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ON STRING SOLUTIONS OF THE BETHE EQUATIONS IN THE $\mathcal{N} = 4$ SUPERSYMMETRIC YANG–MILLS THEORY

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The Bethe equations which arise in description of the spectrum of the dilatation operator for the $su(2)$ sector of the $\mathcal{N}$=4 supersymmetric Yang–Mills theory are considered in the anti-ferromagnetic regime. These equations are a deformation of those for the Heisenberg XXX magnet. We prove that in the thermodynamic limit, roots of the deformed equations group into strings. We prove that the corresponding Yang's action is convex, which implies the uniqueness of a solution for centers of the strings. The state formed by strings of length $(2n+1)$ is considered, and the density of their distribution is found. It is shown that the energy of such a state decreases as n grows. It is observed that the nonanalyticity of the left-hand sides of the Bethe equations leads to an additional contribution to the density and energy of strings of even length. We conclude that the structure of the anti-ferromagnetic vacuum is determined by the behavior of exponential corrections to string solutions in the thermodynamic limit and, possibly, involves strings of length 2. Bibliography: 14 titles.

1. Introduction

Integrable models, in particular, spin chains appear in several problems of high-energy physics as effective models of interaction. For example, the Hamiltonian of the XXX spin chain with a noncompact representation of spin $s=-1$ arises in description of scattering of hadrons at high energies [1, 2] and also in description of mixing of composite operators under renormalization in QCD [3].

Mixing of composite operators under renormalization in the super-symmetric Yang–Mills theory also gives rise to an XXX-chain, but with a compact representation. In this theory, one considers locally invariant operators of the form

$$\mathcal{O} = \mathrm{tr}\,(Z^{J_1} W^{J_2} + \text{permutations}), \tag{1}$$

where Z and W are two complex scalar fields from the supermultiplet. The conformal dimensions Δ of these operators comprise the spectrum of the dilatation operator D. It is convenient to describe mixing of operators (1) under renormalization with the help of an analogy with the quantum spin chain of length $L = J_1 + J_2$, where each occurrence of Z is represented by a spin up and each occurrence of W is is represented by a spin down. For example, the state $ZZZWWZWZ$ corresponds to the following spin chain: $\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\,|\rangle$. An important observation made in [4] states that, in the $su(2)$-sector of the theory in the one-loop approximation (i.e., in the first order in $\lambda = g^2_{\mathrm{YM}}N$, where g_{YM} is the Yang–Mills coupling constant and N is the number of colors), the dilatation operator D can be expressed via the XXX Hamiltonian of spin $s=\frac{1}{2}$ for the chain described above as follows:

$$D = \mathrm{const} - \lambda\, H_{\mathrm{XXX}} + O(\lambda^2). \tag{2}$$

Therefore, determination of the spectrum of D in this approximation is reduced to investigation of the Bethe equations for the Heisenberg magnet (see [5–7]):

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k\neq j} \frac{u_j - u_k + i}{u_j - u_k - i}, \tag{3}$$

where the u_i are rapidities of elementary excitations.

High-loop corrections to (2) were found in papers which followed [4], and it was shown there that the corresponding expressions are also integrable Hamiltonians for the spin chain (that include interactions between several nearest sites). In these approximations, the spectrum of D is determined not by Eqs. (3) but by their "deformations" which explicitly contain parameter λ in left-hand sides. Assuming that integrability of D takes place in all orders, Beisert, Dippel, and Staudacher [8] argued that the exact Bethe equations determining the spectrum of D look as follows:

$$\left(\frac{x(u_j + i/2)}{x(u_j - i/2)}\right)^L = \prod_{k\neq j} \frac{u_j - u_k + i}{u_j - u_k - i}, \tag{4}$$

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where

$$x(u)=\frac{1}{2}\left(u+\sqrt{u^2-\chi^2}\right),\quad \chi^2\equiv\frac{\lambda}{4\pi^2}. \tag{5}$$

