## AN ANSATZ FOR $\mathrm{sl}_{2}$-INVARIANT $\boldsymbol{R}$-MATRICES

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#### Abstract

We study spectral decomposition of regular $\mathrm{sl}_{2}$-invariant $R$-matrices $R(\lambda)$ by the method of reduction of the YangBaxter equation to subspaces of a given spin. Restrictions on the possible structure of several leading coefficients in the spectral decomposition are derived. The origin and structure of the exceptional solution for spin $s=3$ are explained. A similar analysis is performed for constant $R$-matrices. In particular, it is shown that the permutation matrix $\mathbb{P}$ is a "rigid" solution. Bibliography: 6 titles.


## 1. InTRODUCTION

The Yang-Baxter equation plays a key role in the quantum inverse scattering method (see, e.g., the reviews $[1,2]$ ). The braid group form of this equation looks as follows:

$$
\begin{equation*}
R_{12}(\lambda) R_{23}(\lambda+\mu) R_{12}(\mu)=R_{23}(\mu) R_{12}(\lambda+\mu) R_{23}(\lambda) \tag{1}
\end{equation*}
$$

In this paper, we consider the Yang-Baxter equation (1) on the space $V_{s}^{\otimes 3}$, where $V_{s}$ is an irreducible finitedimensional representation of the algebra $\mathrm{sl}_{2}$. The dimension of the representation $V_{s}$ is $(2 s+1)$, where $s$ is either a positive integer or a semi-integer number (called spin below). Here and in what follows, we use the standard notation: lower indices of $R(\lambda)$ indicate tensor components of $V_{s}^{\otimes 3}$ where $R(\lambda)$ acts nontrivially.

An operator-valued function $R(\lambda): \mathbb{C} \mapsto$ End $V_{s}^{\otimes 2}$ that satisfies Eq. (1) is called an $R$-matrix. We consider $\mathrm{sl}_{2}$-invariant $R$-matrices, i.e., matrices that have spectral decomposition of the form

$$
\begin{equation*}
R(\lambda)=\sum_{j=0}^{2 s} r_{j}(\lambda) P^{j} \tag{2}
\end{equation*}
$$

In (2), $P^{j}$ is the projection onto $V_{j}$, the subspace of $\operatorname{spin} j$ in $V_{s}^{\otimes 2}$, and $r_{j}(\lambda)$ is a scalar function. Additionally, we assume that $R$-matrices under consideration are regular, unitary, and normalized, i.e., the following relations are satisfied:

$$
\begin{equation*}
r_{j}(0)=1, \quad r_{j}(\lambda) r_{j}(-\lambda)=1, \quad \text { and } \quad r_{2 s}(\lambda)=1 \tag{3}
\end{equation*}
$$

Let us remark that unitarity is a consequence of regularity and normalization [3].
Since regular $R$-matrices can be used to construct local integrals of motion for lattice models, in particular for spin chains, the problem of finding all the solutions of the Yang-Baxter equation with properties (3) is important for the quantum inverse scattering method. At present, four series of nonequivalent $\mathrm{sl}_{2}$-invariant regular solutions and one exceptional solution for $s=3$ are known (see [4] and references therein). A computer-based check [4] led to a conjecture that this list of solutions is exhaustive. However, the corresponding classification theorem has not been proven yet. In the present paper, we apply the approach developed in [3] and make some progress in this direction. In particular, we explain the origin and structure of the exceptional solution for $s=3$.

The paper is organized as follows. Section 2 contains analysis of one ansatz for an $R$-matrix. Although the results presented here are well known, we provide all the necessary technical details because our aim is to develop a similar technique for a more general case. In Sec. 3, we recall briefly the main details of the approach developed in [3] for analysis of $\mathrm{sl}_{2}$ - and $U_{q}\left(\mathrm{sl}_{2}\right)$-invariant $R$-matrices. Here we also prove one useful additional relation (Lemma 3). In Sec. 4.1, we demonstrate that analysis of some number of leading coefficients in the spectral decomposition of an $R$-matrix can be performed in a way which closely resembles the analysis described in Sec. 2. In Secs. 4.2-4.3, we give details of this analysis. In particular, it turns out that the exceptional solution arises as a consequence of degeneration of a certain set of matrices. In Sec. 5, we perform a similar analysis for constant $R$-matrices. In particular, it is shown that the permutation $\mathbb{P}$ is a "rigid" solution. The Conclusion summarizes the main results.
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## 2. Analysis of one ansatz for an $R$-matrix

Let $\mathbb{E}$ denote the identity operator on $V_{s}^{\otimes 2}$. For $s \geq 1$, let us consider $R$-matrices of the following form:

$$
\begin{equation*}
R(\lambda)=\frac{1}{1+f(\lambda)}\left(\mathbb{E}+f(\lambda) \mathbb{P}+g(\lambda) P^{0}\right) \tag{4}
\end{equation*}
$$

Here $\mathbb{P}$ is the permutation operator on $V_{s} \otimes V_{s}$. Recall that this operator can be expressed in terms of projections:

$$
\begin{equation*}
\mathbb{P}=\sum_{j=0}^{2 s}(-1)^{2 s-j} P^{j} \tag{5}
\end{equation*}
$$

If the scalar functions $f(\lambda)$ and $g(\lambda)$ satisfy the condition $f(0)=g(0)=0$, then (4) is an ansatz for a solution of the Yang-Baxter equation in the class (3). It turns out that all the $R$-matrices of that type can be described explicitly.
Lemma 1. The following relations hold on $V_{s}^{\otimes 3}$ :

$$
\begin{gather*}
P_{l}^{0} P_{l}^{0}=P_{l}^{0} \mathbb{P}_{l}, \quad \mathbb{P}_{l}=\mathbb{E}, \quad P_{l}^{0} \mathbb{P}_{l}=\mathbb{P}_{l} P_{l}^{0}=\xi P_{l}^{0},  \tag{6}\\
\mathbb{P}_{l} \mathbb{P}_{l^{\prime}} \mathbb{P}_{l}=\mathbb{P}_{l^{\prime}} \mathbb{P}_{l} \mathbb{P}_{l^{\prime}},  \tag{7}\\
P_{l}^{0} \mathbb{P}_{l^{\prime}} \mathbb{P}_{l}=\mathbb{P}_{l^{\prime}} \mathbb{P}_{l} P_{l^{\prime}}^{0}, \quad \mathbb{P}_{l} P_{l^{\prime}}^{0} \mathbb{P}_{l}=\mathbb{P}_{l^{\prime}} P_{l}^{0} \mathbb{P}_{l^{\prime}},  \tag{8}\\
P_{l}^{0} \mathbb{P}_{l^{\prime}} P_{l}^{0}=\eta P_{l}^{0}, \quad P_{l}^{0} P_{l^{\prime}}^{0} P_{l}^{0}=\eta^{2} P_{l}^{0},  \tag{9}\\
P_{l}^{0} P_{l^{\prime}}^{0} \mathbb{P}_{l}=\xi \eta P_{l}^{0} \mathbb{P}_{l^{\prime}}, \quad \mathbb{P}_{l} P_{l^{\prime}}^{0} P_{l}^{0}=\xi \eta \mathbb{P}_{l^{\prime}} P_{l}^{0}, \tag{10}
\end{gather*}
$$

where $l=\{12\}, l^{\prime}=\{23\}$ or $l=\{23\}, l^{\prime}=\{12\}$, and $\xi$ and $\eta$ are scalar constants:

$$
\begin{equation*}
\xi=(-1)^{2 s} \quad \text { and } \quad \eta=\frac{1}{2 s+1} \tag{11}
\end{equation*}
$$

