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HALDANE–WU STATISTIC AND ROGERS DILOGARITHM

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The Haldane–Wu exclusion statistic is considered from the generalized extensive statistics point of view and certain related mathematical aspects are investigated. A series representation for the corresponding generating function is obtained. Equivalence of two formulas for the central charge derived for the Haldane–Wu statistic via the thermodynamic Bethe ansatz is established. As a corollary, a series representation with a free parameter for the Rogers dilogarithm is found. It is shown that the generating function, entropy, and central charge for the Gentile statistic majorize those for the Haldane–Wu statistic (under an appropriate choice of parameters). This fact is applied in derivation of a dilogarithm inequality. Bibliography: 14 *titles.*

1. Introduction

Consider a $(1+1)$-dimensional system of relativistic particles on an interval of length L. If the particle interaction is described by a factorizable scattering matrix, then the boundary condition for the wave function of a particle has the following form:

$$\exp(iLm_k \sinh\theta_k) \prod_{l\neq k}^{N} S_{kl}(\theta_k - \theta_l) = \varsigma_k\,, \qquad k = 1, \dots, N, \tag{1}$$

where θ_k and m_k are the rapidity and mass of a particle, $S_{kl}(\theta)$ is the two-particle scattering matrix, and N is the total number of particles. The phases ς_k can be different for different particles (their exact values are not relevant for our purposes). For simplicity, we consider the case where all of the particles belong to the same species and have mass m.

Analysis of the multiparticle system (1) in the thermodynamic limit ($L \to \infty$ while the density N/L remains finite) is based on the thermodynamic Bethe ansatz [1]. In addition to system (1), this ansatz uses the thermodynamic equilibrium condition, i.e., the condition of minimum of the free energy $\mathcal{F}$ ($\mathcal{F} = \mathcal{E} - T\mathcal{S}$, where T is the temperature, $\mathcal{E}$ is the total energy, and $\mathcal{S}$ is the entropy of the system). Thus, the initial data for the thermodynamic Bethe ansatz consist of the two-particle scattering matrix $S(\theta)$, spectrum of particle masses, and statistic which governs filling of states in the momentum space. This statistic, called the *exclusion statistic*, determines the exact form of the entropy of the system.

For one-dimensional systems, the exclusion statistics are not necessarily of fermion or boson type but can depend nontrivially on the number of particles already present at a given state. For instance, a generalized *extensive* statistic is defined by a choice of generating function $f(t)$ such that

$$\big(f(t)\big)^N = \sum_{n\geq 0} W(N,n)\, t^n, \tag{2}$$

where $W(N,n)$ is the number of possible ways for n identical particles to occupy N states. It is natural to impose the condition $f(0) = 1$ which implies that vacuum is realized with probability one independently of the size of the system.

The thermodynamic Bethe ansatz allows one to obtain certain information about the ultra-violet (i.e., high temperature) limit of the system under consideration. In particular, it allows one to find the effective central charge for the corresponding conformal model. For instance, in the case of a generalized extensive statistic, the effective central charge is given by the following formula [2]:

$$c = \frac{6}{\pi^2}\Big[\int_0^{x_0} \frac{dt}{t} \log f(t) - \frac{1}{2}\log x_0 \log f(x_0)\Big]. \tag{3}$$

Here x_0 is the positive root of the equation

$$\log x_0 + \Phi\, \log f(x_0) = 0, \tag{4}$$

which is unique if $f(t)$ is monotonically increasing and $\Phi \geq 0$. From the physical point of view, Φ is related to the asymptotics of the scattering matrix, $2\pi i\Phi = \log S(-\infty) - \log S(\infty)$, but we treat Φ just as a free nonnegative parameter.

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2. Haldane–Wu statistic

The Haldane–Wu statistic [3, 4] is one of the most studied cases of an exotic statistic (see, e.g., [4–9]). It has applications, for instance, in the quantum Hall effect theory. For this statistic, the number of possible ways for n identical particles to occupy N states is given by the formula

$$W_g(N,n) = \frac{(N+(1-g)n+g-1)!}{n!\,(N-gn+g-1)!}, \tag{5}$$

where $0 \le g \le 1$. The Haldane–Wu statistic interpolates between fermions ($g=1$) and bosons ($g=0$).