For a spin chain of length L, the spectrum of the Hamiltonian is bounded by the energies of the ferromagnetic and antiferromagnetic vacua. Therefore, the spectrum of the operator D (i.e., the dimensions of operators (1)) belongs to the interval

$$L\leqslant\Delta\leqslant\Delta_{\max}. \tag{6}$$

The lower bound in (6) follows obviously from the structure of the ferromagnetic vacuum (all spins up), while determination of the upper bound requires a nontrivial evaluation of the energy of the antiferromagnetic vacuum. The value of $\Delta_{\max}$ in the thermodynamic limit has been found in [9] and [10] by means of the standard techniques of passing to the limit as $L\to\infty$ in the Bethe equations. In this procedure, it was assumed that, similarly to the case of Eqs. (3), the antiferromagnetic vacuum is formed by strings of length 1, i.e., by real roots of Eqs. (4). However, an example of an XXX spin chain of higher spin [11, 12] shows that the antiferromagnetic vacuum for Bethe equations whose left-hand sides differ from (3) may have a different structure; for instance, it can be filled by strings of greater length.

The aim of the present work is to prove the existence of solutions corresponding to strings of length greater than 1 for Eqs. (4) in the thermodynamic limit and to check the validity of the assumption on the structure of the antiferromagnetic vacuum.

2. Existence of string solutions

To make an analytic single-valued function from $x(u)$, we fix in $\mathbb{C}$ a cut $[-\chi,\chi]$ and define $x(u)$ on $\mathbb{C}/[-\chi,\chi]$ as follows:

$$x(u)=\frac{1}{4}\left(\sqrt{r_1}\,e^{\frac{i}{2}\theta_1}+\sqrt{r_2}\,e^{\frac{i}{2}\theta_2}\right)^2, \tag{7}$$

where r_1, $r_2\in\mathbb{R}_+$, and θ_1, $\theta_2\in\,]-\pi,\pi]$ are determined from the relations $u=\chi+r_1e^{i\theta_1}=-\chi+r_2e^{i\theta_2}$. Note that the signs of the imaginary parts of u and $x(u)$ coincide, i.e., $x(u)$ maps a point from the upper/lower half-plane to the upper/lower half-plane, respectively.

Let us prove that the following relations:

$$\left|\frac{x(u+i\delta)}{x(u-i\delta)}\right|\quad\begin{cases}>1 & \text{for}\ \ \operatorname{Im}u>0,\\ =1 & \text{for}\ \ \operatorname{Im}u=0,\\ <1 & \text{for}\ \ \operatorname{Im}u<0,\end{cases} \tag{8}$$

hold for every $\delta>0$.

Let us denote $s=\operatorname{Re}(u)$, $t=\operatorname{Im}(u)$, $a=\operatorname{Re}\big(x(u)\big)$, and $b=\operatorname{Im}\big(x(u)\big)$. It follows from (7) that $|x(u)|$ is a function that is continuous in u (in particular, even when u crosses the cut), and $|x(\overline{u})|=|x(u)|$. Therefore, if s is fixed, then the function $|x(s+it)|$ is continuous and symmetric in t. We prove that this function is convex, which implies (8) as an obvious consequence.

It follows from (5) that the function inverse to $x(u)$ is given by $u(x)=x+\frac{\chi^2}{4x}$. Thus, it is easy to derive the following relations:

$$s=\left(1+\frac{\chi^2}{a^2+b^2}\right)a\quad\text{and}\quad t=\left(1-\frac{\chi^2}{a^2+b^2}\right)b. \tag{9}$$

Let ∂_t denote the partial derivative w.r.t t (i.e., $\partial_t s=0$). Taking this derivative of (9) and solving the system of equations for $\partial_t a$ and $\partial_t b$, we see that

$$\partial_t a=\frac{2\chi^2ab}{D}\quad\text{and}\quad\partial_t b=\frac{1}{D}\left((a^2+b^2)^2+\chi^2(b^2-a^2)\right), \tag{10}$$

where $D=((\chi-a)^2+b^2)((\chi+a)^2+b^2)$. Hence,

$$\partial_t|x(u)|^2=2\,\frac{a^2+b^2}{D}\,(a^2+b^2+\chi^2)\,b. \tag{11}$$

Due to the remark after Eq. (7), t and b are of the same sign. Therefore, expression (11) is positive/negative in the upper/lower half-plane, respectively. Hence, $|x(u)|$ is convex in $\operatorname{Im}(u)$, which completes the proof of (8).

