e., $P_{K}^{I J}=\sum \epsilon_{n} \otimes \epsilon_{n}^{t}$ and $\epsilon_{m}^{t} \epsilon_{n}=$ $\delta_{m n}$.

Formula (11) resembles the fusion formula for $R$-matrices [10]:

$$
\begin{equation*}
\stackrel{1.32}{R}_{ \pm}^{L K}=\stackrel{23}{P_{K}^{I J}}{ }^{13} R_{ \pm}^{L J}{ }^{12} R_{ \pm}^{L I} P_{K}^{23}{ }_{K}^{I J} . \tag{13}
\end{equation*}
$$

Here we use the notation from [10], and on the left-hand side of (13) we have $R_{ \pm}^{L K}$ written in the basis of $V^{L} \otimes V^{I} \otimes V^{J}$. Of course, the origin of both (11) and (13) is the Hopf structure of $\mathcal{G}$.

The fusion formulas given above are of direct practical use since they allow one to construct the corresponding objects (generating matrices and $R$-matrices) for higher representations, starting with those for the fundamental irreducible representations. For later use, we rewrite (12)-(13) as follows:

$$
\begin{align*}
U^{K} & =C[I J K] F^{I J} U^{J}{ }^{1} U^{I} C^{\prime}[I J K]  \tag{14}\\
R_{ \pm}^{L K} & =\stackrel{23}{C}[I J K] \stackrel{13}{R_{ \pm}^{L J}} \stackrel{12}{R}_{ \pm}^{L I} \stackrel{23}{C}[I J K] . \tag{15}
\end{align*}
$$

Here we used the so-called Clebsch-Gordan maps $C[I J K]: V^{I} \otimes V^{J} \mapsto V^{K}$ and $C^{\prime}[I J K]: V^{K} \mapsto V^{I} \otimes V^{J}$, which are constructed according to the rules

$$
\begin{equation*}
C[I J K]=\sum_{n=1}^{\operatorname{dim} \rho^{K}} \widehat{\epsilon}_{n} \otimes \epsilon_{n}^{\prime}, \quad C^{\prime}[I J K]=\sum_{n=1}^{\operatorname{dim} \rho^{K}} \epsilon_{n} \otimes \hat{\epsilon}_{n}^{t}, \tag{16}
\end{equation*}
$$

where $\widehat{\epsilon}_{n}$ is the vector $\epsilon_{n}$ rewritten in the basis of the space $V^{K}$. The main properties of the $C G$ maps are as follows:

$$
\begin{align*}
& C[I J K]\left(\rho^{I} \otimes \rho^{J}\right) \Delta(\xi) C^{\prime}[I J K]=\rho^{K}(\xi) \quad \text { for any } \quad \xi \in \mathcal{G},  \tag{17}\\
& \sum_{K^{\prime}} C^{\prime}[I J K] C[L M K]=\delta_{I L} \delta_{J M}, \quad C D[I J K] C[I J L]=\delta_{K L} . \tag{18}
\end{align*}
$$

## §1.3. Exact generating matrices

Let $\mathcal{C}$ denote a commutative *-algebra of functions on the weight space of $U_{q}(\mathcal{J})$. It is convenient to parametrize the coordinate on $\mathcal{C}$ by the vector $\vec{p}=2 J+\rho$, where $J$ runs over all highest weights, and $\rho$ is the sum of the simple roots of $C J$. Thus, $\mathcal{C}$ is an algebra of functions depending on the "variable" $p_{i}$-components of $\vec{p}$.

Let $\mathcal{J}$ be a Lie algebra of rank $n$, and let $|q|=1$. Generating matrices for $\mathcal{G}=U_{q}(\mathcal{J})$ can be constructed by the following method $[4,5,11,12]$. Define $D=q^{2 \vec{H} \otimes \vec{p}} \in \mathcal{G} \otimes \mathcal{C}$ and $\Omega=q^{4 \vec{H} \otimes \vec{H}} \in \mathcal{G} \otimes \mathcal{G}$, where $\vec{A} \otimes \vec{B}$ is understood as $\sum_{i=1}^{n} A_{i} \otimes B_{i}$, and $H_{i}$ are the basic generators of the Cartan subalgebra of $\mathcal{G}$. Next, introduce the map

$$
\begin{equation*}
\sigma: \mathcal{C} \mapsto \otimes \mathcal{C}, \quad \sigma(\vec{p})=\epsilon \otimes \vec{p}+2 \vec{H} \otimes \epsilon \tag{19}
\end{equation*}
$$

Let $\mathcal{R}_{ \pm}(\vec{p}) \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{C}$ be solutions (related, as usual, by $\left.\mathcal{R}_{-}(\vec{p})=\left(\mathcal{R}_{+}^{\prime}(\vec{p})\right)^{-1}\right)$ of the equations

$$
\begin{align*}
& \stackrel{12}{\mathcal{R}}_{ \pm}(\vec{p}) \stackrel{13}{R_{ \pm}}\left(\vec{p}_{2}\right) \stackrel{23}{\mathcal{R}}_{ \pm}(\vec{p})=\stackrel{23}{\mathcal{R}}_{ \pm}\left(\vec{p}_{1}\right) \stackrel{13}{\mathcal{R}}_{ \pm}(\vec{p}) \stackrel{12}{\mathcal{R}}_{ \pm}\left(\vec{p}_{3}\right),  \tag{20}\\
& {\left[\mathcal{R}_{ \pm}(\vec{p}), q^{H_{i}} \otimes q^{H_{i}}\right]=0 \text { for all } i,}  \tag{21}\\
& \mathcal{R}_{-}(\vec{p}) \stackrel{2}{D}=\stackrel{2}{D} \mathcal{R}_{+}(\vec{p}),  \tag{22}\\
& \mathcal{R}_{ \pm}^{*}(\vec{p})=\mathcal{R}_{ \pm}^{-1}(\vec{p}) . \tag{23}
\end{align*}
$$

The subscripts in (20) denote the shifts of the corresponding arguments ${ }^{1}$ according to (19). It is easy to verify that the unitary property ${ }^{2}(23)$ of $\mathcal{R}_{ \pm}(\vec{p})$ is consistent with (20)-(22) for $|q|=1$.

In general, for a given $\mathcal{J}$, there exists a family of solutions of Eq. (20). However, in the case $\mathcal{J}=s l(n)$, additional conditions (21)-(23) fix the solution uniquely [5, 13], and it is very likely that the same picture holds for any (deformed) semisimple Lie algebra $\mathcal{J}$. Moreover, such a solution has the following remarkable property $[4,5,20]$ : the entries of $\mathcal{R}_{ \pm}^{I J}(\vec{p})$ are nothing but the corresponding (deformed) $6 j$-symbols (involving the weights $I, J$, and all $K$ admissible by the triangle inequality).