Proof. The third relation in (6) follows from (5). Equalities (7) and (8) are obvious. Relations (9) follow from the well-known relation

$$
\begin{equation*}
P_{12}^{0} P_{23}^{j} P_{12}^{0}=\frac{2 j+1}{(2 s+1)^{2}} P_{12}^{0} \tag{12}
\end{equation*}
$$

(see, e.g., [3]). Relation (10) can be derived as follows:

$$
\begin{equation*}
P_{12}^{0} P_{23}^{0} \mathbb{P}_{12}=P_{12}^{0} \mathbb{P}_{12} P_{13}^{0} \stackrel{(6)}{=} \xi P_{12}^{0} P_{13}^{0}=\xi P_{12}^{0} \mathbb{P}_{23} P_{12}^{0} \mathbb{P}_{23} \stackrel{(9)}{=} \xi \eta P_{12}^{0} \mathbb{P}_{23} . \tag{13}
\end{equation*}
$$

Substituting (4) in (1) and using the relations of Lemma 1, it is not difficult to check that the Yang-Baxter equation for the ansatz under consideration is equivalent to the following equation:

$$
\begin{equation*}
F_{\lambda, \mu} \mathrm{F}+G_{\lambda, \mu} \mathrm{G}+H_{\lambda, \mu} \mathrm{H}+H_{\mu, \lambda} \tilde{\mathrm{H}}=0, \tag{14}
\end{equation*}
$$

where

$$
\begin{array}{ll}
\mathrm{F}=\mathbb{P}_{12}-\mathbb{P}_{23}, & \mathrm{G}=P_{12}^{0}-P_{23}^{0}, \\
\mathrm{H}=P_{12}^{0} \mathbb{P}_{23}-\mathbb{P}_{12} P_{23}^{0}, & \tilde{\mathrm{H}}=\mathbb{P}_{23} P_{12}^{0}-P_{23}^{0} \mathbb{P}_{12}, \tag{15}
\end{array}
$$

$$
\begin{align*}
F_{\lambda, \mu} & =f(\lambda)+f(\mu)-f(\lambda+\mu)  \tag{16}\\
G_{\lambda, \mu} & =g(\lambda)+g(\mu)-g(\lambda+\mu)+\xi f(\lambda) g(\mu)+\xi g(\lambda) f(\mu)+g(\lambda) g(\mu)+\eta g(\lambda) g(\mu) f(\lambda+\mu)+\eta^{2} g(\lambda) g(\mu) g(\lambda+\mu), \tag{17}
\end{align*}
$$

and

$$
\begin{equation*}
H_{\lambda, \mu}=g(\lambda) f(\lambda+\mu)-f(\lambda) g(\lambda+\mu)+\xi \eta g(\lambda) f(\mu) g(\lambda+\mu) . \tag{18}
\end{equation*}
$$

Lemma 2. For $s \geq 1$, the matrices $\mathrm{F}, \mathrm{G}, \mathrm{H}$, and $\tilde{\mathrm{H}}$ in (15) are linearly independent.
The proof is given in Appendix B.
As a consequence of Lemma 2, we see that Eqs. (14) are equivalent to the following system of functional equations:

$$
\begin{align*}
F_{\lambda, \mu} & =0  \tag{19}\\
G_{\lambda, \mu} & =0  \tag{20}\\
H_{\lambda, \mu}=H_{\mu, \lambda} & =0 \tag{21}
\end{align*}
$$

Analysis of system (19)-(21) is fairly simple. There are the following three nontrivial cases:
(1) $f(\lambda) \neq 0$ and $g(\lambda)=0$. In this case, it is obvious from (16) that $f(\lambda)$ is a linear function. Without loss of generality, one may choose $f(\lambda)=\lambda$.
(2) $f(\lambda)=0$ and $g(\lambda) \neq 0$. In this case, there remains a single equation for $g(\lambda)$ :

$$
\begin{equation*}
g(\lambda)+g(\mu)-g(\lambda+\mu)+g(\lambda) g(\mu)+\eta^{2} g(\lambda) g(\mu) g(\lambda+\mu)=0 \tag{22}
\end{equation*}
$$

which has (for $\eta \neq \frac{1}{2}$ ) the following solution:

$$
\begin{equation*}
g(\lambda)=b \frac{1-e^{\gamma \lambda}}{e^{\gamma \lambda}-b^{2}}, \quad b+b^{-1}=\eta^{-1} \tag{23}
\end{equation*}
$$

Here $\gamma$ is an arbitrary finite constant, which can be chosen equal one without loss of generality.
(3) $f(\lambda) \neq 0$ and $g(\lambda) \neq 0$. Again, one may take $f(\lambda)=\lambda$. Introducing a new function $h(\lambda)=f(\lambda) / g(\lambda)$, we can rewrite Eq. (21) in the following form:

$$
\begin{equation*}
h(\lambda+\mu)=h(\lambda)-\xi \eta f(\mu)=h(\mu)-\xi \eta f(\lambda) \tag{24}
\end{equation*}
$$

Hence, $h(\lambda)$ is a linear function. Therefore, the solution for $g(\lambda)$ is as follows:

$$
\begin{equation*}
g(\lambda)=\frac{\lambda}{\beta-\xi \eta \lambda} \tag{25}
\end{equation*}
$$

The function (25) solves Eq. (20) provided that the following restrictions are imposed:

$$
\begin{equation*}
\xi^{2}=1 \quad \text { and } \quad \beta=\eta-\xi / 2 \tag{26}
\end{equation*}
$$

$R$-matrices that correspond to cases (1), (2), and (3) are known as the $R$-matrices of Yang, Baxter, and Zamolodchikov, respectively. The above analysis shows clearly that there are no other solutions of the form (4). It is remarkable that the ansatz (4) covers three of the four known series of $\mathrm{sl}_{2}$-invariant regular $R$-matrices. Therefore, it is natural to study a generalization of the ansatz which could be analyzed in a similar way.