Relations (8) allow us to adapt the analysis of complex roots of Eqs. (3) (see [5, 7]) to the case of Eqs. (4). Namely, it follows from (8) that, in the $L \to \infty$ limit, the absolute value of the left-hand side of (4) tends to ∞ if $\operatorname{Im}(u_j) > 0$ and to 0 if $\operatorname{Im}(u_j) < 0$. This implies that the right-hand side has a pole or zero, respectively, i.e., there must also exist a root $u_j - i$ in the first case and a root $u_j + i$ in the second case. Thus, like in the case of the XXX magnet, roots of Eqs. (4) group in the thermodynamic limit into "strings," which are complexes of the form $u_{j,m} = u_j + im$, where $u_j \in \mathbb{R}$ and $2m \in \mathbb{Z}$.

3. Strings of odd length

3.1. Equation for centers of strings

Now we turn to the following problem: which state has the maximal energy in the case where the vacuum is filled with strings of length $2n+1$, where n is integer? The number of these strings ν_n is fixed by the condition that $(2n+1)\nu_n = L/2$. Following [5–7], we multiply the Bethe equations (4) along a string of length $2n+1$. Since the right-hand sides of these equations are the same as in the "undeformed" Bethe equations, strings will have the same form, i.e., $u_j = u_j^n + im$, $m \in \mathbb{Z}$. Further, considering the thermodynamic limit (as $L \to \infty$), we see that the centers of strings are arranged along the real axis with some density that satisfies a certain integral equation. Having found this density, one can compute the energy of the ground state (see [5, 6]).

Thus, we obtain the following equation for the centers of strings:

$$\frac{i}{2} L \log \frac{x\big(u_j^n + (2n+1)\frac{i}{2}\big)}{x\big(u_j^n - (2n+1)\frac{i}{2}\big)} = \pi Q_j^n + \sum_{k=1}^{\nu_n} \Phi_{n,n}(u_j^n - u_k^n), \tag{12}$$

where

$$\Phi_{n,n}(u) = \arctan \frac{u}{2n+1} + 2 \sum_{m=0}^{2n-1} \arctan \frac{u}{m+1}. \tag{13}$$

3.2. Yang's action

To prove the uniqueness of a solution to Eqs. (12) for a given set of integer numbers Q_j^n, we use Yang's action. As in the case of the XXX magnet [7], there exists a functional S (called Yang's action) such that Eqs. (12) are the conditions of its extremum: $\partial_{u_\alpha} S = 0$. Let us consider the quadratic form for the matrix of second derivatives of S:

$$\sum_{\alpha,\beta} v_\alpha \frac{\partial^2 S}{\partial_{u_\alpha} \partial_{u_\beta}} v_\beta = \frac{i}{2} L \sum_\alpha \partial_{u_\alpha} \log \frac{x(u_\alpha + \frac{i}{2}(2n+1))}{x(u_\alpha - \frac{i}{2}(2n+1))} v_\alpha^2 + \sum_{\alpha>\beta} \frac{1}{(u_\alpha - u_\beta)^2 + 1} (v_\alpha - v_\beta)^2, \tag{14}$$

where $v_\alpha \in \mathbb{R}$. It is obvious that the second term is always positive. Let us prove the positivity of the first one:

$$\frac{i}{2} \partial_s \log \frac{x(s+it)}{x(s-it)} = \partial_s \arctan \frac{a}{b} = \frac{b \partial_s a - a \partial_s b}{(a^2+b^2)} = \frac{a \partial_t a + b \partial_t b}{(a^2+b^2)} = \frac{\partial_t |x(s+it)|^2}{2(a^2+b^2)}. \tag{15}$$

Here we use the same notation as in Sec. 2 and apply the Cauchy equations for derivatives of an analytic function. Relation (11) shows that expression (15) is positive for $b > 0$. Therefore, the quadratic form (14) is positive definite; consequently, the action S has a unique minimum.