Now we consider the element $U \in \mathcal{G} \otimes$ End $\mathcal{H}$ which satisfies the equations

$$
\begin{gather*}
\mathcal{R}_{ \pm}(\vec{p}) \stackrel{2}{U} \stackrel{1}{U}=\stackrel{1}{U}{ }^{2} R_{ \pm}  \tag{24}\\
\stackrel{1}{U} \stackrel{2}{D}=\Omega \stackrel{2}{D} \stackrel{1}{U},  \tag{25}\\
U^{-1} D U=q^{2 C(\vec{p})} L_{+} L_{-}^{-1}, \tag{26}
\end{gather*}
$$

where $C(\vec{p})=\frac{1}{4}(\vec{p}+\rho)(\vec{p}-\rho)$ is the Casimir element. It can be shown $[4,5,11,12]$ that such a $U$ (if it exists) is a generating matrix ${ }^{3}$ for $\mathcal{G}$. Note that (20)-(21) are nothing but consistency conditions for (24) and (25), and (21) are the same symmetry conditions which are known ${ }^{4}$ for the standard $R$-matrices of $U_{q}(\mathcal{J})$. Relation (25) is a matrix form of the equation

$$
\begin{equation*}
U \vec{p}=\sigma(\vec{p}) U . \tag{27}
\end{equation*}
$$

Equation (26) is a kind of normalization condition.

[^1]Notice that, from the group of transformations (10), only the transformations

$$
\begin{equation*}
U \mapsto D^{\alpha} U, \quad \alpha \in \mathbb{C}, \tag{28}
\end{equation*}
$$

survive for the solution $U$ of Eqs. (24)-(26). The validity of (28) can easily be checked with the help of (21) and (22). Furthrmore, it is easy to verify that the rescaling

$$
\begin{equation*}
U \mapsto(\epsilon \otimes f(\vec{p})) U\left(\epsilon \otimes f^{-1}(\vec{p})\right) \tag{29}
\end{equation*}
$$

with an arbitrary function $f(\vec{p}) \in \mathcal{C}$ is allowed.
Let us explain why the generating matrix satisfying (24)-(27) is of special interest from the viewpoint of the theory of tensor operators. Notice that property (27) ensures that if such a $U$ exists, then its rows are linearly independent tensor operators. In other words, if for a given irreducible representation $\rho^{I}$ and a given vector $|I, m\rangle$ in the model space $\mathcal{M}$ of $\mathcal{G}$, we consider a set of vectors $U_{i j}^{J}|I, m\rangle i, j=1, \ldots, \operatorname{dim} \rho^{J}$, then all nonvanishing vectors in this set are pairwise linearly independent. In particular, if $J$ is a fundamental irreducible representation, then the entries of $U^{J}$ provide a set of basic shifts on $\mathcal{M}$. Thus, the solution of (24)-(26) presents a very special but, in fact, the most interesting case of the generating matrix. We shall refer to it as the exact generating matrix.

It is worth mentioning that the matrix elements $\left\langle K, m^{\prime \prime}\right| U_{i j}^{J}\left|I, m^{\prime}\right\rangle$ coincide with the Clebsch-Gordan coefficients $\left\{\begin{array}{ccc}I & J & K \\ m & m^{\prime} & m^{\prime \prime}\end{array}\right\}_{q}$ (up to some $p_{i}$-dependent factors allowed by (28) and (29)) which appear in the decomposition of the tensor product $\rho^{I} \otimes \rho^{J}$ (with the weights $I, J$, and $K$ restricted by the triangle inequality). This property of exact generating matrices makes them especially important from the practical point of view.

As for the physical content of the relations given above, Eq. (20) has appeared in various forms in studies of quantum versions of the Liouville [14], Toda [16], and Calogero-Moser [17] models. In these models, $\mathcal{R}(\vec{p})$ is regarded as a dynamical $R$-matrix. From the viewpoint of the theory of tensor operators, relations (19)(27) are most closely connected with the quantization of the WZW model $[4,5,11]$. Here $\mathcal{R}(9 \vec{p})$ plays the role of the braiding matrix, and Eqs. (24)-(26) with appropriate dependence on the spatial coordinate (or its discretized version) describe the vertex operators. Let us mention that, in the WZW theory, the quantum-group parameter of $\mathcal{G}=U_{q}(\mathcal{J})$ is given by $q=\epsilon^{i \gamma \hbar}$, where $\hbar>0$ is the Planck constant, and the deformation parameter $\gamma>0$ is interpreted as a coupling constant. That is why we consider the case $|q|=1$.

## II. Fusion of exact generating matrices

## §2.1. FORMULATION OF THE PROBLEM

Suppose we are given generating matrices $U^{I}$ and $U^{J}$ for some irreducible representations of $\mathcal{G}$. Then, using relation (11), we can build up the generating matrices $U^{K}$ for every irreducible representation $p^{K}$ which appears in the decomposition of $\rho^{I} \otimes \rho^{J}$. For brevity, we shall call them descendant matrices. As explained above, it is natural to deal not with all possible generating matrices but only with the exact ones, i.e., with those satisfying additional equations (24)-(27) with $\mathcal{R}_{ \pm}(\vec{p}), D$, and $\Omega$ introduced above. Here we face the problem of finding a matrix $F^{I J}$ such that the descendant matrix $U^{K}$ obtained by formula (11) is also an exact generating matrix.

Let us point out that this problem would not arise if the left-hand side of Eq. (24) contained the standard $R$-matrix instead of $\mathcal{R}(\vec{p})$. Indeed, for an operator-valued matrix $g^{J} \in \operatorname{End} V^{J} \otimes \mathcal{G}$ (it may be regarded as an $L$-operator type object ${ }^{5}$ ) which satisfies the usual quadratic relations ${ }^{6}$

$$
\begin{equation*}
R^{I J}{ }_{g}^{2}{ }_{g}^{1}{ }_{g}^{I}={ }_{g}^{1} I_{g}^{2}{ }^{J} R^{I J}, \tag{30}
\end{equation*}
$$

[^2]the fusion formula is well known (Eq. (13) is its specific realization):
\[

$$
\begin{equation*}
\left(g^{K}\right)_{m n}=\epsilon_{m}^{t} \stackrel{2}{g}^{J} \stackrel{1}{g}^{I} \epsilon_{n} \tag{31}
\end{equation*}
$$

\]

where, as before, $\epsilon_{n}, n=1, \ldots, \operatorname{dim} \rho^{K}$, are the eigenvectors of the projector $P_{K}^{I J}$. For example, in the case $\mathcal{G}=U_{q}(s l(2))$, we start with $g^{\frac{1}{2}}$ and, applying (31) sufficiently many times, obtain the matrix $g^{J}$ for any $\operatorname{spin} J$ :

$$
\begin{gather*}
g^{0}=1, g^{\frac{1}{2}}=\left(\begin{array}{cc}
a & b \\
c & d
\end{array}\right) \\
g^{1}=\left(\begin{array}{ccc}
a^{2} & q^{-\frac{1}{2}} \sqrt{[2]} a b & b^{2} \\
q^{-\frac{1}{2}} \sqrt{[2]} a c & a d+q^{-1} b c & q^{-\frac{1}{2}} \sqrt{[2]} b d \\
c^{2} & q^{-\frac{1}{2}} \sqrt{[2]} c d & d^{2}
\end{array}\right), \ldots \tag{32}
\end{gather*}
$$

For generating matrices, the fusion problem is more complicated because $\mathcal{R}(\vec{p})$ in Eqs. (20) and (24) is an attribute not of a Hopf algebra but of a quasi-Hopf algebra. In this section, we discuss some general aspects of the fusion problem in the quasi-Hopf case. In the next section, we consider, as an example, the case $U_{q}(s l(2))$.