## 3. Reduced Yang-Baxter equation

Let us recall the main details of the approach developed in [3] to analyze $U_{q}\left(\mathrm{sl}_{2}\right)$-invariant $R$-matrices (we take into account from the very beginning that $q=1$ in our case). Denote by $\lfloor t\rfloor$ the entire part of a number $t$. For $n=0,1, \ldots,\lfloor 3 s\rfloor$, the subspace $W_{n}^{(s)} \subset V_{s}^{\otimes 3}$ is defined as the linear span of the highest weight vectors of $\operatorname{spin}(3 s-n)$, i.e.,

$$
\begin{equation*}
W_{n}^{(s)}=\left\{\psi \in V_{s}^{\otimes 3} \quad \mid \quad S_{123}^{+} \psi=0, \quad S_{123}^{z} \psi=(3 s-n) \psi\right\} . \tag{27}
\end{equation*}
$$

For a given $R$-matrix of the form (2), we construct a set of diagonal matrices $D^{(n)}(\lambda)$ as follows:

$$
D_{k k^{\prime}}^{(n)}(\lambda)=\delta_{k k^{\prime}} r_{2 s-k}(\lambda), \quad \text { where } \quad \begin{cases}0 \leq k \leq n & \text { for } 0 \leq n \leq 2 s  \tag{28}\\ n-2 s \leq k \leq 4 s-n & \text { for } 2 s \leq n \leq\lfloor 3 s\rfloor\end{cases}
$$

Set $\widehat{D}^{(n)}(\lambda) \equiv A^{(s, n)} D^{(n)}(\lambda) A^{(s, n)}$, where $A^{(s, n)}$ is a certain special matrix with properties described below. Then the condition that the Yang-Baxter equation (1) is fulfilled on the subspace $W_{n}^{(s)}$ can be written as the following matrix equation:

$$
\begin{equation*}
D^{(n)}(\lambda) \widehat{D}^{(n)}(\lambda+\mu) D^{(n)}(\mu)=\widehat{D}^{(n)}(\mu) D^{(n)}(\lambda+\mu) \widehat{D}^{(n)}(\lambda) \tag{29}
\end{equation*}
$$

which we call the reduced Yang-Baxter equation (of level $n$ ). The initial Eq. (29) is equivalent to the system of reduced Eqs. (29) with $n=0,1, \ldots,\lfloor 3 s\rfloor$.

The matrix $A^{(s, n)}$, which plays an important role in the outlined approach, has the following basic properties. Entries of the matrix are expressed (for $q=1$ ) in terms of $6-j$ symbols of the algebra $\mathrm{sl}_{2}$ as follows (see also Appendix A):

$$
A_{k k^{\prime}}^{(s, n)}=(-1)^{2 s-n} \sqrt{(4 s-2 k+1)\left(4 s-2 k^{\prime}+1\right)}\left\{\begin{array}{ccc}
s & s & 2 s-k^{\prime}  \tag{30}\\
s & 3 s-n & 2 s-k
\end{array}\right\},
$$

where $k$ and $k^{\prime}$ take the same values as in (28). The matrix $A^{(s, n)}$ is orthogonal, symmetric, and coincides with its own inverse ( $t$ stands for matrix transposition):

$$
\begin{equation*}
A^{(s, n)}=\left(A^{(s, n)}\right)^{t}=\left(A^{(s, n)}\right)^{-1} \tag{31}
\end{equation*}
$$

For the purpose of the present work, we need one more property which we formulate as follows.
Lemma 3. For all $n=0, \ldots,\lfloor 3 s\rfloor$, the following matrix relation holds:

$$
\begin{equation*}
A^{(s, n)} D_{0}^{(n)} A^{(s, n)}=(-1)^{n} D_{0}^{(n)} A^{(s, n)} D_{0}^{(n)} \tag{32}
\end{equation*}
$$

where the diagonal matrix $D_{0}^{(n)}$ has the form

$$
\begin{equation*}
\left(D_{0}^{(n)}\right)_{k k^{\prime}}=(-1)^{k} \delta_{k k^{\prime}} \tag{33}
\end{equation*}
$$

and $k$ and $k^{\prime}$ take the same values as in (28).
Proof. Let us write out matrix entries of (32) taking into account that $A^{(s, n)}$ is symmetric:

$$
\begin{equation*}
\sum_{m}(-1)^{m} A_{k m}^{(s, n)} A_{k^{\prime} m}^{(s, n)}=(-1)^{n+k+k^{\prime}} A_{k k^{\prime}}^{(s, n)} \tag{34}
\end{equation*}
$$

Now, taking into account formula (30), it is easy to see that relation (34) can be reduced to the Racah identity for 6- $j$ symbols (see, e.g., [5]):

$$
\sum_{p}(-1)^{p}(2 p+1)\left\{\begin{array}{ccc}
r_{1} & r_{3} & l  \tag{35}\\
r_{2} & r_{4} & p
\end{array}\right\}\left\{\begin{array}{ccc}
r_{1} & r_{2} & l^{\prime} \\
r_{3} & r_{4} & p
\end{array}\right\}=(-1)^{l+l^{\prime}}\left\{\begin{array}{ccc}
r_{3} & r_{1} & l \\
r_{2} & r_{4} & l^{\prime}
\end{array}\right\}
$$

where we have to set $r_{1}=r_{2}=r_{3}=s, r_{4}=3 s-n, l=2 s-k, l^{\prime}=2 s-k^{\prime}$, and $p=2 s-m$.
It is obvious from (5) that $D_{0}^{(n)}$ and $\widehat{D}_{0}^{(n)} \equiv A^{(s, n)} D_{0}^{(n)} A^{(s, n)}$ correspond to the restrictions of the operators $\mathbb{P}_{12}$ and $\mathbb{P}_{23}$ to $W_{n}^{(s)}$. In particular, reduction of Eq. (7) on the subspace $W_{n}^{(s)}$ leads to the following relation:

$$
\begin{equation*}
D_{0}^{(n)} A^{(s, n)} D_{0}^{(n)} A^{(s, n)} D_{0}^{(n)}=A^{(s, n)} D_{0}^{(n)} A^{(s, n)} D_{0}^{(n)} A^{(s, n)} D_{0}^{(n)} A^{(s, n)} ; \tag{36}
\end{equation*}
$$

the correctness of (36) follows immediately from the statement of Lemma 3. One more corollary of Lemma 3 is that $(-1)^{n} A^{(s, n)}$ corresponds to the restriction of the operator $\mathbb{P}_{13}=\mathbb{P}_{12} \mathbb{P}_{23} \mathbb{P}_{12}$ to $W_{n}^{(s)}$.

## 4. Partial analysis of the general ansatz

### 4.1. Derivation of the equations

We note that any $\mathrm{sl}_{2}$-invariant $R$-matrix of $\operatorname{spin} s \geq 1$ can be represented by the following ansatz:

$$
\begin{equation*}
R(\lambda)=\frac{1}{1+f(\lambda)}\left(\mathbb{E}+f(\lambda) \mathbb{P}+g(\lambda) P^{2 s-m}+\sum_{j=0}^{2 s-m^{\prime}} \tilde{r}_{j}(\lambda) P^{j}\right) \tag{37}
\end{equation*}
$$

where $2 \leq m \leq 2 s$ and $m<m^{\prime}$ (if $m=2 s$, then the last sum in (37) is omitted). Below we assume that $g(\lambda) \neq 0$, since otherwise (37) belongs to the known case (1) of Sec. 2. The regularity requirement imposes the condition

$$
\begin{equation*}
f(0)=g(0)=\tilde{r}_{j}(0)=0 \tag{38}
\end{equation*}
$$

Let $\pi^{(m, n)}$ denote the matrix such that $\left(\pi^{(m, n)}\right)_{k k^{\prime}}=\delta_{k m} \delta_{k^{\prime} m}, k=0, \ldots, n$. Then $\pi^{(m, n)}$ and $\widehat{\pi}^{(m, n)} \equiv$ $A^{(s, n)} \pi^{(m, n)} A^{(s, n)}$ correspond to the restrictions of the operators $P_{12}^{2 s-m}$ and $P_{23}^{2 s-m}$ to $W_{n}^{(s)}$. Note that $\pi^{(m, n)}$ and $\widehat{\pi}^{(m, n)}$ are projections of rank 1 .