3.3. Thermodynamic limit

Now we pass to the thermodynamic limit. Letting $L \to \infty$ and differentiating (12) with respect to u, we get the following expressions for the left-hand side:

$$l.h.s. = \frac{i}{2} \left(\frac{1}{\sqrt{\big(\frac{(2n+1)}{2} i + u\big)^2 - \chi^2}} - \frac{1}{\sqrt{\big(\frac{(2n+1)}{2} i - u\big)^2 - \chi^2}} \right). \tag{16}$$

Now we introduce the root density $\rho(u)$:

$$\rho(u) = \frac{1}{\left(\dfrac{du}{dq}\right)_{q=q(u)}}, \tag{17}$$

where $q(u) = Q_j/L$. $\rho(u)$ plays the role of density of the numbers $q(u)$ on the interval du. Having introduced this density, we can rewrite the left-hand side of (12) as follows:

$$r.h.s. = \pi\rho(u) \int\limits_{-\infty}^{\infty} d\mu\, \rho(\mu) \left[\frac{2n+1}{(2n+1)^2 + (u-\mu)^2} + 2 \sum_{m=0}^{2n-1} \frac{m+1}{(m+1)^2 + (u-\mu)^2} \right]. \tag{18}$$

This integral equation can be solved by means of the Fourier transform. To this end, we first compute

$$\int\limits_{-\infty}^{\infty} du\, \frac{\mathrm{e}^{iku}}{\sqrt{(u+il)^2 - \chi^2}} = \theta(-kl)\mathrm{e}^{-|kl|} \oint \frac{dq}{q} \exp\left(ik \left(q + \frac{\chi^2}{4q} \right) \right)$$

$$= \mathrm{sign}\,(-l)\theta(-kl)\mathrm{e}^{-|kl|} \int\limits_{0}^{2\pi} id\varphi \mathrm{e}^{ik\chi\cos\varphi} = 2\pi i\, \mathrm{sign}\,(-l)\theta(-kl)\mathrm{e}^{-|kl|} J_0(\chi k), \tag{19}$$

where $J_0(k)$ is the Bessel function of the first kind. Thus, after the Fourier transform, the left-hand side of our integral equation acquires the following form:

$$F[l.h.s] = \pi \mathrm{e}^{-\frac{2n+1}{2}|k|} J_0(\chi k). \tag{20}$$

On the right, the Fourier integral is decomposed into the following terms:

$$\int\limits_{-\infty}^{\infty} d\mu\, \rho(\mu) \int\limits_{-\infty}^{\infty} \mathrm{e}^{iku} \frac{A}{A^2 + (u-\mu)^2} du = \pi \int\limits_{-\infty}^{\infty} d\mu\, \rho(\mu) \mathrm{e}^{i\mu k} \mathrm{e}^{-|k|A} = \pi \mathrm{e}^{-|k|A} \widetilde{\rho}(k), \tag{21}$$

where $\widetilde{\rho}(k)$ stands for the Fourier transform of the density $\rho(u)$. These terms can be summed up:

$$\pi\widetilde{\rho}(k) \left(2 \sum_{m=0}^{2n-1} \mathrm{e}^{-|k|(m+1)} + \mathrm{e}^{-|k|(2n+1)} \right) = \pi\widetilde{\rho}(k) \left(\frac{2 - \mathrm{e}^{-2n|k|} - \mathrm{e}^{-(2n+1)|k|}}{\mathrm{e}^{|k|} - 1} \right). \tag{22}$$

As a result, we obtain the following expression for the Fourier transformation of the density $\widetilde{\rho}(k)$:

$$\widetilde{\rho}(k) = J_0(\chi k) \frac{\mathrm{e}^{|k|} - 1}{\mathrm{e}^{-\frac{2n+1}{2}|k|}(1 + \mathrm{e}^{|k|})(\mathrm{e}^{(2n+1)|k|} - 1)} = \frac{J_0(\chi k) \tanh \frac{|k|}{2}}{2 \sinh \left(n + \frac{1}{2}\right) |k|}. \tag{23}$$

This yields a solution of the integral equation:

$$\rho(u) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{J_0(\chi k) \tanh \frac{|k|}{2}}{2 \sinh \left(n + \frac{1}{2}\right) |k|} \mathrm{e}^{iku} dk. \tag{24}$$

For $n = 0$ (i.e., for strings of length 1), this expression coincides with the expression for the density that was obtained in [9, 10].