It should also be noted that the fusion problem (in the form stated above) does not appear if one uses the language of universal objects (see, e.g., $[19,11]$ ) instead of the language of operator-valued matrices. For example, instead of the set of matrices $g^{J} \in$ End $V^{J} \otimes \mathcal{G}$ satisfying (30), we can introduce the element $g \in \mathcal{G} \otimes \mathcal{G}$ and fix its functoriality relation as follows:

$$
\begin{equation*}
(\Delta \otimes i d)(g)=\stackrel{21}{g} g \tag{33}
\end{equation*}
$$

Then both quadratic relations (30) and fusion formula (31) can be obtained from (33) with the help of the axioms of the quasi-triangular Hopf algebra. In fact, in this approach we do not even need the fusion formula because each $g^{J}$ can simply be obtained by evaluation of $g$ ina fixed representation: $g^{J}=\left(\rho^{J} \otimes i d\right) g$.

Similarly, we can introduce the universal object $U \in \mathcal{G} \otimes$ End $\mathcal{H}$ with the functoriality relation [11]

$$
\begin{equation*}
(\Delta \otimes i d)(U)=\mathcal{F} \stackrel{2}{U} \stackrel{1}{U} \tag{34}
\end{equation*}
$$

where $\mathcal{F}$ satisfies certain axioms. Then quadratic relations (24) (with $\mathcal{R}(\vec{p})$ constructed from $\mathcal{F}$ and $R$ according to (41)) are consequences of (34). Again, fixing a representation of the $\mathcal{G}$-part of the universal element $U$, we obtain a generating matrix $U^{J}=\left(\rho^{J} \otimes i d\right) U$ and, therefore, we do not need the fusion formula.

Although the language of universal objects is more convenient in abstract theoretical constructions, in practice we usually do not have explicit formulas for the universal objects involved (for instance, the universal $R$-matrices are known only for $U_{q}(s l(2))$ and $U_{q}(s l(3))$. Therefore, in the present paper, we intentionally adopted the matrix language to discuss how to construct exact generating matrices for different representations from those of given representations if explicit universal formulas are unknown.

## §2.2. QUASI-HOPF FEATURES

Let us recall that an associative algebra $\mathcal{G}$ is said to be a quasi-Hopf algebra [7] if its co-multiplication is "quasicoassociative," i.e., if, for all $\xi \in \mathcal{G}$, the following relations hold:

$$
\begin{gather*}
((i d \otimes \Delta) \Delta(\xi)) \Phi=\Phi((\Delta \otimes i d) \Delta(\xi))  \tag{35}\\
(\epsilon \otimes i d) \Delta(\xi)=(i d \otimes \epsilon)=(i d \otimes \epsilon) \Delta(\xi)=\xi
\end{gather*}
$$

Here $\Phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$ is an invertible element (co-associator) which must satisfy certain equations. For the definition of a quasitriangular quasi-Hopf algebra, one postulates, in addition, the existence of an invertible $\mathcal{R} \in \mathcal{G} \otimes \mathcal{G}$ (twisted $R$-matrix) such that

$$
\begin{equation*}
\mathcal{R} \Delta(\xi)=\Delta^{\prime}(\xi) \mathcal{R} \quad \text { for all } \quad \xi \in \mathcal{G} \tag{36}
\end{equation*}
$$

$$
\begin{gather*}
(\Delta \otimes i d) \mathcal{R}=\Phi_{312} \stackrel{13}{\mathcal{R}} \Phi_{132}^{-1} \stackrel{23}{\mathcal{R}} \Phi_{123}, \quad(i d \otimes \Delta) \mathcal{R}=\Phi_{231}^{-1} \stackrel{13}{\mathcal{R}} \Phi_{213} \stackrel{12}{\mathcal{R}} \Phi_{123}^{-1},  \tag{37}\\
(\epsilon \otimes i d) \mathcal{R}=(i d \otimes \epsilon) \mathcal{R}=\epsilon . \tag{38}
\end{gather*}
$$

An analog of the Yang-Baxter equation for $\mathcal{R}$ follows from (36) and (37) and appears as

$$
\begin{equation*}
{ }^{12} \Phi_{312}{\stackrel{13}{\mathcal{R}} \Phi_{132}^{-1}}_{\stackrel{23}{\mathcal{R}} \Phi_{123}=\Phi_{321}{ }^{23} \mathcal{R} \Phi_{231}^{-1} \stackrel{13}{\mathcal{R}} \Phi_{213} \stackrel{12}{\mathcal{R}} . . .} \tag{39}
\end{equation*}
$$

A crucial observation $[5,11,18]$ is that the construction used in $\S \S 2$ and 3 for the description of exact generating matrices involves the quasi-Hopf algebra (where $\mathcal{R}_{+}(\vec{p})$ plays the role of the element $\mathcal{R}$ ) which is obtained as a twist of the quasitriangular Hopf algebra $\mathcal{G}$. More precisely, there exists an invertible element $\mathcal{F}(\vec{p}) \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$ whereby one can construct the objects below (which satisfy the axioms of the quasi-Hopf algebra)

$$
\begin{gather*}
\Delta_{\mathcal{F}}(\cdot)=\mathcal{F}^{-1}(\vec{p}) \Delta(\cdot) \mathcal{F}(\vec{p}),  \tag{40}\\
\mathcal{R}_{ \pm}(\vec{p})=\left(\mathcal{F}^{\prime}(\vec{p})\right)^{-1} R_{ \pm} \mathcal{F}(\vec{p}),  \tag{41}\\
\Phi(\vec{p})_{123}=\mathcal{F}^{12}\left(\vec{p}_{3}\right) \stackrel{12}{\mathcal{F}}(\vec{p}) \tag{42}
\end{gather*}
$$

from the standard comultiplication and $R$-matrices (which satisfy the axioms of the Hopf-algebra) ${ }^{7}$. In particular, (20) provides a realization of the abstract form (39) of the twisted Yang-Baxter equation.

The fact that $\mathcal{R}_{ \pm}(\vec{p})$ introduced in (20)-(22) admit decompositions of type (41) is very important in the context of the fusion problem for exact generating matrices. Indeed, suppose we are given two exact generating matrices, $U^{I}$ and $U^{J}$, which satisfy (24)-(27) with certain $\mathcal{R}_{ \pm}^{I J}(\vec{p})$. Applying formula (11) with some matrix $F^{I J}(\vec{p})$ (it can be $\mathcal{C}$-valued) to these $U^{I}$ and $U^{J}$, we obtain a new matrix $U_{K}^{I J}$ which automatically satisfies (9). Moreover, it is easy to verify that the exchange relations between $U_{K}^{I J}$ and any exact generating matrix $U^{L}$ are again of the form (24) but with a new $R$-matrix on the left-hand side which in the basis of $V^{L} \otimes V^{I} \otimes V^{J}$ looks like

$$
\begin{equation*}
\stackrel{1,32}{\mathcal{R}}_{ \pm}^{L K}(\vec{p})=\stackrel{23}{P I J}_{P^{I J}} \vec{p}_{1} \stackrel{R}{\mathcal{R}}_{ \pm}^{L J}(\vec{p})^{12} \mathcal{R}_{ \pm}^{L I}\left(\vec{p}_{3}\right)\left({ }^{23}{ }^{I J}(\vec{p})\right)^{-1}{ }^{23} P_{K}^{I J} . \tag{43}
\end{equation*}
$$