For $n<m^{\prime}$, the matrices $D^{(n)}(\lambda)$ and $\widehat{D}^{(n)}(\lambda)$ corresponding to the $R$-matrix (27) look as follows:
and

$$
\begin{align*}
& D^{(n)}(\lambda)=\frac{1}{1+f(\lambda)}\left(\mathbb{E}+f(\lambda) D_{0}^{(n)}+\theta_{m n} g(\lambda) \pi^{(m, n)}\right)  \tag{39}\\
& \widehat{D}^{(n)}(\lambda)=\frac{1}{1+f(\lambda)}\left(\mathbb{E}+f(\lambda) \widehat{D}_{0}^{(n)}+\theta_{m n} g(\lambda) \widehat{\pi}^{(m, n)}\right)
\end{align*}
$$

where $\theta_{m n}=0$ for $n<m$ and $\theta_{m n}=1$ for $m \leq n<m^{\prime}$.
The following observation is a key place of the present work: analysis of the reduced Yang-Baxter equation (29) for the ansatz (39) is absolutely analogous (except for one special case) to analysis of Eq. (1) for the ansatz (4) given in Sec. 2. This observation is based on the following assertion.

Lemma 4. Relations (6)-(10) of Lemma 1 remain true after the replacement

$$
\begin{equation*}
\mathbb{P}_{l} \rightarrow D_{0}^{(n)}, \quad \mathbb{P}_{l^{\prime}} \rightarrow \widehat{D}_{0}^{(n)}, \quad P_{l}^{0} \rightarrow \pi^{(m, n)}, \quad P_{l^{\prime}}^{0} \rightarrow \widehat{\pi}^{(m, n)} \tag{40}
\end{equation*}
$$

as well as after the replacement

$$
\begin{equation*}
\mathbb{P}_{l} \rightarrow \widehat{D}_{0}^{(n)}, \quad \mathbb{P}_{l^{\prime}} \rightarrow D_{0}^{(n)}, \quad P_{l}^{0} \rightarrow \widehat{\pi}^{(m, n)}, \quad P_{l^{\prime}}^{0} \rightarrow \pi^{(m, n)} \tag{41}
\end{equation*}
$$

The corresponding scalar constants $\xi$ and $\eta$ become dependent on $m$ and $n$ :

$$
\begin{equation*}
\xi_{m}=(-1)^{m} \quad \text { and } \quad \eta_{m, n}=(-1)^{n} A_{m m}^{(s, n)} \tag{42}
\end{equation*}
$$

Proof. The analogs of relations (6) follow from the definition of the matrices $D_{0}^{(n)}, \widehat{D}_{0}^{(n)}, \pi^{(m, n)}$, and $\widehat{\pi}^{(m, n)}$ and from the property $\left(A^{(s, n)}\right)^{2}=\mathbb{E}$. The analog of relation (7) is identity (36), which we have established above. The analogs of relations (8) can be reduced to the identity

$$
\begin{equation*}
\pi^{(m, n)} A^{(s, n)} D_{0}^{(n)} A^{(s, n)} D_{0}^{(n)} A^{(s, n)}=A^{(s, n)} D_{0}^{(n)} A^{(s, n)} D_{0}^{(n)} A^{(s, n)} \pi^{(m, n)} \tag{43}
\end{equation*}
$$

which is easily verified with the help of relation (32) and the analogs of relations (6). The analogs of relations (9) and (10) are derived as follows:

$$
\begin{aligned}
\pi^{(m, n)} \widehat{D}_{0}^{(n)} \pi^{(m, n)} & =\pi^{(m, n)} A^{(s, n)} D_{0}^{(n)} A^{(s, n)} \pi^{(m, n)} \stackrel{\stackrel{(32)}{=}}{=}(-1)^{n} \pi^{(m, n)} D_{0}^{(n)} A^{(s, n)} D_{0}^{(n)} \pi^{(m, n)} \\
& =(-1)^{n} \pi^{(m, n)} A^{(s, n)} \pi^{(m, n)}=(-1)^{n} A_{m m}^{(s, n)} \pi^{(m, n)}=\eta_{m, n} \pi^{(m, n)}, \\
\pi^{(m, n)} \widehat{\pi}^{(m, n)} \pi^{(m, n)} & =\pi^{(m, n)} A^{(s, n)} \pi^{(m, n)} A^{(s, n)} \pi^{(m, n)}=\left(A_{m m}^{(s, n)}\right)^{2} \pi^{(m, n)}=\eta_{m, n}^{2} \pi^{(m, n)}, \\
\pi^{(m, n)} \widehat{\pi}^{(m, n)} D_{0}^{(n)} & =\pi^{(m, n)} A^{(s, n)} \pi^{(m, n)} A^{(s, n)} D_{0}^{(n)} \\
& =A_{m m}^{(s, n)} \pi^{(m, n)} A^{(s, n)} D_{0}^{(n)}\left(A^{(s, n)}\right)^{2} \stackrel{(32)}{=} \eta_{m, n} \pi^{(m, n)} D_{0}^{(n)} A^{(s, n)} D_{0}^{(n)} A^{(s, n)} \\
& =\xi_{m} \eta_{m, n} \pi^{(m, n)} A^{(s, n)} D_{0}^{(n)} A^{(s, n)}=\xi_{m} \eta_{m, n} \pi^{(m, n)} \widehat{D}_{0}^{(n)} .
\end{aligned}
$$

Let us emphasize that Lemma 3 plays a key role in this proof.
The derivation of Eq. (14) is based only on relations of Lemma 1. Therefore, Lemma 4 implies that for the $R$-matrix (37), the reduced Yang-Baxter equation at levels $n<m^{\prime}$ leads to the same Eq. (14). The only difference is that the scalar coefficients (apart from $F_{\lambda, \mu}$ ) now depend on $m$ and $n$ :

$$
\begin{align*}
F_{\lambda, \mu} & =f(\lambda)+f(\mu)-f(\lambda+\mu)  \tag{44}\\
G_{\lambda, \mu}^{(m, n)} & =\theta_{m, n}\left(g(\lambda)+g(\mu)-g(\lambda+\mu)+\xi_{m} f(\lambda) g(\mu)+\xi_{m} g(\lambda) f(\mu)\right. \\
& \left.+g(\lambda) g(\mu)+\eta_{m, n} g(\lambda) g(\mu) f(\lambda+\mu)+\eta_{m, n}^{2} g(\lambda) g(\mu) g(\lambda+\mu)\right)  \tag{45}\\
H_{\lambda, \mu}^{(m, n)} & =\theta_{m, n}\left(g(\lambda) f(\lambda+\mu)-f(\lambda) g(\lambda+\mu)+\xi_{m} \eta_{m, n} g(\lambda) f(\mu) g(\lambda+\mu)\right), \tag{46}
\end{align*}
$$

and the matrices $F, G, H, \tilde{H}$ are given by formula (15) after substitution (40), i.e.,

$$
\begin{array}{ll}
\mathrm{F}^{(m, n)}=D_{0}^{(n)}-\widehat{D}_{0}^{(n)}, & \mathrm{G}^{(m, n)}=\pi^{(m, n)}-\widehat{\pi}^{(m, n)}, \\
\mathrm{H}^{(m, n)}=\pi^{(m, n)} \widehat{D}_{0}^{(n)}-D_{0}^{(n)} \widehat{\pi}^{(m, n)}, & \tilde{\mathrm{H}}^{(m, n)}=\widehat{D}_{0}^{(n)} \pi^{(m, n)}-\widehat{\pi}^{(m, n)} D_{0}^{(n)}
\end{array}
$$