The Haldane–Wu statistic is *asymptotically* extensive in the following sense. For a generalized extensive statistic (2), the entropy density is defined as follows:

$$s(\mu) = \lim_{N\to\infty} \frac{1}{N} \log W(N, \mu N). \tag{6}$$

One can show that (see, e.g., [2])

$$s(\mu) = \log f(x) - \mu \log x, \tag{7}$$

where $x \equiv x(\mu)$ is the positive root of the equation (the prime denotes a derivative)

$$x\, f'(x) = \mu\, f(x). \tag{8}$$

It follows that

$$f\bigl(x(\mu)\bigr) \equiv f(\mu) = \exp\{s(\mu) - \mu\,\partial_\mu s(\mu)\}. \tag{9}$$

In the case of the Haldane–Wu statistic, application of the Stirling formula to formula (5) yields the relation

$$s_g(\mu) = (1+\mu(1-g))\log(1+\mu(1-g)) - \mu\log\mu - (1-g\mu)\log(1-g\mu). \tag{10}$$

Comparison of formulas (9) and (10) shows that

$$f_g(\mu) = \frac{1+(1-g)\mu}{1-g\mu}; \tag{11}$$

therefore, Eq. (8) takes the following form:

$$\bigl(g f_g(t) + 1 - g\bigr)\, t\, f_g'(t) = f_g^2(t) - f_g(t). \tag{12}$$

Determining the integration constant from the condition $f_g(0)=1$, we see that

$$f_g(t) - 1 = t\,\bigl(f_g(t)\bigr)^{1-g}. \tag{13}$$

If f^{1-g} on the right-hand side is understood as $\exp[(1-g)\log f]$, where $\Im(\log f) = 0$ for $f>0$, then for $0 \le g \le 1$, Eq. (13) has a unique positive solution. Equations (11) and (13) are well known in the context of exotic exclusion statistics [4–6]. Note that the solution to (13) satisfies a duality relation:

$$f_g(t)\, f_{1-g}(-t) = 1. \tag{14}$$

Furthermore, it follows from (12) that $t f_g'/(f_g - 1) > 0$, i.e., $f_g(t)$ is a monotonically increasing function. From (13) we infer (applying inequality (48) with $g>1$) that

$$f_g(t) < \frac{1}{g} + t^{\frac{1}{g}} \tag{15}$$

for nonnegative t. In fact, the right-hand side of (15) gives an asymptotic of $f_g(t)$ for large t.

Using Eq. (13), we can compute derivatives of f_g at $t=0$ in a recursive way:

$$f_g^{(n)}(0) = n\,\partial_t^{n-1}(f_g^{1-g})|_{t=0}. \tag{16}$$

The first few values allow us to conjecture that f_g is given by the following Taylor series:

$$f_g(t) = 1 + t + \sum_{n=2}^{\infty}\Bigl(\prod_{k=2}^{n}\bigl(1-\frac{gn}{k}\bigr)\Bigr)\, t^n. \tag{17}$$

This series for f_g has been suggested in [6]; some confirming combinatorial arguments were given in [8] (for positive integer values of g). Furthermore, it was also suggested in [10, 5, 8] that the logarithm of f_g is given by the series

$$\log f_g(t) = t + \sum_{n=2}^{\infty}\Bigl(\frac{1}{n}\prod_{k=1}^{n-1}\bigl(1-\frac{gn}{k}\bigr)\Bigr)\, t^n. \tag{18}$$

Let us prove the following statement.

Proposition 1. *Series* (17) *and* (18) *are absolutely convergent for*

$$\log|t| < \log t_0 = -g\log g - (1-g)\log(1-g). \tag{19}$$

On this interval, series (17) *and* (18) *are the positive solution of Eq.* (13) *and its logarithm, respectively. In addition, for an integer* m*,*

$$\big(f_g(t)\big)^m = 1 + mt + \sum_{n=2}^{\infty}\Big(m\prod_{k=2}^{n}\big(1+\frac{m-1-gn}{k}\big)\Big)\, t^n \tag{20}$$

on the same interval.