This is an analog of fusion formula (13) for the standard $R$-matrices. Since the new generating matrix $U_{K}^{I J}$ is exact, and, in particular, satisfies (24), we see that expression (43) rewritten in the basis of $V^{I} \otimes V^{J}$ coincides with $\mathcal{R}_{ \pm}^{L K}(\vec{p})$. Taking into account that $\mathcal{R}_{ \pm}(\vec{p})$ satisfies (41) for some matrix $\mathcal{F}$, we obtain the equation

$$
\begin{align*}
& \mathcal{R}_{ \pm}^{I J}(\vec{p}) \\
&=\stackrel{23}{C}[I J K]{ }^{23} F^{I J}\left(\vec{p}_{1}\right) \stackrel{13}{\mathcal{R}}_{ \pm}^{L J}(\vec{p})\left(\stackrel{\mathcal{F}}{ }_{I L}^{I L}\right)^{-1}\left(\vec{p}_{3}\right) \stackrel{21}{\mathcal{F}}^{I L}(\vec{p}) \stackrel{12}{\mathcal{R}}_{ \pm}^{L I}(\vec{p})\left(\stackrel{12}{\mathcal{F}}^{L I}\right)^{-1} \stackrel{12}{\mathcal{F}}^{L I}\left(\vec{p}_{3}\right)\left(\stackrel{23}{ }^{I J}\right)^{-1}(\vec{p}) C^{23}[I J K] . \tag{44}
\end{align*}
$$

The latter is equivalent (due to (17) and (40)) to the identity

$$
\begin{aligned}
\left(\rho^{L} \otimes \rho^{I} \otimes \rho^{J}\right)\left(i d \otimes \Delta_{\mathcal{F}}\right) \mathcal{R} & \\
& \left.=\left(F^{23}(\vec{p})\right)^{-1} \stackrel{F}{F}^{I J}\left(\vec{p}_{1}\right)\right)_{ \pm}^{13}(\vec{p})\left(\mathcal{F}^{I L}\right)^{-1}\left(\vec{p}_{3}\right) \stackrel{\mathcal{F}}{ }_{21}^{I L}(\vec{p}) \mathcal{R}_{ \pm}^{L I}(\vec{p})\left(\mathcal{F}^{L I}\right)^{-1}(\vec{p}) \mathcal{F}^{L I}\left(\vec{p}_{3}\right),
\end{aligned}
$$

which, as we see from (37) and (42), takes place only if $F^{I J}=\mathcal{F}^{I J}(\vec{p})$.

[^3]
## §2.3. Properties of the twisting element

The practical summary of the previous section is as follows. If we are given exact generating matrices $U^{I}$ and $U^{J}$ (and, hence, we know $\mathcal{R}_{ \pm}^{I J}(\vec{p})$ ), then to construct a new exact matrix $U^{K}$, we must substitute the matrix $F^{I J}=\left(\rho^{I} \otimes \rho^{J}\right) \mathcal{F}(\vec{p})$, where $\mathcal{F}(\vec{p})$ is the twisting element of the quasi-Hopf algebra introduced above, in the fusion formula (11). An obstacle to the application of this prescription is that usually an explicit universal expression for $\mathcal{F}(\vec{p})$ is unknown. However, assuming that such an $\mathcal{F}(\vec{p})$ exists, we can look for $\mathcal{F}^{I J}(\vec{p})$ as a matrix satisfying the following conditions:

1. $\mathcal{F}^{I J}(\vec{p})$ is a solution of Eq. (41) for given $\mathcal{R}_{ \pm}^{I J}(\vec{p})$, which can be rewritten in the following form, which is more convenient in practice:

$$
\begin{equation*}
R_{ \pm}^{I J} \mathcal{F}^{I J}(\vec{p})=\left(\mathcal{F}^{I J}(\vec{p})\right)^{\prime} \mathcal{R}_{ \pm}^{I J}(\vec{p}) ; \tag{45}
\end{equation*}
$$

2. $\mathcal{F}^{I J}(\vec{p})$ satisfies the symmetry condition

$$
\begin{equation*}
\left[\mathcal{F}^{I J}(\vec{p}), q^{H_{i}^{I}} \otimes q^{H_{i}^{J}}\right]=0 \quad \text { for all } \quad i ; \tag{46}
\end{equation*}
$$

3. $\mathcal{F}^{I J(\vec{p})}$ is such that, for any weight $M$, all entries of the matrix ${ }^{8}$

$$
\begin{equation*}
U_{0}^{I J}=P_{0}^{I J} \mathcal{F}^{I J}(\vec{p}) \stackrel{2}{U^{J}} \stackrel{1}{U}^{I} P_{0}^{I J} \tag{47}
\end{equation*}
$$

or, equivalently, of the matrix

$$
\begin{equation*}
U^{0}=C[I J 0] \mathcal{F}^{I J}(\vec{p}) \stackrel{2}{U^{J}}{ }^{1} U^{I} C^{\prime}[I J 0], \tag{48}
\end{equation*}
$$

commute with all entries of the generating matrix $U^{M}$;
4. $\mathcal{F}^{I J}(\vec{p})\left(\mathcal{F}^{I J}(\vec{p})\right)^{*}$ is a $p$-independent object, or, in other words, $\mathcal{F}^{I J}(\vec{p})\left(\mathcal{F}^{I J}(\vec{p})^{*}=\chi\right.$, where $\chi$ is an element of $\mathcal{G} \otimes \mathcal{G}$.

Let us comment on these conditions. The necessity of the first of them was explained in the previous section. Since, in general, Eq. (45) possesses a family of solutions, the above condition is not sufficient. In principle, we could separate the right solution in this family verifying whether the substitution of this solution in fusion formulas (43) or (44) yields matrices $\mathcal{R}(\vec{p})$ satisfying (20)-(23). However, such a verification would be quite tedious in practice.

The second condition ensures that the descendant matrix $U^{K}$ satisfies (27), and, consequently, (25). This can easily be checked by applying (27) to (11). Notice also that (46) implies that, for this specific quasi-Hopf algebra, the comultiplication on the Cartan subalgebra is not deformed and, hence, it is the same as for $U_{q}(\mathcal{J})$ and $C J$.

The third condition follows from (38) and the same property $(\epsilon \otimes i d) \mathcal{R}=(i d \otimes \epsilon) \mathcal{R}=\epsilon$ known for the standard $R$-matrices (recall that $\epsilon$ is the trivial one-dimensional representation of $\mathcal{G}$ ). Indeed, applying $(\epsilon \otimes i d)$ or $(i d \otimes \epsilon)$ to (24), we conclude that $U^{0}=(\epsilon \otimes i d) U$ commutes with all entries of $U^{J}$ for any $J$. Therefore, if the trivial representation $\rho^{0} \equiv \epsilon$ appears in the decomposition of the product $\rho^{I} \otimes \rho^{J}$ (e.g., in the case where both irreducible representations coincide with the fundamental one), then the left-hand side of Eqs. (47) and (48) do not vanish and represent a tensor operator of zero weight (i.e., a scalar) with respect to the adjoint action of $\mathcal{G}$. In this case, the third condition is nontrivial because a tensor operator of zero weight can be $p$-dependent and, hence, in general, this operator does not commute with the other tensor operators.