### 4.2. Analysis of the equations in the case $f(\lambda)=0$

Assuming that $g(\lambda) \neq 0$ in (37), let us first consider the case $f(\lambda)=0$. In this case, Eq. (14) at level $n=m$ is equivalent to the equation $G_{\lambda, \mu}^{(m, m)} \mathrm{G}^{(m, m)}=0$, i.e., to Eq. (22) for $g(\lambda)$, where $\eta$ has the form

$$
\begin{equation*}
\eta_{m, m}=(-1)^{m} A_{m m}^{(s, m)}=\frac{(2 s)!}{(2 s-m)!} \frac{(4 s-2 m+1)!}{(4 s-m+1)!} . \tag{48}
\end{equation*}
$$

For $2 \leq m \leq 2 S$ and $S \geq 1,\left|A_{m m}^{(s, m)}\right|<A_{11}^{(s, 1)}=\frac{1}{2}$. Therefore, $g(\lambda)$ is given by (23), where $\eta=\eta_{m, m}$.
The further analysis of the case $f(\lambda)=0$ leads naturally to the following question: is it possible to have $m^{\prime}>(m+1)$ for the ansatz (37)? The inequality above is possible only if the already found function $g(\lambda)$ solves Eq. (22) at level $n=m+1$, i.e., only if $\eta^{2}$ takes the same value for levels $n=m$ and $n=m+1$. According to (42), the condition $\eta_{m, m}^{2}=\eta_{m, m+1}^{2}$ is equivalent to the requirement

$$
\begin{equation*}
\left|A_{m m}^{(s, m)}\right|=\left|A_{m m}^{(s, m+1)}\right| . \tag{49}
\end{equation*}
$$

However, it is easy to derive from formula (82) that

$$
\begin{equation*}
A_{m m}^{(s, m+1)}=\frac{m^{2}-m-3 m s+s}{2 s} A_{m m}^{(s, m)} \tag{50}
\end{equation*}
$$

Since $m^{2}-m-3 m s+3 s<0$ for $2 \leq m \leq 2 s$, we infer that (49) cannot hold for these values of $m$. Thus, we conclude that $m^{\prime}=m+1$.

### 4.3. Analysis of the equations in the case $f(\lambda) \neq 0$

Let us now turn to the case $f(\lambda) \neq 0$. Equations (14) at levels $n=1, \ldots, m-1$ are equivalent to the equation $F_{\lambda, \mu} \mathrm{F}^{(m, n)}=0$, i.e., to Eq. (19) for $f(\lambda)$. Therefore, without loss of generality we may choose $f(\lambda)=\lambda$.

In order to analyze Eq. (14) for $n \geq m$, it is important to note that the analog of Lemma 2 is not true, in general. That is, the matrices (47) can be linearly dependent. Note that the matrices $\mathrm{F}^{(m, n)}$ and $\mathrm{G}^{(m, n)}$ are symmetric and, obviously, linearly independent, whereas the matrices $\mathbf{H}^{(m, n)}$ and $\tilde{\mathbf{H}}^{(m, n)}$ are transposed to each other: $\tilde{\mathrm{H}}^{(m, n)}=\left(\mathrm{H}^{(m, n)}\right)^{t}$. It turns out that the following relations:

$$
\begin{equation*}
\tilde{\mathbf{H}}^{(m, n)}=\mathbf{H}^{(m, n)} \tag{51}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{H}^{(m, n)}+\tilde{\mathbf{H}}^{(m, n)}=\beta \mathrm{G}^{(m, n)}, \tag{52}
\end{equation*}
$$

where $\beta$ is a scalar constant, can hold only simultaneously. The case where these relations hold is called exceptional.
Lemma 5. For $m \geq 2$, each of relations (51) and (52) holds only for $m=3$ and $n=4$. In this case, relation (52) holds in the following form:

$$
\begin{equation*}
\mathrm{H}^{(3,4)}+\tilde{\mathrm{H}}^{(3,4)}=2 \mathrm{G}^{(3,4)} . \tag{53}
\end{equation*}
$$

The proof is given in Appendix B.
In the generic case, $\mathrm{H}^{(m, n)} \neq \tilde{\mathrm{H}}^{(m, n)}$. Therefore, the antisymmetric matrix part of Eq. (14) imposes the following condition:

$$
\begin{equation*}
H_{\lambda, \mu}^{(m, n)}=H_{\mu, \lambda}^{(m, n)}, \tag{54}
\end{equation*}
$$

and the symmetric part of this equation looks as follows:

$$
\begin{equation*}
F_{\lambda, \mu} \mathrm{F}^{(m, n)}+G_{\lambda, \mu}^{(m, n)} \mathrm{G}^{(m, n)}+\frac{1}{2}\left(H_{\lambda, \mu}^{(m, n)}+H_{\mu, \lambda}^{(m, n)}\right)\left(\mathrm{H}^{(m, n)}+\tilde{\mathrm{H}}^{(m, n)}\right)=0 \tag{55}
\end{equation*}
$$

If $\mathrm{F}^{(m, n)}, \mathrm{G}^{(m, n)}$, and $\left(\mathrm{H}^{(m, n)}+\tilde{\mathrm{H}}^{(m, n)}\right)$ are linearly independent, then Eqs. (54)-(55) lead to the system of functional equations (19)-(21). If, however, $\mathbf{H}^{(m, n)}+\tilde{\mathbf{H}}^{(m, n)}=\beta \mathbf{G}^{(m, n)}+\tilde{\beta} \mathbf{F}^{(m, n)}$, where $\tilde{\beta} \neq 0$, then (55) is equivalent to the following system:

$$
\begin{align*}
2 F_{\lambda, \mu}+\tilde{\beta}\left(H_{\lambda, \mu}^{(m, n)}+H_{\mu, \lambda}^{(m, n)}\right) & =0,  \tag{56}\\
2 G_{\lambda, \mu}^{(m, n)}+\beta\left(H_{\lambda, \mu}^{(m, n)}+H_{\mu, \lambda}^{(m, n)}\right) & =0 . \tag{57}
\end{align*}
$$

Since our choice $f(\lambda)=\lambda$ has already ensured the equality $F_{\lambda, \mu}=0$, we infer that Eqs. (54), (56)-(57) lead again to system (19)-(21). Thus, we conclude that the analysis of the generic case is absolutely analogous to the analysis of case (3) in Sec. 2.