Proof. Let f_n and w_n, $n = 0, 1, 2, \ldots$, denote the coefficients at t^n in series (17) and (18), respectively (so that $w_0 = 0$ and $f_0 = f_1 = w_1 = 1$). Note that these coefficients can be written in terms of the gamma–function:

$$f_n = \frac{\Gamma(1+(1-g)n)}{n!\,\Gamma(2-gn)} = -\frac{\sin\pi gn}{\pi\, n!}\,\Gamma(1+(1-g)n)\,\Gamma(gn-1) \tag{21}$$

and

$$w_n = \frac{\Gamma((1-g)n)}{n!\,\Gamma(1-gn)} = \frac{\sin\pi gn}{\pi\, n!}\,\Gamma((1-g)n)\,\Gamma(gn)\,. \tag{22}$$

Introduce the following notation: $\tilde{f}_n = f_n/\sin\pi gn$ and $\tilde{w}_n = w_n/\sin\pi gn$. Applying the Stirling formula (for large z and $\delta \ll z$) in the form $\log\Gamma(z+\delta) - \log\Gamma(z) = \delta\log z + o(1)$, we see that

$$\lim_{n\to\infty}\log\Big|\frac{\tilde{f}_{n+1}}{\tilde{f}_n}\Big| = \lim_{n\to\infty}\log\Big|\frac{\tilde{w}_{n+1}}{\tilde{w}_n}\Big| = g\log g + (1-g)\log(1-g). \tag{23}$$

Thus, the series $\sum_{n\geq 1}\tilde{f}_n t^n$ and $\sum_{n\geq 1}\tilde{w}_n t^n$ (hence, the series (17) and (18)) converge absolutely on interval (19).

To prove the second assertion of the proposition, we note that Eq. (12) multiplied by f_g^{m-2} acquires the following form:

$$(1-g)t\,(\log f_g)' = f_g - 1 - gt\,f_g' \quad \text{for } m=1 \tag{24}$$

and

$$f_g^m - \frac{g}{m}\,t\left(f_g^m\right)' = f_g^{m-1} + \frac{(1-g)}{m-1}\,t\left(f_g^{m-1}\right)' \quad \text{for } m \neq 0, 1. \tag{25}$$

Similarly, for the function $h_g(t) = f_g(t) - 1$, Eq. (12) yields the relation

$$h_g^m - \frac{g}{m}\,t\left(h_g^m\right)' = -h_g^{m-1} + \frac{1}{m-1}\,t\left(h_g^{m-1}\right)' \quad \text{for } m \neq 0, 1. \tag{26}$$

From formulas (24)–(26), we derive the following relations between the Taylor coefficients:

$$w_n = \frac{1-gn}{(1-g)n}\,f_n\,, \qquad n = 1, 2, \ldots, \tag{27}$$

$$f_n^{[m]} = \frac{m(m-1+(1-g)n)}{(m-1)(m-gn)}\,f_n^{[m-1]}\,, \qquad n = 0, 1, \ldots, \tag{28}$$

and

$$h_n^{[m]} = \frac{m(n+1-m)}{(m-1)(m-gn)}\,h_n^{[m-1]}\,, \qquad n \geq m = 2, 3, \ldots. \tag{29}$$

Here $f_n^{[m]}$ and $h_n^{[m]}$ are the Taylor coefficients of the series $\big(f_g(t)\big)^m = \sum_{n\ge 0} f_n^{[m]} t^n$ and $\big(h_g(t)\big)^m = \sum_{n\ge m} h_n^{[m]} t^n$, respectively. Solving Eqs. (27)–(29), we find the relations

$$f_n^{[m]} = m\, f_n\, \frac{\Gamma(2-gn)\,\Gamma(m+(1-g)n)}{\Gamma(1+(1-g)n)\,\Gamma(m+1-gn)} \tag{30}$$

and

$$h_n^{[m]} = m\, h_n^{[1]}\, \frac{(n-1)!\,\Gamma(2-gn)}{(n-m)!\,\Gamma(m+1-gn)}\,. \tag{31}$$