3.4. Energy of strings of odd length

Now we compute the energy (or, equivalently, the maximal dimension of the dilatation operator $\Delta_{\max}$) for the state filled with strings of length $2n+1$, where n is integer. The dispersion in the considered theory differs from that of the XXX magnet; the corresponding energy density is as follows (see [9]):

$$\frac{\Delta_{\max}}{L} = 1 + \frac{i\lambda}{8\pi^2} \int\limits_{-\infty}^{\infty} du\, \rho(u) \left(\frac{1}{x\left(u + \frac{2n+1}{2} i\right)} - \frac{1}{x\left(u - \frac{2n+1}{2} i\right)} \right). \tag{25}$$

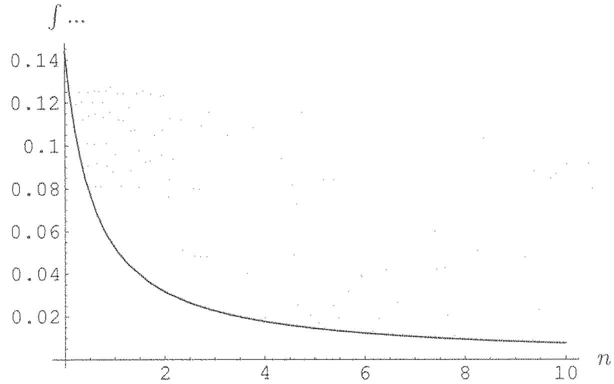

FIG. 1

Substituting into (25) expression (24) for the density, we see that

$$\frac{\Delta_{\max}}{L} = 1 + \frac{\sqrt{\lambda}}{\pi}\int\limits_0^\infty \frac{dk}{k}\frac{J_0(\chi k)J_1(\chi k)\tanh\frac{k}{2}}{\mathrm{e}^{(2n+1)k}-1}. \tag{26}$$

Here we have used the following relation:

$$\frac{\chi^2}{2}\left(\frac{i}{\sqrt{(u-il)^2-\chi^2}} - \frac{i}{\sqrt{(u+il)^2-\chi^2}}\right) = \frac{1}{2}\chi\partial_\chi\left[\frac{\chi^2}{2i}\left(\frac{1}{x(u+il)} - \frac{1}{x(u-il)}\right)\right]. \tag{27}$$

In Fig. 1, we provide a plot which shows how the second term in (26) depends on n for a fixed value of λ.

It is apparent that the maximum of the integral (thus, the maximum of $\Delta_{\max}$) is attained at $n = 0$. In the Appendix, we give a strict proof of monotone decrease of $\Delta_{\max}$ as n grows.

Thus, we conclude that the state with maximal energy in the sector of strings of odd length indeed corresponds to strings of length 1 (as was assumed in [9, 10]). However, in the next section we show that the real anti-ferromagnetic vacuum may have a more complicated structure and possibly corresponds to strings of length 2.

4. Strings of even length

Unlike the absolute value $|x(u)|$, the phase of $x(u)$ is not a continuous function. Its value changes by a finite amount when u crosses the cut. Using formula (7), it is easy to show that if $u \in [-\chi, \chi]$, then the following relation holds:

$$\lim_{\epsilon\to 0}\log\frac{x(u-i\epsilon)}{x(u+i\epsilon)} = 2i\,\nu_\epsilon\,\arctan\frac{\sqrt{\chi^2-u^2}}{u}, \tag{28}$$

where $\nu_\epsilon = -1$ if ϵ tends to zero from the right and $\nu_\epsilon = 1$ if ϵ tends to zero from the left.