By Lemma 5, the level $n=m$ corresponds to the generic case. Hence, the function $g(\lambda)$ is determined by system (19)-(21) uniquely and has the following form:

$$
\begin{equation*}
g(\lambda)=\frac{\lambda}{\eta_{m, m}-\xi_{m} / 2-\xi_{m} \eta_{m, m} \lambda} \tag{58}
\end{equation*}
$$

where $\xi_{m}$ and $\eta_{m, m}$ are given by formula (42).
The further analysis of the case $f(\lambda)=\lambda$ leads to the following question: is it possible to have $m^{\prime}>(m+1)$ for the ansatz (37)? If $m \neq 3$, then the level $n=m+1$ corresponds to the generic case. It is easy to check that the function (58) can satisfy system (19)-(21) for $n=m+1$ only if $\eta_{m, m}=\eta_{m, m+1}$. The last equality is impossible, as was shown in Sec. 4.2. However, for $m=3$, this level corresponds to the exceptional case. In this case, Eq. (55) is equivalent to the equation

$$
\begin{equation*}
G_{\lambda, \mu}^{(3,4)}+H_{\lambda, \mu}^{(3,4)}+H_{\mu, \lambda}^{(3,4)}=0 \tag{59}
\end{equation*}
$$

whilst condition (54) is not imposed (since Eq. (14) has no antisymmetric part). Substituting (58) into (59), it is easy to verify that Eq. (59) is true if $\left(\eta_{3,3}-\eta_{3,4}\right)\left(2 \eta_{3,4}-1\right)=0$. It is interesting to note that the last condition is satisfied for all $s \geq \frac{3}{2}$ since

$$
\begin{equation*}
\eta_{3,4}=A_{33}^{(s, 4)}=1 / 2 \tag{60}
\end{equation*}
$$

according to (48) and (50). Thus, $m^{\prime}=4$ or $m^{\prime}=5$ in the ansatz (37) for $m=3$ and for all $s \geq \frac{3}{2}$. In fact, $m^{\prime}>5$ is possible only for $s=3$. Indeed, the level $n=5$ corresponds to the generic case, and hence, a necessary condition to have $m^{\prime}=6$ is the equality

$$
\begin{equation*}
A_{33}^{(s, 3)}=A_{33}^{(s, 5)} \tag{61}
\end{equation*}
$$

However, it is not difficult to derive from (82) the following relation:

$$
\begin{equation*}
A_{33}^{(s, 5)}=\frac{10 s^{2}-32 s+21}{s(4 s-7)} A_{33}^{(s, 3)} \tag{62}
\end{equation*}
$$

which shows that (61) can hold only if $(s-3)(6 s-7)=0$. Finally, since $A_{33}^{(3,6)} \neq A_{33}^{(3,3)}$, we conclude that $m^{\prime} \leq 6$ for $s=3$.

## 5. Analysis of constant $R$-matrices

We call a matrix $R \in$ End $V_{s}^{\otimes 2}$ a constant $R$-matrix if this matrix solves the following Yang-Baxter equation:

$$
\begin{equation*}
R_{12} R_{23} R_{12}=R_{23} R_{12} R_{23} \tag{63}
\end{equation*}
$$

We consider $\mathrm{sl}_{2}$-invariant $R$-matrices, i.e., matrices that have spectral decomposition of the form

$$
\begin{equation*}
R=\sum_{j=0}^{2 s} r_{j} P^{j} \tag{64}
\end{equation*}
$$

where the $r_{j}$ are scalar constants. In addition, we assume that

$$
\begin{equation*}
r_{2 s}=1 \tag{65}
\end{equation*}
$$

The technique of analyzing the spectral decomposition described in Secs. 3-4 is applicable to the case of constant $R$-matrices as well. In particular, the following remark explains why condition (65) is natural.

Lemma 6. There exist no nontrivial $\mathrm{sl}_{2}$-invariant constant $R$-matrices such that $r_{2 s}=0$.
Proof. If such an $R$-matrix exists, then by a suitable normalization, this matrix can be reduced to the form $R=$ $P^{2 s-m}+\sum_{j=0}^{2 s-m-1} r_{j} P^{j}$, where $0<m \leq 2 s$. Then the corresponding reduced Yang-Baxter equation (29) at level $n=m$ reads $\left(A^{(s, m)} \pi^{m, m}\right)^{3}=\left(\pi^{m, m} A^{(s, m)}\right)^{3}$, which is equivalent to the relation $\left(A_{m, m}^{(s, m)}\right)^{2}\left[A^{(s, m)}, \pi^{m, m}\right]=0$. However, this relation cannot hold for $m>0$ since $A_{k, m}^{(s, m)} \neq 0$ for all $k$.

We note that any $\mathrm{sl}_{2}$-invariant constant $R$-matrix of $\operatorname{spin} s \geq 1$ that satisfies (65) and has $r_{2 s-1} \neq-1$ can be represented by the following ansatz:

$$
\begin{equation*}
R=\frac{1}{1+f}\left(\mathbb{E}+f \mathbb{P}+g P^{2 s-m}+\sum_{j=0}^{2 s-m^{\prime}} \tilde{r}_{j} P^{j}\right) \tag{66}
\end{equation*}
$$

where $2 \leq m \leq 2 s$ and $m<m^{\prime}$ (if $m=2 s$, then the last sum in (66) is omitted).
Applying the same arguments as in Sec. 3, we can use Lemma 4 to show that the reduced Yang-Baxter equation for the $R$-matrix (66) at levels $n<m^{\prime}$ is equivalent to the same Eq. (14), where the matrices F, G, H , and $\tilde{\mathrm{H}}$ are given by formula (47), and the scalar coefficients are obtained from (44)-(46) by converting the functions $f(\lambda)$ and $g(\lambda)$ into constants $f$ and $g$, i.e.:

$$
\begin{align*}
F & =f  \tag{67}\\
G^{(m, n)} & =\theta_{m, n}\left(g+2 \xi_{m} f g+g^{2}+\eta_{m, n} g^{2} f+\eta_{m, n}^{2} g^{3}\right) \tag{68}
\end{align*}
$$

and

$$
\begin{equation*}
H^{(m, n)}=\theta_{m, n} \xi_{m} \eta_{m, n} g^{2} f \tag{69}
\end{equation*}
$$

Equation (14) at levels $n<m$ is equivalent to the equation $F \mathrm{~F}^{(m, n)}=0$, which can hold only for $f=0$. As a result, $G^{(m, n)}$ acquires the following form:

$$
\begin{equation*}
G^{(m, n)}=\theta_{m, n}\left(g+g^{2}+\eta_{m, n}^{2} g^{3}\right) \tag{70}
\end{equation*}
$$

and then Eq. (14) at levels $m \leq n<m^{\prime}$ yields the equation $G^{(m, n)} \mathrm{G}^{(m, n)}=0$, i.e., the following quadratic equation for $g$ :

$$
\begin{equation*}
1+g+\eta_{m, n}^{2} g^{2}=0 \tag{71}
\end{equation*}
$$

Hence, for $n=m$ we find $g=\frac{1}{2}\left(1 \pm \sqrt{1-4 \eta_{m, m}^{2}}\right)$, where $\eta_{m, m}$ is given by formula (48). It was shown in Sec. 4.2 that $\eta_{m, m}^{2} \neq \eta_{m, m+1}^{2}$ for $2 \leq m \leq 2 s$. Hence, the obtained value of $g$ cannot satisfy (71) for $n>m$. Thus, we conclude that $m^{\prime}=m+1$ in (66).