Substituting $m = n$ into (31) and taking into account that $h_n^{[1]} = f_n$ and $h_n^{[n]} = 1$ for all $n \ge 1$, we obtain precisely formula (21) for the coefficients of series (17). The assertion that series (18) is the logarithm of series (17) follows now from relation (27). Finally, combining relations (30) and (21), we obtain the formula

$$f_n^{[m]} = \frac{m\,\Gamma(m+(1-g)n)}{n!\,\Gamma(m+1-gn)}\,, \qquad n = 1, 2, \dots\,; \tag{32}$$

the latter formula yields the series expansion (20). Analysis of absolute convergence of this series on interval (19) is performed in the same way as for series (17) and (18). Although we have considered only positive values of m, an easily verified relation $(-1)^n f_{1-g,n}^{[m]} = f_{g,n}^{[-m]}$ together with the duality relation (14) shows that formula (20) holds for negative m as well. $\square$

Note that if we assume the validity of formula (30) for $m = 1-g$, then we can use the relation $f_{n+1} = f_n^{[1-g]}$ (following from (13)) to obtain a recurrence relation. Solution of this relation coincides with expression (21). Thus, formula (20) holds also for noninteger m. Another evidence of the latter fact is that series (18) and (20) are consistent in the sense that $\lim_{m\to 0}(f_g^m - 1)/m = \log f_g$.

3. Central charge for the Haldane–Wu statistic

Strictly speaking, formula (5) for counting states in the Haldane–Wu statistic needs some refinements for finite n and N. However, this formula is sufficient for constructing the corresponding thermodynamic Bethe ansatz along the same lines as in the case of the ordinary statistic. The latter approach does not use the explicit form of f_g and leads to the following expression for the effective central charge [9]:

$$c_g = \frac{6}{\pi^2} L(y_0)\,, \tag{33}$$

where y_0 is the positive root of the equation

$$\log y_0 = (\Phi + g)\, \log(1 - y_0)\,. \tag{34}$$

The right-hand side of (33) contains the Rogers dilogarithm defined as follows:

$$L(x) = -\frac{1}{2}\int_0^x dt\,\Big(\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\Big) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2}\log x \log(1-x)\,. \tag{35}$$

On the other hand, since the Haldane–Wu statistic is asymptotically extensive, the corresponding effective central charge c_g should also be given by the general formula (3) if we substitute $f = f_g$. Thus, we have two expressions (rather different at the first sight) for the effective central charge in the Haldane–Wu statistic. Since the derivation of the formula for an effective central charge in the thermodynamic Bethe ansatz involves a nontrivial passage to the limit and uses some additional assumptions, it seems to be instructive to provide a direct proof of the equivalence of the two expressions for c_g.

Proposition 2. *Let $0 \le g \le 1$, let $\Phi \ge 0$, and let $f_g(t)$ be the positive solution of Eq.* (13). *Then the following equality holds:*

$$\int_0^{x_0} \frac{dt}{t} \log f_g(t) - \frac{1}{2} \log x_0 \log f_g(x_0) = L\Big(1 - \frac{1}{f_g(x_0)}\Big) = L(y_0)\,, \tag{36}$$

where y_0 is the positive root of Eq. (34) *and x_0 is the positive root of the equation*

$$\log x_0 + \Phi \, \log f_g(x_0) = 0\,. \tag{37}$$

Proof. Note that, since $f_g(t)$ increases monotonically, Eq. (37) has a unique positive solution x_0. Furthermore, $x_0 \le 1$ since $f_g(0) = 1$.

Consider the function $y(t) = 1 - 1/f_g(t)$. For this function, Eq. (13) takes the form $t = y(1-y)^{-g}$. Therefore,

$$\begin{aligned} &\int \frac{dt}{t} \log f_g(t) - \frac{1}{2} \log t \log f_g(t) = -\int d\big(\log y - g \log(1-y)\big) \, \log(1-y) \\ &+\frac{1}{2}\big(\log y - g\log(1-y)\big) \, \log(1-y) = -\int \frac{dy}{y} \log(1-y) + \frac{1}{2} \log y \log(1-y)\,. \end{aligned} \tag{38}$$

Comparison of the latter expression with formula (35) yields the first equality in (36). Further, Eqs. (13) and (37) imply that

$$\begin{aligned} \log y(x_0) &= \log(f_g(x_0) - 1) - \log f_g(x_0) = \log x_0 - g \log f_g(x_0) \\ &= -(\Phi + g) \log f_g(x_0) = (\Phi + g) \log(1 - y(x_0))\,. \end{aligned} \tag{39}$$

Since Eq. (34) has a unique positive solution for $(\Phi + g) \ge 0$, we conclude that $y(x_0) = y_0$, which proves the second equality in (36). $\square$

Let us formulate a mathematical corollary of Propositions 1 and 2.