For a chain of a large but finite length L, roots of the Bethe equations group into strings only up to exponential deviations: $u_{j,m} = u_j + i(m+\epsilon_m)$, where m is integer or half-integer and $\epsilon_m = O(e^{-\omega_m L})$, $\omega_m > 0$. The total energy of a string is given by $\mathcal{E}_j = \frac{i}{2}\sum\limits_m\left(\frac{1}{x(u_{j,m}+\frac{i}{2})} - \frac{1}{x(u_{j,m}-\frac{i}{2})}\right)$. Computing this expression for a string of even length and taking relation (28) into account, we observe that the roots

$$u_j + i\Big(\frac{1}{2}+\epsilon\Big) \quad\text{and}\quad u_j - i\Big(\frac{1}{2}+\epsilon\Big) \tag{29}$$

give us additional contributions,

$$\mathcal{E}_j = \lim_{\epsilon\to 0}\frac{i}{2}\left(\frac{1}{x(u_j - i\epsilon)} - \frac{1}{x(u_j+i\epsilon)} + \frac{1}{x(u_j+\frac{2n+1}{2}i)} - \frac{1}{x(u_j-\frac{2n+1}{2}i)}\right). \tag{30}$$

It is important to note that for the regime corresponding to $\nu_\epsilon = -1$, the first two terms in (30) give a negative contribution to $\mathcal{E}_j$. Furthermore, it can be shown that in this regime, $\mathcal{E}_j$ is positive not everywhere on the real axis. As a consequence, the anti-ferromagnet vacuum in this case has a more complicated structure: it has to be filled with strings only in the intervals where $\mathcal{E}_j > 0$. In the present work, we only consider the regime corresponding to $\nu_\epsilon = 1$. Let us note that in this case, the proof of convexity of Yang's action given in Sec. 3.2 remains valid.

Taking the product of the left-hand sides of the Bethe equations (4) along a string of even length and taking into account additional contributions due to roots (29), we conclude that the left-hand side of the equation for centers of strings looks as follows:

$$\lim_{\epsilon\nearrow 0}\left(\frac{x(u_j-i\epsilon)}{x(u_j+i\epsilon)}\right)^L\left(\frac{x(u_j+\frac{i}{2}(2n+1))}{x(u_j-\frac{i}{2}(2n+1))}\right)^L. \tag{31}$$

Taking logarithm of this expression, we obtain Eqs. (12) but with an additional term on the left. Differentiating (28) in u, applying the Fourier transformation,

$$F\Big[\partial_u \lim_{\epsilon\to 0}\log\frac{x(u-i\epsilon)}{x(u+i\epsilon)}\Big] = -2i\,\nu_\epsilon\int_{-\chi}^{\chi}\frac{e^{iku}\,du}{\sqrt{\chi^2-u^2}} = -2\pi i\,\nu_\epsilon\,J_0(\chi k), \tag{32}$$

and comparing the result with formula (19), we see that, for strings of even length in the regime $\nu_\epsilon = 1$, Eq. (20) acquires the following form:

$$F[l.h.s] = \pi\left(e^{-\frac{2n+1}{2}|k|}+1\right)J_0(\chi k). \tag{33}$$

Repeating the procedure of computation of the Fourier image of the density of Sec. 3.3, we find

$$\rho_\epsilon(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{J_0(\chi k)\big(1+e^{\frac{2n+1}{2}|k|}\big)\tanh\frac{|k|}{2}}{2\sinh\left(n+\frac{1}{2}\right)|k|}\,e^{iku}dk. \tag{34}$$

Formula (33) shows that additional contributions from roots (29) yield the same effect as if the vacuum was filled with mixture of strings of length $(2n{+}1)$ and strings of length 0 (which formally corresponds to $n = -\frac{1}{2}$). In this context, we remark that expressions of type (31) appear in left-hand sides of the Bethe equations for chains with alternating spins (see, e.g., [13, 14]).

Formula (30) implies that, in order to compute the total energy of a chain, we have to replace (25) by

$$\frac{\Delta}{L} = 1+\frac{i\chi^2}{2}\lim_{\epsilon\nearrow 0}\int_{-\infty}^{\infty}du\,\rho_\epsilon(u)\left(\frac{1}{x(u-i\epsilon)}-\frac{1}{x(u+i\epsilon)}+\frac{1}{x\left(u+\frac{2n+1}{2}i\right)}-\frac{1}{x\left(u-\frac{2n+1}{2}i\right)}\right). \tag{35}$$