The ansatz (66) covers not all of $\mathrm{sl}_{2}$-invariant constant $R$-matrices of spin $s \geq 1$ that satisfy (65). Namely, if such an $R$-matrix has $r_{2 s-1}=-1$, then this matrix can be represented by the following ansatz:

$$
\begin{equation*}
R=\mathbb{P}+g P^{2 s-m}+\sum_{j=0}^{2 s-m^{\prime}} \tilde{r}_{j} P^{j} \tag{72}
\end{equation*}
$$

where $2 \leq m \leq 2 s$ and $m<m^{\prime}$ (if $m=2 s$, then the last sum in (72) is omitted).
Using relations of Lemma 4, it is not difficult to check that the reduced Yang-Baxter equation for the $R$-matrix (72) at level $n=m$ is equivalent to the following equation:

$$
\begin{equation*}
g^{2}\left(1+\eta_{m, m} g\right) \mathrm{G}+\xi_{m} g^{2}(\mathrm{H}+\tilde{\mathrm{H}})=0 \tag{73}
\end{equation*}
$$

where $G, H$, and $\tilde{H}$ are given by formula (47). Since the level $n=m$ corresponds to the generic case (cf. Sec. 4.3), we infer that the only solution of (73) is $g=0$. Thus, the ansatz (72) is a solution for (63) only if $g=r_{j}=0$. We have shown that the permutation $\mathbb{P}$ is a "rigid" solution, which does not admit a "deformation" of its spectral decomposition in order $2 s-2$ and lower orders.

## Conclusion

The results of analysis carried out in Secs. 4-5 can be formulated as the following restrictions on the structure of spectral decompositions of $R$-matrices.
Proposition 1. Let $R$ be an $\mathrm{sl}_{2}$-invariant solution of Eq. (63) on $V_{s}^{\otimes 3}$ for an integer or half-integer spin $s \geq 1$ that satisfies condition (65). Then either $r_{2 s-1}=1$ or $r_{2 s-1}=-1$.
(I) In the first case,

$$
\begin{equation*}
R=\mathbb{E}+g P^{2 s-m}+\sum_{j=0}^{2 s-m-1} \tilde{r}_{j} P^{j} \tag{74}
\end{equation*}
$$

where $g$ is a solution of Eq. (71) and $2 \leq m \leq 2 s$. If $m<2 s$, then $\tilde{r}_{2 s-m-1} \neq 0$.
(II) In the second case,

$$
\begin{equation*}
R=\mathbb{P} \tag{75}
\end{equation*}
$$

Proposition 2. Let $R(\lambda)$ be an $\mathrm{sl}_{2}$-invariant solution of Eq. (1) on $V_{s}^{\otimes 3}$ for an integer or half-integer spin $s \geq 1$ that satisfies conditions (3). Then either $r_{2 s-1}(\lambda)=1$ or $r_{2 s-1}(\lambda)=\frac{1-\gamma \lambda}{1+\gamma \lambda}$.
(I) In the first case,

$$
\begin{equation*}
R(\lambda)=\mathbb{E}+g(\lambda) P^{2 s-m}+\sum_{j=0}^{2 s-m-1} \tilde{r}_{j}(\lambda) P^{j} \tag{76}
\end{equation*}
$$

where $2 \leq m \leq 2 s$. If $m<2 s$, then $\tilde{r}_{2 s-m-1}(\lambda) \not \equiv 0$. The function $g(\lambda)$ has the form

$$
\begin{equation*}
g(\lambda)=b \frac{1-e^{\gamma \lambda}}{e^{\gamma \lambda}-b^{2}}, \quad b+b^{-1}=\frac{1}{\eta_{m, m}} \tag{77}
\end{equation*}
$$

where $\eta_{m, m}$ is given by (48), and $\gamma$ is a finite constant.
(II) In the second case, either

$$
\begin{equation*}
R(\lambda)=\frac{1}{1+\gamma \lambda}(\mathbb{E}+\gamma \lambda \mathbb{P}) \tag{78}
\end{equation*}
$$

or

$$
\begin{equation*}
R(\lambda)=\frac{1}{1+\gamma \lambda}\left(\mathbb{E}+\gamma \lambda \mathbb{P}+\frac{\lambda}{\eta_{m, m}\left(1-(-1)^{m} \gamma \lambda\right)-\frac{(-1)^{m}}{2}} P^{2 s-m}+\sum_{j=0}^{2 s-m^{\prime}} \tilde{r}_{j}(\lambda) P^{j}\right) \tag{79}
\end{equation*}
$$

where $\eta_{m, m}$ is given by (48), $\gamma$ is a finite constant, $2 \leq m \leq 2 s$ and $m<m^{\prime}$. If $m<2 s$, then $\tilde{r}_{2 s-m^{\prime}}(\lambda) \not \equiv 0$, and, moreover,

$$
\begin{array}{ll}
m^{\prime}=m+1 & \text { if } \quad m \neq 3 \\
m^{\prime} \leq 5 & \text { if } \quad m=3, \quad s \neq 3
\end{array}
$$

and

$$
m^{\prime} \leq 6 \quad \text { if } \quad m=3, \quad s=3
$$

The constant $\gamma$ may be set unity without loss of generality.
Let us make several brief remarks concerning Propositions 1 and 2.
According to Lemma 6, if all the coefficients of the spectral decomposition of $R(\lambda)$ tend to certain limit values as $\lambda \rightarrow \infty$ in some direction in the complex plane, then these values are finite. It follows from Propositions 1 and 2 that the corresponding limit $R(\infty)$ has the form (74) only for solutions of type (76). In other cases, $R(\infty)=\mathbb{P}$.

Concerning Proposition 2, we note that for $s=3$, a solution with $m^{\prime}=6$ really exists (see [4]):

$$
\begin{equation*}
R(\lambda)=P^{6}+\frac{1-\lambda}{1+\lambda} P^{5}+P^{4}+\frac{4-\lambda}{4+\lambda} P^{3}+P^{2}+\frac{1-\lambda}{1+\lambda} P^{1}+\frac{1-\lambda}{1+\lambda} \frac{6-\lambda}{6+\lambda} P^{0} \tag{80}
\end{equation*}
$$

It is easy to see that the coefficient at $P^{3}$ agrees with formula (79). Apart from this case, it is not known whether there exist $R$-matrices of the form (79) with $2<m<2 s$.