To clarify condition 4 , we first recall that, for $|q|=1$, the standard comultiplication has the following property with respect to the conjugation in $\mathcal{G}: \Delta^{*}(\xi)=\Delta^{\prime}\left(\xi^{*}\right)$. On the other hand, relations (23) imply the equation (see also [11])

$$
\begin{equation*}
\left(\Delta_{\mathcal{F}}(\xi)\right)^{*}=\Delta_{\mathcal{F}}\left(\xi^{*}\right) . \tag{49}
\end{equation*}
$$

[^4]Hence, the self-conjugate element $\chi=\mathcal{F}(\vec{p}) \mathcal{F}^{*}(\vec{p})$ satisfies the relation $\chi \Delta=\Delta^{\prime} \chi$. Moreover, we obtain from (23) that

$$
\begin{equation*}
R_{ \pm} \chi(\vec{p})=(\chi(\vec{p}))^{\prime} R_{\mp}^{-1} \tag{50}
\end{equation*}
$$

Next, we notice that, from (23), (37), and (49), the following unitary property of the coassociator follows:

$$
\begin{equation*}
\Phi^{*}(\vec{p})_{123}=\Phi^{-1}(\vec{p})_{123} \tag{51}
\end{equation*}
$$

According to (42), the latter equation leads to the condition $\chi^{12}(\vec{p})={ }_{\chi}^{12}\left(\vec{p}_{3}\right)$, which, due to the possibility of applying (27) to (51) arbitrarily many times, implies the $p$-independence of $\chi$. It should be mentioned that the universal expression for the element $\chi \in \mathcal{G} \otimes \mathcal{G}$ was found in [11].

## III. The case of $U_{q}(s l(2))$

In this section, we illustrate the preceding discussion by some explicit calculations. Although solutions of the twisted Yang-Baxter equation (20) are known [5, 13, 14, 16] for the fundamental representations of $\mathcal{G}=U_{q}(s l(n))$, here we consider only the case of $U_{q}(s l(2))$. However, we emphasize that, in the more general case of $U_{q}(s l(n))$, the computations are essentially the same.

## §3.1. $\mathcal{R}(\vec{p})$ AND $U$ IN THE FUNDAMENTAL REPRESENTATION

As mentioned above, the twisted Yang-Baxter equation possesses a family of solutions. This takes place even in the simplest case of the fundamental representation of $U_{q}(s l(2))$. Imposing additional conditions (21)-(23), we obtain a unique solution $[5,13]$ which depends on the single variable $p=2 J+1$, where $J$ is the spin. In particular, the fundamental matrices $\mathcal{R}_{ \pm}(p)$ (i.e., with both auxiliary spaces of spin $1 / 2$ ) are given by (all nonspecified entries are zeros)

$$
\mathcal{R}_{+}^{\frac{1}{2} \frac{1}{2}}=P\left(\mathcal{R}_{-}^{\frac{1}{2} \frac{1}{2}}\right)^{-1} P=q^{-\frac{1}{2}}\left(\begin{array}{cccc}
q & \frac{\sqrt{[p+1][p-1]}}{[p]} & \frac{q^{p}}{[p]} &  \tag{52}\\
& -\frac{q^{-p}}{[p]} & \frac{\sqrt{[p+1][p-1]}}{[p]} & q
\end{array}\right)
$$

where $[x]=\left(q^{x}-q^{-x}\right) /\left(q-q^{-1}\right)$ is, as usual, the $q$-number. The entries of matrices (52) coincide with values of certain $6 j$-symbols for $U_{q}(s l(2))$ [20].

Note that the asymptotics

$$
\begin{equation*}
\mathcal{R}_{ \pm}^{\frac{1}{2} \frac{1}{2}}(p) \rightarrow R_{ \pm}, \quad \mathcal{R}_{ \pm}^{\frac{1}{2} \frac{1}{2}}(p) \rightarrow R_{\mp}^{-1} \tag{53}
\end{equation*}
$$

hold in the formal limits $q^{p} \rightarrow+\infty$ and $q^{-p} \rightarrow+\infty$, respectively; that is, here we return to the case of the Hopf algebra, in particular, the coassociator becomes trivial, $\Phi_{123}=\epsilon \otimes \epsilon \otimes \epsilon$. Moreover, (53), together with (41) and (50), allows us to add the following condition to the list presented in §2.3.
5. (Asymptotic behavior)

$$
\begin{equation*}
\mathcal{F}\left(q^{p} \rightarrow+\infty\right)=\epsilon \otimes \epsilon, \quad \mathcal{F}\left(q^{-p} \rightarrow+\infty\right)=\chi, \tag{54}
\end{equation*}
$$

where $\chi \in \mathcal{G} \otimes \mathcal{G}$ was described at the end of $\S 2.3$. It should be noted that this additional condition is obtained only for $\mathcal{J}=s l(2)$. It would be interesting to find its generalization for the case $\mathcal{J}=s l(n)$, where
we have the vector $\vec{p}$ instead of a single variable.


Fig. 1. The action of the operators $U_{i}$ on the model space
Now let us turn to the solution of Eqs. (24)-(26) for $\mathcal{R}^{\frac{1}{2} \frac{1}{2}}(p)$ given by (52) and considered in different contexts in $[4,5,12]$. It was shown that this solution is unique up to transformations (28), (29) and, in particular, can be written in terms of the operators of multiplication and shift (difference derivative) of two complex variables [12]:

$$
U^{\frac{1}{2}}=\left(\begin{array}{cc}
U_{1} & U_{2}  \tag{55}\\
U_{3} & U_{4}
\end{array}\right)=\left(\begin{array}{cc}
z_{1} q^{\frac{1}{2} z_{2} \partial_{2}} & z_{2} q^{-\frac{1}{2} z_{1} \partial_{1}} \\
-z_{2}^{-1}\left[z_{2} \partial_{2}\right] q^{-\frac{1}{2}\left(z_{1} \partial_{1}+1\right)} & z_{1}^{-1}\left[z_{1} \partial_{1}\right] q^{\frac{1}{2}\left(z_{2} \partial_{2}+1\right)}
\end{array}\right) \frac{1}{\sqrt{[p]}} .
$$

Here $p=z_{1} \partial_{1}+z_{2} \partial_{2}+1$, and rthe ight-hand side of (55) is a realization of the exact generating matrix of spin $1 / 2$. This means, in agreement with the general description of $\S 1.3$, that the entries of $U^{\frac{1}{2}}$ act on the model space $\mathcal{M}=\otimes_{J=0}^{\infty} \mathcal{H}_{J}$ as the basic shifts (see Fig. 1). This can be directly verified [12] if we realize the model space as the space $D_{q}\left(z_{1}, z_{2}\right)$ of holomorphic functions of two complex variables with a scalar product (a deformation of the standard one) such that the monomials $|J, m\rangle=\frac{z_{1}^{J}+m}{[J+m)!!(J-m)!}$ form an orthonormal basis. Moreover, for specific realization (55), the matrix elements $\left\langle J^{\prime}, m\right| U_{i}\left|J^{\prime \prime}, m^{\prime \prime}\right\rangle$ (evaluated on $D_{q}\left(z_{1}, z_{2}\right)$ ) coincide with the Clebsch-Gordan coefficients $\left\{\begin{array}{ccc}J^{\prime} & 1 / 2 & J^{\prime \prime} \\ m^{\prime} & \pm 1 / 2 & m^{\prime \prime}\end{array}\right\}_{q}$ (four of them do not vanish); we call this property "preciseness."