Proposition 3. *Let $0 < g < 1$ and $\Phi \ge 0$ and let y_0 be the positive root of Eq.* (34). *Then*

$$\sum_{n=1}^{\infty} \sin \pi g n \, \frac{\Gamma((1-g)n)\,\Gamma(gn)}{\pi\, n\, n!} \Big(y_0(1-y_0)^{-g}\Big)^n + \frac{\Phi}{2}\big(\log(1-y_0)\big)^2 = L(y_0) \tag{40}$$

if $t = y_0(1-y_0)^{-g}$ satisfies condition (19).

Proof. By Proposition 1, we can substitute series (17) into the integral on the left-hand side of (36) and integrate term-wise. The resulting series converges to the value of the integral if the condition (19) of absolute convergence is satisfied. The quantity x_0 at the left-hand side of (36) is the solution to Eqs. (13) and (37); the latter equations are equivalent, after the change of variable $y_0 = 1 - 1/f_g(x_0)$, to Eq. (34) and the relation $y_0 = x_0(1-y_0)^g$. $\square$

An interesting feature of identity (40) is the following one: though the left-hand side of (40) depends on g and Φ in essentially different ways, its right-hand side depends only on the value of $\nu \equiv (g + \Phi)$. Thus, for a fixed y_0, identity (40) provides a representation for the dilogarithm $L(y_0)$ as a series with a free parameter. As an example, consider three special cases, namely, $\nu = 2, 1, \frac{1}{2}$. For these values, $y_0 = 1 - \rho, \frac{1}{2}, \rho$, respectively, where $\rho = (\sqrt{5} - 1)/2$. It is known (see, e.g., [11]) that this is a complete list of algebraic points of the interval $(0, 1)$ at which $\frac{6}{\pi^2} L(y_0)$ takes rational values (these values are $\frac{2}{5}$, $\frac{1}{2}$, $\frac{3}{5}$, respectively). Thus, keeping g as a free parameter, we obtain the following identities for the special values of ν:

$$\sum_{n=1}^{\infty} \sin \pi g n \, \frac{\Gamma((1-g)n)\,\Gamma(gn)}{\pi\, n\, n!} \, \rho^{(2-g)n} + \frac{2-g}{2} (\log \rho)^2 = \frac{\pi^2}{15}\,, \tag{41}$$

$$\sum_{n=1}^{\infty} \sin \pi g n \, \frac{\Gamma((1-g)n)\,\Gamma(gn)}{\pi\, n\, n!} \, 2^{(g-1)n} + \frac{1-g}{2} (\log 2)^2 = \frac{\pi^2}{12}\,, \tag{42}$$

and

$$\sum_{n=1}^{\infty} \sin \pi g n \, \frac{\Gamma((1-g)n)\,\Gamma(gn)}{\pi\, n\, n!} \, \rho^{(1-2g)n} + (1-2g)(\log \rho)^2 = \frac{\pi^2}{10}\,. \tag{43}$$

In (41)–(42), $0 < g < 1$, while the upper bound for g in (43) is determined from the convergence condition (19) (approximately, $g < 0.88$).

4. Gentile statistics

Another interesting case of an extensive statistic (which had appeared in [12] and is sometimes called the Gentile statistic) arises if we take the following generating function in (2):

$$F_G(t) = 1 + t + t^2 + \ldots + t^G; \tag{44}$$

this function also interpolates between fermions ($G = 1$) and bosons ($G = \infty$). For this statistic, the general formula (3) for the effective central charge takes the following form [2]:

$$\tilde{c}_G = \frac{6}{\pi^2}\Big[L(x_0) - \frac{1}{G+1} L\left(x_0^{G+1}\right)\Big], \tag{45}$$

where x_0 is the positive root of Eq. (4) for $f(t) = F_G(t)$.