For $m=2$, the three leading order coefficients in (79),

$$
\begin{equation*}
R(\lambda)=P^{2 s}+\frac{1-\lambda}{1+\lambda} P^{2 s-1}+\frac{1-\lambda}{1+\lambda} \frac{1-\frac{2 s}{2 s-1} \lambda}{1+\frac{2 s}{2 s-1} \lambda} P^{2 s-2}+\cdots \tag{81}
\end{equation*}
$$

coincide with the corresponding coefficients of the Kulish-Reshetikhin-Sklyanin $R$-matrix [6]. Let us mention that it follows from Proposition 2 and results of [3] that only $R$-matrices of the form (76) and (81) can have $U_{q}\left(\mathrm{sl}_{2}\right)$-invariant analogs.

## Appendix A. The matrix $A^{(s, n)}$

Expression (30) for entries of the matrix $A^{(s, n)}$ can be rewritten in a more explicit form:

$$
\begin{align*}
A_{k k^{\prime}}^{(s, n)} & =F_{k}^{s} F_{k^{\prime}}^{s} \sum_{l=6 s-n-\min \left(k, k^{\prime}\right)}^{6 s-\max \left(n, k+k^{\prime}\right)}(-1)^{l}(l+1)!\left((l-4 s+k)!\left(l-4 s+k^{\prime}\right)!\right. \\
& \left.\times(l-6 s+n+k)!\left(l-6 s+n+k^{\prime}\right)!(6 s-n-l)!\left(6 s-k-k^{\prime}-l\right)!\left(8 s-n-k-k^{\prime}-l\right)!\right)^{-1} \tag{82}
\end{align*}
$$

where $k$ and $k^{\prime}$ take the same values as in (28), and

$$
\begin{equation*}
F_{k}^{s}=(2 s-k)!\left(\frac{(k)!(n-k)!(2 s-n+k)!(4 s-n-k)!}{(4 s-k+1)!(6 s-n-k+1)!}\right)^{\frac{1}{2}} . \tag{83}
\end{equation*}
$$

The sum in (82) is taken over those $l$ for which the arguments of factorials are nonnegative, and it is understood that $0!=1$.

## Appendix B. Proofs of Lemma 2 and Lemma 5

Proof of Lemma 2. Using the relations

$$
\begin{equation*}
\operatorname{tr}_{a} \mathbb{E}_{a}=2 s+1, \quad \operatorname{tr}_{a} P_{a b}^{j}=\frac{2 j+1}{2 s+1} \mathbb{E}_{b}, \quad \text { and } \quad \operatorname{tr}_{a} \mathbb{P}_{a b}=\mathbb{E}_{b} \tag{84}
\end{equation*}
$$

$(a, b=1,2,3)$, we take the trace over the third tensor component of $\boldsymbol{F}, \mathrm{G}, \mathrm{H}$, and $\tilde{\mathrm{H}}$, which yields:

$$
\operatorname{tr}_{3} \mathrm{~F}=\eta^{-1} \mathbb{P}-\mathbb{E}, \quad \operatorname{tr}_{3} \mathrm{G}=\eta^{-1} P^{0}-\eta \mathbb{E}, \quad \text { and } \quad \operatorname{tr}_{3} \mathrm{H}=\operatorname{tr}_{3} \tilde{\mathrm{H}}=P^{0}-\eta \mathbb{P}
$$

Since $\mathbb{E}, \mathbb{P}$, and $P^{0}$ are linearly independent for $s \geq 1$, we conclude that $\mathrm{F}, \mathrm{G}, \mathrm{H}$, and $\tilde{\mathrm{H}}$ can be linearly dependent only if the following equality holds:

$$
\begin{equation*}
\eta^{2} \mathrm{~F}-\eta \mathrm{G}+\alpha \mathrm{H}+\tilde{\alpha} \tilde{\mathrm{H}}=0, \quad \alpha+\tilde{\alpha}=1 \tag{86}
\end{equation*}
$$

Multiply (86) by $P_{12}^{0}$ from the left, take into account relations (6), and take the trace over the first tensor component. Using again the linear independence of $\mathbb{E}, \mathbb{P}$, and $P^{0}$, we deduce that (86) can hold only if $\alpha=\xi \eta$. Multiplying (86) by $P_{12}^{0}$ from the right, we deduce analogously that $\tilde{\alpha}=\xi \eta$. Thus, (86) can hold only if $\xi \eta=1 / 2$, which is impossible as seen from (11).
Proof of Lemma 5. Let us write out entries of the matrices H, $\tilde{H}$, and G explicitly:

$$
\begin{align*}
H_{k k^{\prime}} & =(-1)^{m+k^{\prime}} \delta_{k m} A_{k k^{\prime}}^{(s, n)}-(-1)^{k} A_{k m}^{(s, n)} A_{m k^{\prime}}^{(s, n)},  \tag{87}\\
\tilde{H}_{k k^{\prime}} & =(-1)^{m+k} \delta_{k m} A_{k k^{\prime}}^{(s, n)}-(-1)^{k^{\prime}} A_{k m}^{(s, n)} A_{m k^{\prime}}^{(s, n)}, \tag{88}
\end{align*}
$$

and

$$
\begin{equation*}
G_{k k^{\prime}}=\delta_{k m} \delta_{k^{\prime} m}-A_{k m}^{(s, n)} A_{m k^{\prime}}^{(s, n)} \tag{89}
\end{equation*}
$$

Recall that $m \geq 2$ and $k, k^{\prime}=0,1, \ldots, n$. Comparing (87)-(89) for $k=k^{\prime}=0$, we note that (52) can hold only for $\beta=2$. Further, considering (87)-(89) for $k, k^{\prime} \neq m$, it is easy to see that each of relations (51) and (52) can hold only if

$$
\begin{equation*}
A_{k m}^{(s, n)} A_{m k^{\prime}}^{(s, n)}=0 \tag{90}
\end{equation*}
$$

for all values of $k$ and $k^{\prime}$ such that $k, k^{\prime} \neq m$ and $(-1)^{k}+(-1)^{k^{\prime}} \neq 2$. In particular, (90) must hold for $k=k^{\prime}=1$, which implies that $A_{1 m}^{(s, n)}=0$. As can be seen from (82), the latter equality is possible only if the following condition is satisfied (for $m<n$ ):

$$
\begin{equation*}
2 m^{2}-2 m+n^{2}-n=8 m s-6 n s \tag{91}
\end{equation*}
$$

Note that $A_{k n}^{(s, n)} \neq 0$ for all $k$. Therefore, (90) implies that $m \neq n$ and also that $n$ is an even number. Furthermore, if $m \neq n-1$, then (90) must hold for $k=k^{\prime}=n-1$, which implies that $A_{1, n-1}^{(s, n)}=0$. We apply (82) once more to show that the last equality is possible only if the following condition is fulfilled:

$$
\begin{equation*}
m^{2}-m=4 m s-n s \tag{92}
\end{equation*}
$$

It is easy to see that conditions (91) and (92) are incompatible since they imply the equality $n^{2}-n+4 n s=0$. Thus, the only remaining possibility is the case $m=n-1$. In this case, condition (91) is satisfied only for $n=4$. A direct check shows that relations (51) and (52) indeed hold for $m=3$ and $n=4$.

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