Now we encounter the simplest version of the fusion problem, namely, the problem of constructing the exact generating matrix of spin 1 from $U^{\frac{1}{2}}$. To this end, we must find an explicit form of the corresponding twisting element $\mathcal{F}$ in the fundamental representation.

Before proceeding with the computations, recall that in [15] a universal formula (i.e., applicable for representations of any spin) for the solution $\widetilde{\mathcal{R}}(p)$ of Eq. (20) was found, and a universal expression for $\widetilde{\mathcal{F}}(p)$ satisfying (41) with given $\tilde{\mathcal{R}}(p)$ was obtained. However, $\widetilde{\mathcal{R}}(p)$ does not satisfy (23) and, therefore, taken, e.g., in the fundamental representation, this solution differs from (52). Thus, solutions $\widetilde{U}$ of (24) with such $\tilde{\mathcal{R}}(p)$ are not exact generating matrices in our sense. In particular, the solution for spin $1 / 2$ differs from that given by ( 55 ) and, therefore, does not have the remarkable properties discussed above.

Let us stress that these new matrices $\widetilde{U}$ are still the generating matrices in the sense of definition (9). Therefore, one can examine whether these matrices can be transformed into exact generating matrices by means of the transformation $\widetilde{U}=M(p) U$ with $M(p) \in \mathcal{G} \otimes \mathcal{C}$. If such an $M(p)$ exists, then the following relations hold:

$$
\begin{aligned}
\tilde{\mathcal{F}}(p) & =(\Delta \otimes i d) M(p) \mathcal{F}\left(\stackrel{1}{M}\left(p_{2}\right) \stackrel{2}{M}(p)\right)^{-1} \\
\widetilde{\mathcal{R}}_{ \pm}(p) & =\stackrel{2}{M}\left(p_{1}\right) \stackrel{1}{M}(p) \mathcal{R}_{ \pm}(p)\left(\stackrel{1}{M}\left(p_{2}\right) \stackrel{2}{M}(p)\right)^{-1}
\end{aligned}
$$

and we can construct our $\mathcal{F}(p)$ from $\widetilde{\mathcal{F}}(p)$ and $M(p)$. However, bearing in mind possible extensions to the cases where no universal formulas for $\mathcal{R}(p)$ are known, we prefer to give more direct computations of $\mathcal{F}(p)$ instead of the seeking such an $M(p)$.

## §3.2. The Computation of $\mathcal{F}^{\frac{1}{2} \frac{1}{2}}(p)$ and $U^{1}$

The matrix $\mathcal{F}^{\frac{1}{2} \frac{1}{2}}$ must satisfy the conditions presented in $\S 2.3$. First of all, this matrix must be a solution of Eq. (45), where $\mathcal{R}_{ \pm}(p)$ on the right-hand side are given by (52), and the standard $R$-matrices on the left-hand side are

$$
\begin{align*}
& \mathcal{R}_{+}^{\frac{1}{2} \frac{1}{2}}=q^{-1 / 2}\left(\begin{array}{cccc}
q & & & \\
& 1 & \omega & \\
& & 1 & \\
& & & q
\end{array}\right) \\
& \mathcal{R}_{-}^{\frac{1}{2} \frac{1}{2}}=q^{1 / 2}\left(\begin{array}{cccc}
q^{-1} & & & \\
& 1 & & \\
& -\omega & 1 & \\
& & & \\
& q^{-1}
\end{array}\right), \tag{56}
\end{align*}
$$

with $\omega=q-q^{-1}$. Symmetry condition 2 dictates that one look for the solution of Eq. (45) in the following form:

$$
\mathcal{F}(p)=\left(\begin{array}{cccc}
1 & & &  \tag{57}\\
& \alpha(p) & \beta(p) & \\
& \gamma(p) & \delta(p) & \\
& & & 1
\end{array}\right)
$$

A straightforward check shows that only two of the functions $\alpha(p), \beta(p), \gamma(p)$, and $\delta(p)$ are independent, and we can express, say, the entries of the third line in (57) via those of the second one. The result reads

$$
\begin{align*}
& \gamma(p)=\frac{q^{-p}}{[p]} \alpha(p)+\frac{\sqrt{[p+1][p-1]}}{[p]} \beta(p), \\
& \delta(p)=\frac{\sqrt{[p+1][p-1]}}{[p]} \alpha(p)+\frac{q^{p}}{[p]} \beta(p) . \tag{58}
\end{align*}
$$

Now we should employ condition 3. To this end, we use the following formulas for the fundamental $R$-matrices of $U_{q}(s l(n))$ (see [ 9$]$ for details):

$$
\begin{equation*}
P_{ \pm}=\frac{q^{\frac{1}{n} \pm 1} \widehat{R}_{+}-q^{-\frac{1}{n} \mp 1} \widehat{R}_{-}}{q^{2}-q^{-2}} \tag{59}
\end{equation*}
$$

where $\widehat{R}_{ \pm}=P R_{ \pm}$and $P_{+}, P_{\text {- }}$ are the projectors in $\mathbb{C}^{n} \otimes \mathbb{C}^{n}$ (the $q$-symmetrizer and the $q$-antisymmetrizer) of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$, respectively. In the case of $U_{q}(s l(2))$, these projectors are

$$
P_{ \pm}=\left(\begin{array}{cccc}
1 & & &  \tag{60}\\
& q^{ \pm 1} \lambda & \pm \lambda & \\
& \pm \lambda & q^{ \pm 1} \lambda & \\
& & & 1
\end{array}\right)
$$

where $\lambda=\frac{1}{[2]}=\left(q+q^{-1}\right)^{-1}$. It is easy to find their eigenvectors $\vec{x}_{i}$ such that $\vec{x}_{i}^{t} \vec{x}_{j}=\delta_{i j}$,

$$
\begin{gather*}
P_{+}=\sum_{i=1}^{3} \vec{x}_{i} \otimes \vec{x}_{i}^{t}, \quad \vec{x}_{1}=\left(\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right), \\
\vec{x}_{2}=\sqrt{\lambda}\left(\begin{array}{c}
0 \\
q^{-1 / 2} \\
q^{1 / 2} \\
0
\end{array}\right), \quad \vec{x}_{3}=\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right), \tag{61}
\end{gather*}
$$

$$
P_{-}=\vec{x}_{0} \otimes \vec{x}_{0}^{t}, \quad \vec{x}_{0}=\sqrt{\lambda}\left(\begin{array}{c}
0  \tag{62}\\
q^{1 / 2} \\
-q^{-1 / 2} \\
0
\end{array}\right)
$$

According to (16), we can construct the following $C G$ maps from these vectors:

$$
\begin{align*}
& C\left[\frac{1}{2} \frac{1}{2} 0\right]=\sqrt{\lambda}\left(0, q^{1 / 2},-q^{-1 / 2}, 0\right), \\
& C\left[\frac{1}{2} \frac{1}{2} 1\right]=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \sqrt{\lambda} q^{-1 / 2} & \sqrt{\lambda} q^{1 / 2} & 0 \\
0 & 0 & 0 & 1
\end{array}\right) . \tag{63}
\end{align*}
$$