The maximal value of μ for which Eq. (8) has a positive root is interpreted (since $\mu = n/N$ in formula (6)) as the maximal occupation number for a single state. It is easy to understand that this number equals $\mu_{\max} = G$ for the Gentile statistic and $\mu_{\max} = 1/g$ for the Haldane–Wu statistic. In both cases, the entropy density $s(\mu)$ is a concave function such that $s(0) = s(\mu_{\max}) = 0$. Therefore, it is natural to compare properties of the Gentile statistic with parameter G and the Haldane–Wu statistic with parameter $g = 1/G$. It has been conjectured in [2] that the former statistics majorizes the latter one. We prove the following statement.

Proposition 4. *Let $1 < G < \infty$ and let $g = 1/G$. Then the Gentile statistic majorizes the Haldane–Wu statistic in the following sense:*

$$F_G(t) > f_g(t) \tag{46}$$

for $t > 0$.

Proof. To prove this assertion, it is again reasonable to use the function $y(t) = 1 - 1/f_g(t)$; in this case, Eq. (13) takes the form $t = y(1-y)^{-g}$. Hence,

$$(1-t)\big(F_{\frac{1}{g}}(t) - f_g(t)\big) = 1 - t^{1+\frac{1}{g}} + (t-1)\, f_g(t) = y\,(1-y)^{-g-1}\,\phi_g(y)\,, \tag{47}$$

where $\phi_g(y) = 1 - y^{\frac{1}{g}} - (1-y)^g$. We claim that $\phi_g(y) > 0$ for $0 < t < 1$, i.e., for $0 < y < y_0$, where y_0 is the positive root of the equation $y_0 = (1-y_0)^g$. The inequality

$$(1-y)^g < 1 - gy\,, \tag{48}$$

which holds for $0 < g < 1$ and $0 < y \le 1$, leads us to the estimate $\phi_g(y) > gy - y^{\frac{1}{g}}$. Consequently, $\phi_g(y) > 0$ for $0 < y \le \tilde{y}$, where $\tilde{y}$ is the positive root of the equation $g\tilde{y} = \tilde{y}^{\frac{1}{g}}$. If $y > \tilde{y}$, then

$$\phi_g'(y) = g(1-y)^{g-1} - \frac{1}{g} y^{\frac{1}{g}-1} < g(1-y)^{g-1} - 1 < -\frac{1-g}{1-y}\big(1 - y(1+g)\big)\,, \tag{49}$$

where we use inequality (48) once more. On the other hand, if $y \le y_0$, then it follows from (48) that $y < y_0 < \frac{1}{1+g}$. Therefore, $\phi_g'(y) < 0$ on the interval $\tilde{y} < y < y_0$. Since $\phi_g(y_0) = 1 - y_0^{\frac{1}{g}} - (1-y_0)^g = 1 - y_0^{\frac{1}{g}} - y_0 = 0$, we conclude that $\phi_g(y) > 0$ on this interval as well. Thus, the right-hand side of (47) is positive for $0 < t < 1$. Since $\phi_g(y) = \phi_{\frac{1}{g}}(1-y)$ (note that inequality (48) reverses for $g > 1$), the same reasoning shows that the right-hand side of (47) is negative for $t > 1$. Finally, if $t = 1$, then (cf. (15))

$$f_g(1) = \frac{1}{1-y_0} < 1 + \frac{1}{g} = F_{\frac{1}{g}}(1)\,, \tag{50}$$

which completes the proof. □

Let us note that, as is seen from the proof, the value G in (46) is not necessarily integer if we write $F_G(t)$ as $(1-t^{G+1})/(1-t)$. Following this way, we can consider also the case $0 < G < 1$. In this case, inequality (46) reverses; this can be shown by a proper modification of the above proof. However, for a physical interpretation, the case of a noninteger G is less natural.

Proposition 4 can be used to establish inequalities between physical quantities related to the statistic in question. For example, the following statement holds.

Proposition 5. *Let $\tilde{s}_{G}(\mu)$ and $\tilde{c}_{G}$ be the entropy density and effective central charge for the Gentile statistic. Let $s_g(\mu)$ and c_g be the entropy density and effective central charge for the Haldane–Wu statistic. If $1 < G < \infty$ and $g = 1/G$, then the following inequalities hold:*

$$\tilde{s}_{G}(\mu) > s_g(\mu) \tag{51}$$

and

$$\tilde{c}_{G} > c_g\,, \tag{52}$$

where $0 < \mu < G$ in (51).