Substituting (34) into (35) and making the same computations in the Sec. 3.4, we see that

$$\frac{\Delta}{L} = 1+2\chi\int_0^{\infty}\frac{dk}{k}\,J_0(\chi k)J_1(\chi k)\,\tanh\frac{k}{2}\,\coth\frac{(2n{+}1)k}{4}. \tag{36}$$

Using the method described in the Appendix, one can show that for $n \geq \frac{1}{2}$, expression (36) decreases monotonically as n grows. The maximal value of (36) at $n = \frac{1}{2}$ is given by

$$\frac{\Delta_{\max}}{L} = 1+2\chi\int_0^{\infty}\frac{dk}{k}\,J_0(\chi k)J_1(\chi k). \tag{37}$$

In order to compare this expression with the value of $\frac{\Delta_{\max}}{L}$ corresponding to strings of length 1, one can subtract the value of (26) for $n = 0$ from (37). The resulting expression has the form of the integral in (38) with a function $f(k)$ that is monotonically decreasing in k. As is shown in the Appendix, such an integral is positive.

Thus, filling the vacuum with strings of length 2 in the regime $\nu_\epsilon = 1$ yields a larger value for $\frac{\Delta_{\max}}{L}$ than filling it with strings of length 1. However, which regime is indeed realized for strings of even length in the thermodynamic limit, remains at present an open problem. Its solution requires a quite subtle analysis of the exponential corrections ϵ in (29) as $L \to \infty$.

Appendix

Let us denote $\alpha = (2n+1)$ and $\chi = \sqrt{\lambda}/(2\pi)$ and make in (26) a substitution $k' = \chi k$. Then

$$\partial_\alpha \Big(\frac{\Delta_{\max}}{L} \Big) = -2 \int_0^\infty dk\, J_0(k)\, J_1(k)\, f(k), \tag{38}$$

where

$$f(k) = \left(\frac{1}{e^{\frac{k}{\chi}} + 1} \right) \left(\frac{e^{\frac{\alpha k}{\chi}}}{e^{\frac{\alpha k}{\chi}} - 1} \right) \left(\frac{e^{\frac{k}{\chi}} - 1}{e^{\frac{\alpha k}{\chi}} - 1} \right). \tag{39}$$

In this form, it is obvious that $f(k)$ decreases monotonically as k grows for all $\alpha > 1$ and $\chi > 0$. Using this fact, we show that the integral in (38) is positive.

Let $0 < t_1 < t_3 < t_5 \ldots$ be the ordered set of roots of $J_0(t)$, and let $0 = t_0 < t_2 < t_4 \ldots$ be the ordered set of roots of $J_1(t)$. It follows from the relation

$$\partial_t J_0(t) = -J_1(t) \tag{40}$$

that $t_{2n} < t_{2n+1} < t_{2n+2}$. Since $J_0(t)J_1(t)$ is positive on $]t_{2n}, t_{2n+1}[$ and negative on $]t_{2n+1}, t_{2n+2}[$ and the function $f(k)$ is positive and monotonically decreasing for all $k > 0$, we obtain the following estimate:

$$\int_0^\infty dk\; J_0(k)\, J_1(k)\, f(k) = \sum_{n=0}^\infty \int_{t_{2n}}^{t_{2n+2}} dk\, J_0(k)\, J_1(k)\, f(k) > \sum_{n=0}^\infty \Big[f(t_{2n+1}) \int_{t_{2n}}^{t_{2n+2}} dk\, J_0(k)\, J_1(k) \Big]$$

$$= \frac{1}{2} \sum_{n=0}^\infty \Big[f(t_{2n+1}) \left(J_0^2(t_{2n}) - J_0^2(t_{2n+2}) \right) \Big]. \tag{41}$$

In the last equality, we have used relation (40). Now, since roots of $J_1(t)$ are points of local extrema for $J_0(t)$ and the values of $|J_0(t)|$ at these points form a decreasing sequence, we conclude that the sum on the right in (41) (and hence the initial integral) is positive. Thus, $\partial_\alpha \big(\frac{\Delta_{\max}}{L} \big) < 0$, which implies that $\frac{\Delta_{\max}}{L}$ decreases monotonically as n grows.

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