Now, substituting (60) into (48), we can compute $U^{0}$, which, in this simplest case, is not a matrix but a single operator. To be able to use condition 3 , we must compare $U^{0}$ with the central element of the algebra $\mathcal{U}$ generated by the entries $U_{i}$ of the matrix $U^{\frac{1}{2}}$ and the spin operator $p$. As was shown in [12], the only nontrivial central element of the algebra $\mathcal{U}$ is given by the following analog of the determinant of $U^{\frac{1}{2}}$ :

$$
\begin{equation*}
\operatorname{Det} U^{\frac{1}{2}}=\left(U_{1} U_{4}-q U_{2} U_{3}\right) \sqrt{\frac{[p]}{[p-1]}}=\left(q U_{4} U_{1}-U_{3} U_{2}\right) \sqrt{\frac{[p]}{[p+1]}} \tag{64}
\end{equation*}
$$

Omitting simple calculations, we state the result: $U^{0}$ coincides (up to a numerical factor) with (64) only if the constraint

$$
\begin{equation*}
\alpha(p) \sqrt{[p+1]}-\beta(p) \sqrt{[p-1]}=\varepsilon \sqrt{[p]} \tag{65}
\end{equation*}
$$

holds and, thus, we have only one independent variable. Here the numerical constant $\varepsilon$ on the right-hand side can be arbitrary (nonzero), but additional condition 5 says that $\varepsilon=q^{1 / 2}$.

Finally, we can use conditions 4 and 5 . To apply condition 4 in practice, we can first consider the nondeformed case ( $q=1$ ), where the entries of $\mathcal{F}(p)$ are self-conjugate, and then extend the solution obtained to the generic $q$ so that condition 5 would be satisfied. After simple calculations, we obtain

$$
\begin{align*}
& \alpha(p)=\delta(p)=\frac{1}{[2]}\left(q^{\frac{1}{2}} \sqrt{\frac{[p+1]}{[p]}}+q^{-\frac{1}{2}} \sqrt{\frac{[p-1]}{[p]}}\right) \\
& \beta(p)=-\gamma(p)=\frac{1}{[2]}\left(q^{-\frac{1}{2}} \sqrt{\frac{[p+1]}{[p]}}-q^{\frac{1}{2}} \sqrt{\frac{[p-1]}{[p]}}\right) . \tag{66}
\end{align*}
$$

Thus, $\mathcal{F}^{\frac{1}{2} \frac{1}{2}}(p)$ is found. Notice that $\operatorname{det} \mathcal{F}^{\frac{1}{2} \frac{1}{2}}(p)=1$. It is instructive to derive the following explicit expression for the element $\chi$ (in the fundamental representation) with the help of (66) and (54):

$$
\chi=\left(\begin{array}{cccc}
1 & & &  \tag{67}\\
& 2 \lambda & -\omega \lambda & \\
& \omega \lambda & 2 \lambda & \\
& & & 1
\end{array}\right)
$$

In the nondeformed limit $q=1$, we have $\chi=\epsilon \otimes \epsilon$, as expected.
Finally, substituting (63) into (14) and using the explicit form of $\mathcal{F} \frac{1}{2} \frac{1}{2}(p)$, we obtain the following exact generating matrix of spin 1 :

$$
U^{1}=\left(\begin{array}{ccc}
U_{1}^{2} & q^{-\frac{1}{2}} \sqrt{[2]} U_{1} U_{2} & U_{2}^{2}  \tag{68}\\
\sqrt{\frac{[2][p]}{[p+1}} U_{1} U_{3} & \sqrt{\frac{[p]}{[p+1]}}\left(q^{\frac{1}{2}} U_{1} U_{4}+\frac{-\frac{1}{2}}{\left.U_{2} U_{3}\right)}\right. & \sqrt{\frac{[2][p]}{[p+1]}} U_{2} U_{4} \\
U_{3}^{2} & q^{-\frac{1}{2}} \sqrt{[2]} U_{3} U_{4} & U_{4}^{2}
\end{array}\right)
$$

Let us briefly discuss this formula. First, as we would expect, in the formal limit $q^{p} \rightarrow+\infty$, it coincides with expression (32) for $g^{1}$. Next, it is easy to see that the second row of (68) coincides (up to rescaling by $\sqrt{\frac{[p]}{[p+1]}}$ ) with the spin 1 tensor operator (7) constructed from the generators of $U_{q}(s l(2))$.

Finally, notice that the elements $U_{i j}^{1}, i, j=1,2,3$, act on the model space $\mathcal{M}$ as shifts from the state $|J, m\rangle$ to the state $|J+(2-i), m+(2-j)\rangle$, which is natural because we applied the fusion scheme to the matrix $U^{\frac{1}{2}}$ whose entries are basic shifts on $\mathcal{M}$. Furthermore, if we realize the operators $U_{i}$ in (68) as on right-hand side of (55), then $U^{1}$ will also have the "preciseness" property. Namely, it can be checked that the matrix elements $\left\langle J^{\prime}, m^{\prime} \mid U_{i j}^{1} J^{\prime \prime}, m^{\prime \prime}\right\rangle$ evaluated on $D_{q}\left(z_{1}, z_{2}\right)$ coincide with the $C G$ coefficients $\left\{\begin{array}{ccc}J^{\prime} & 1 & J^{\prime \prime} \\ m^{\prime} & 2-j & m^{\prime \prime}\end{array}\right\}_{q}$ (nine of them do not vanish). Thus, the fusion procedure preserves the "preciseness" of the exact generating matrices. This observation might be useful for practical computations.

## §3.3. Another construction for $\mathcal{F}$

The computations of the previous section inspire us to introduce $p$-dependent counterparts of the projectors $P_{ \pm}$used above. Indeed, it is obvious from the connection formula $\widehat{\mathcal{R}}_{ \pm}(\vec{p})=\mathcal{F}^{-1} \widetilde{R}_{ \pm} \mathcal{F}$ that the objects $\mathcal{P}_{+}$and $\mathcal{P}_{-}$introduced by the analogs of decomposition formulas (59) below

$$
\begin{equation*}
\mathcal{P}_{ \pm}=\frac{q^{\frac{1}{n} \pm 1} \widehat{\mathcal{R}}_{+}(\vec{p})-q^{-\frac{1}{n} \mp 1} \widehat{\mathcal{R}}_{-}(\vec{p})}{q^{2}-q^{-2}} \tag{69}
\end{equation*}
$$

are projectors of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$, respectively. In the case of $U_{q}(s l(2))$, we find

$$
\mathcal{P}_{ \pm}=\left(\begin{array}{cccc}
1 & &  \tag{70}\\
& \lambda \frac{[p \mp 1]}{[p]} & \pm \lambda \frac{\sqrt{[p+1][p-1]}}{[p]} & \\
& \pm \lambda \frac{\sqrt{[p+1][p-1]}}{[p]} & \lambda \frac{[p \neq 1]}{[p]} & 1
\end{array}\right)
$$