Proof. For a fixed value of μ, Eqs. (7) and (8) define the entropy density as a functional of the generating function, $s = s[f]$. Taking a small variation of the function f (which involves also variation of x due to relation (8)), we obtain the equalities

$$\delta s[f] = \delta(\log f - \mu \log x) = \frac{\delta f}{f} + \frac{f'}{f}\delta x - \frac{\mu}{x}\delta x = \frac{\delta f}{f}\,, \tag{53}$$

where the latter equality takes Eq. (8) into account.

Similarly, for a fixed value of Φ, Eqs. (3) and (4) define a functional $c[f]$. For a small variation of the function f, we have the following relations:

$$\delta\Big(\frac{\pi^2}{6}\,c[f]\Big) = \frac{1}{2}\Big(\frac{\log f(x_0)}{x_0}\,\delta x - \frac{\log x_0}{f(x_0)}\,\delta f - \log(x_0)\frac{f'(x_0)}{f(x_0)}\,\delta x\Big) + \int_0^{x_0}\frac{dt}{t}\delta f(t) = \int_0^{x_0}\frac{dt}{t}\delta f(t). \tag{54}$$

Deriving (54), we use Eq. (4) and the equality $f(x_0)\,\delta x + \Phi\,x_0\,(\delta f + f'(x_0)\,\delta x) = 0$ which follows from (4).

Let $\psi_a(t) = aF_{\frac{1}{g}}(t) + (1-a)f_g(t)$ for $a \in [0,1]$. This function is positive for all t. In addition, by Proposition 4, $\delta\psi_a(t) = \delta a\,\big(F_{\frac{1}{g}}(t) - f_g(t)\big) > 0$ if $\delta a > 0$ and $t > 0$. Combining these relations with (53)–(54), we see that $s[\psi_a]$ and $c[\psi_a]$ are monotonically increasing functions of a; hence, relations (51)–(52) follow. $\square$

Relation (52) gives us an inequality involving the Rogers dilogarithm with specifically chosen arguments. Let us formulate this inequality explicitly.

Proposition 6. *Let $\Phi \geq 0$ and let $0 \leq g \leq 1$. Let x_0 and y_0 be the positive roots of the equations*

$$\log x_0 = \Phi\,\log(1 - x_0) - \Phi\,\log\Big(1 - x_0^{1+\frac{1}{g}}\Big) \tag{55}$$

and

$$\log y_0 = (\Phi + g)\,\log(1 - y_0), \tag{56}$$

respectively. Then

$$L\big(x_0\big) - \frac{g}{1+g}L\Big(x_0^{1+\frac{1}{g}}\Big) \geq L(y_0), \tag{57}$$

and equality takes place if and only if either $g = 0$ or $g = 1$.

Proof. If $g = 0^+$, then $x^{\frac{1}{g}} = 0$. Thus, Eqs. (55)–(56) yield the equality $y_0 = x_0$ and (57) turns into an equality. For $g = 1$, Eqs. (55)–(56) yield the equality $y_0 = \frac{x_0}{1+x_0}$. In this case, (57) becomes an equality due to the Abel identity $L(t^2) = 2L(t) - 2L(\frac{t}{1+t})$, which holds for any t in the interval $[0,1]$.

For $0 < g < 1$, the inequality in (57) follows from relation (52) in Proposition 5, Eqs. (33)–(34), formula (45), and Eq. (4) for $f(t) = (1 - t^{1+\frac{1}{g}})/(1-t)$. $\square$

In the simplest case, $\Phi = 0$, we have the equality $x_0 = 1$, and Proposition 6 reduces to the following estimate:

$$L(y) < \frac{1}{1+g}\,\frac{\pi^2}{6} \qquad \text{for} \quad y = (1-y)^g \quad \text{and} \quad 0 < g < 1\,. \tag{58}$$

The case $\Phi = 1$ can be interpreted as a case related to the A_2 affine Toda model [2] and to the Calogero–Sutherland model with coupling constant $\lambda = g$ [9].

Remark. After completing the manuscript, the author have been informed that multivariable analogs of formulas (18) and (20) have been obtained in [13] and [14] by means of the multivariable Lagrange inversion theorem.

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