Repeating the procedure described in the previous section, we can find the eigenvectors $\vec{x}_{i}$ such that $\vec{x}_{i}^{t} \vec{x}_{i}=\delta_{i j}, \mathcal{P}_{-}=\vec{x}_{0} \otimes \vec{x}_{0}^{t}$, and $\mathcal{P}_{+}=\sum_{i=1}^{3} \vec{x}_{i} \otimes \vec{x}_{i}^{t}$. Next, using the same formulas (16), we can construct $p$-dependent counterparts of the $C G$-maps. They appear as follows:

$$
\begin{align*}
& C_{p}\left[\frac{1}{2} \frac{1}{2} 0\right]=\sqrt{\lambda}\left(0, \sqrt{\frac{[p+1]}{[p]}},-\sqrt{\frac{[p-1]}{[p]}}, 0\right) \\
& C_{p}\left[\frac{1}{2} \frac{1}{2} 1\right]=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \sqrt{\lambda} \sqrt{\frac{[p-1]}{[p]}} & \sqrt{\lambda} \sqrt{\frac{[p+1]}{[p]}} & 0 \\
0 & 0 & 0 & 1
\end{array}\right) . \tag{71}
\end{align*}
$$

Now, a straightforward check shows that the matrix $\mathcal{F}^{\frac{1}{2} \frac{1}{2}}(p)$ found before can be obtained as follows: $\mathcal{F}^{\frac{1}{2} \frac{1}{2}}(p)=C^{\prime}\left[\frac{1}{2} \frac{1}{2} 0\right] C_{p}\left[\frac{1}{2} \frac{1}{2} 0\right]+C^{\prime}\left[\frac{1}{2} \frac{1}{2} 1\right]+C^{\prime}\left[\frac{1}{2} \frac{1}{2} 1\right] C_{p}\left[\frac{1}{2} \frac{1}{2} 1\right]$. Actually, we can give a more general version of this formula,

$$
\begin{equation*}
\mathcal{F}^{I J}(\vec{p})=\sum_{K} C^{\prime}[I J K] C_{p}[I J K] \tag{72}
\end{equation*}
$$

since very similar expressions have already appeared in $[11,18]$, where $C[I J K]$ and $C_{p}[I J K]$ were taken a priori as the $C G$ coefficients and $6 j$-symbols, respectively, with specific dependence on $\vec{p}$. With these objects appropriately defined, one can prove [11] that $\mathcal{F}(\vec{p})$ given by (72) satisfies all axioms for the twisting element.

Thus, we have another way of computing the twisting element in practice, though it is not much simpler than that we used previously. Indeed, to apply it, we must know the values of the $C G$ coefficients and $6 j$ symbols a priori. Here the problem of an appropriate (in the sense of compatibility with the given matrices $R_{ \pm}$and $\mathcal{R}_{ \pm}(\vec{p})$ ) basis and normalization arises. Therefore, our prescription (rather "experimental" because
we have not proved that (72) satisfies the conditions for $\mathcal{F}(\vec{p})$ given above) for constructing $C[I J K]$ and $C_{p}[I J K]$ from the eigenvectors of the projectors $P_{K}^{I J}$ and $\mathcal{P}_{K}^{I J}$ is, possibly, quite useful from the practical point of view.

Finally, let us make the algebraic sense of (72) more transparent. To this end, we note that, since $P_{K}^{I J}=C[I J K] C^{\prime}[I J K]$, we can rewrite the formula for the decomposition of $R$-matrices over projectors (we used its simplest case (59) above) in the following form:

$$
\begin{equation*}
\widehat{R}_{ \pm}^{I J}=\sum_{K} C^{\prime}[I J K] r_{K, \pm}^{I J} C[I J K] \tag{73}
\end{equation*}
$$

where $r_{K, \pm}^{I J}$ are the corresponding eigenvalues (for the fundamental representations of the main series, they are presented in [9]; for arbitrary irreducible representation of $s l(2)$, see [10]). Now, bearing in mind properties (18) of $C G$ maps, we see that, according to (41), (72) transforms expression (73) into a similar one for the twisted $R$-matrices,

$$
\begin{equation*}
\widetilde{\mathcal{R}}_{ \pm}^{I J}(\vec{p})=\sum_{K} C_{p}^{\prime}[I J K] r_{K, \pm}^{I J} C_{p}[I J K] . \tag{74}
\end{equation*}
$$

Thus, the Hopf and quasi-Hopf structures turn out to be identical in terms of projectors.

## Conclusion

In the present paper, we have shown that the theory of (deformed) tensor operators and, in particular, the fusion procedure can most naturally be described by applying the $R$-matrix approach and revealing the underlying quasi-Hopf-algebraic structure. From the practical point of view, the prescription for constructing exact generating matrices can, possibly, be employed for explicit computations, e.g., for calculations of the (deformed) $C G$ coefficients for quantum Lie algebras of higher ranks. On the other hand, the specific quasi-Hopf algebra appearing in this context should certainly be studied in more detail since it provides a nontrivial (and presumably somewhat simplified) realization of the abstract general scheme.

Although the present paper is mainly connected with the mathematical aspect of the theory of tensor operators, we are going to discuss some physical applications in a forthcoming paper.

Finally, we would like to note that it would be interesting to extend the technique developed to the case where $q$ is a root of unity, which will involve truncated quasi-Hopf algebras.

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[^0]:    Published in Zapiski Nauchnykh Seminarov POMI, Vol. 245, 1997, pp. 107-129. Original article submitted April 23, 1996.

[^1]:    ${ }^{1}$ Note that (20) takes values in $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$. The subscript $i=1,2,3$ means that $\vec{H}$-term of (19) appears in the $i^{\text {th }}$ tensor component.
    ${ }^{2}$ The conjugation of an object belonging to the $n$-fold tensor product $\mathcal{G}^{\otimes n}$ is understood as follows: $\left(\xi_{1} \otimes \xi_{2} \ldots \otimes \xi_{n}\right)^{*}=$ $\xi_{1}^{*} \otimes \xi_{2}^{*} \ldots \otimes \xi_{n}^{*}$.
    ${ }^{3}$ To be more precise, $U$ and $\mathcal{R}_{ \pm}(\vec{p})$ are not matrices but the so-called universal objects. If we fix representations of their $\mathcal{G}$-parts, $U^{J}=\left(\rho^{J} \otimes i d\right) U, \mathcal{R}_{ \pm}^{I J}(\vec{p})=\left(\rho^{I} \otimes \rho^{J}\right) \mathcal{R}_{ \pm}(\vec{p})$, we obtain a generating matrix and $\mathcal{C}$-valued counterparts of the standard $R$-matrices.
    ${ }^{4}$ Recall that quantization does not deform the co-multiplication for elements of the Cartan subalgebra.

[^2]:    ${ }^{5}$ For $\mathcal{G}$ replaced by its dual $G^{\prime}$, the matrix $g^{J}$ is regarded as a quantum group-like element. In this case, fusion formulas (31)-(32) are also valid.
    ${ }^{6}$ We prefer this order of auxiliary spaces since it is the same as in (24).

[^3]:    ${ }^{7}$ In fact, here we deal with a generalization of Drinfeld's scheme since $\mathcal{F}(\vec{p}), \mathcal{R}_{ \pm}(\vec{p})$, and $\Phi(\vec{p})$ possess additional $\mathcal{C}$-valued tensor components. However, all Hopf-algebra operations are applied only to $\mathcal{G}$-parts of these objects.

[^4]:    ${ }^{8}$ Formulas (47) and (48) can be regarded [6] as a generalization of the formula for the quantum determinant [9, 10].

