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ON CONSTANT $U_q(sl_2)$-INVARIANT R-MATRICES

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We consider the spectral resolution of a $U_q(sl_2)$-invariant solution R of the constant Yang–Baxter equation in the braid group form. It is shown that if the two highest coefficients in this resolution are not equal, then R is either the Drinfeld R-matrix or its inverse. Bibliography: 13 titles.

§1. Introduction

Recall that the algebra $U_q(sl_2)$ is generated by generators X_+, X_-, q^H, q^{-H} satisfying the following relations (see [9]):

$$[X^+, X^-] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}, \quad q^H X^\pm = q^{\pm 1} X^\pm q^H, \quad \text{and} \quad q^{\pm H} q^{\mp H} = 1. \tag{1}$$

The homomorphism Δ which is defined on the generators as follows:

$$\Delta(X^\pm) = X^\pm \otimes q^{-H} + q^H \otimes X^\pm \quad \text{and} \quad \Delta(q^{\pm H}) = q^{\pm H} \otimes q^{\pm H} \tag{2}$$

turns $U_q(sl_2)$ into a bialgebra (moreover, into a Hopf algebra [12]).

We consider the standard finite-dimensional representation π_s of the algebra $U_q(sl_2)$ in which generators act on basis vectors ω_k of a module V_s ($\dim V_s = (2s+1)$, $2s \in \mathbb{N}$) as follows:

$$\pi_s(X^\pm)\,\omega_k = \sqrt{[s \mp k][s \pm k + 1]}\;\omega_{k\pm 1} \quad \text{and} \quad \pi_s(q^{\pm H})\,\omega_k = q^{\pm k}\,\omega_k, \tag{3}$$

where $[t] \equiv (q^t - q^{-t})/(q - q^{-1})$ and $k = -s, -s+1, \ldots, s$.

The universal R-matrices for algebra (1)–(2) are given by

$$R^\pm = q^{\pm H\otimes H} \sum_{n=0}^{\infty} \frac{q^{\pm\frac{1}{2}(n^2-n)}}{\prod_{k=1}^{n} [k]_q} \left(\pm(q-q^{-1}) X^\mp \otimes X^\pm\right)^n q^{\pm H\otimes H} \tag{4}$$

(see [6]).

Let $\mathbb{P}$ denote the operator which permutes the tensor components in $U_q(sl_2)^{\otimes 2}$. Then the operator

$$R \equiv \mathbb{P}\, R^+ = (R^-)^{-1}\, \mathbb{P}$$

satisfies the Yang–Baxter equation in the braid group form:

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}. \tag{5}$$

The spectral resolution of R in the representation π_s is given by

$$\mathsf{R} \equiv \pi_s^{\otimes 2}(R) = \sum_{k=0}^{2s} \xi_k\, \mathsf{P}^{2s-k} \tag{6}$$

(see [8]), where P^j stands for the projector onto the irreducible submodule V_j in $V_s^{\otimes 2} = \oplus_{j=0}^{2s} V_j$. Here and below, we use the following notation:

$$\xi_k \equiv (-1)^k\, q^{\rho(2s-k) - 2\rho(s)} \quad \text{and} \quad \rho(t) \equiv t(t+1). \tag{7}$$

Consider a $U_q(sl_2)$-invariant solution R' of the Yang–Baxter equation (5). Its spectral resolution in the representation π_s is given by

$$\mathsf{R}' \equiv \pi_s^{\otimes 2}(R') = \sum_{k=0}^{2s} r_k\, \mathsf{P}^{2s-k}, \tag{8}$$

where $r_0 \neq 0$ by Lemma 6 of [4], which applies to the case $q \neq 1$ as well. We prove the following statement.

Proposition 1. *If $r_1 \neq r_0$ in the spectral resolution* (8), *then R' coincides either with R or with R^{-1} up to normalization.*

This statement is a q-analogue of the second part of Proposition 1 of [4], where sl_2-invariant solutions of the Yang–Baxter equation were considered. Note that the limit $q \to 1$ is degenerate in the sense that both operators R and R^{-1} turn into the permutation operator $\mathbb{P}$.

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§2. Reduction to the subspace $W_n^{(s)}$

Let us recall the method of analyzing $U_q(sl_2)$-invariant solutions of the Yang–Baxter equation developed in [3]. Let $\lfloor t \rfloor$ denote the entire part of t. The subspace $W_n^{(s)} \subset V_s^{\otimes 3}$ for $n = 0, 1, \dots, \lfloor 3s \rfloor$ is defined as the span of the highest weight vectors of weight $(3s - n)$, i.e.,

$$W_n^{(s)} = \{ \psi \in V_s^{\otimes 3} \;\; | \;\; \mathsf{X}^+_{123}\psi = 0 \,, \quad q^{\mathsf{H}_{123}}\psi = q^{3s-n}\psi \}. \tag{9}$$

Here and below, for $O \in U_q(sl_2)$ we use the notation $\mathsf{O}_{123} = \pi_s^{\otimes 3}((\Delta \otimes id)\Delta(O))$.

Since $[\mathsf{X}^{\pm}_{123}, \mathsf{R}_{12}] = [\mathsf{X}^{\pm}_{123}, \mathsf{R}_{23}] = 0$, $W_n^{(s)}$ is an invariant subspace for R_{12} and R_{23}; thus, we can consider reductions of these operators to $W_n^{(s)}$. We can choose a basis of $W_n^{(s)}$ in which the operator $\mathsf{R}_{12}\big|_{W_n^{(s)}}$ is represented by a diagonal matrix $D_0^{(n)}$ of the following form:

$$\big(D_0^{(n)}\big)_{kk'} = \delta_{kk'}\, \xi_k ; \tag{10}$$

here $0 \le k \le n$ for $0 \le n \le 2s$ and $(n - 2s) \le k \le (4s - n)$ for $2s \le n \le \lfloor 3s \rfloor$.

In the same basis, the operator $\mathsf{R}_{23}\big|_{W_n^{(s)}}$ is represented by the following matrix:

$$\hat{D}_0^{(n)} = A^{(s,n)}\; D_0^{(n)}\; A^{(s,n)}, \tag{11}$$

where $A^{(s,n)}$ is a matrix with the following properties (see [3]): it is symmetric, orthogonal, equal to its inverse, and self–dual in q:

$$A^{(s,n)} = \big(A^{(s,n)}\big)^t = \big(A^{(s,n)}\big)^{-1} \quad \text{and} \quad A_q^{(s,n)} = A_{q^{-1}}^{(s,n)}. \tag{12}$$

Entries of this matrix are expressed in terms of 6–j symbols of the algebra $U_q(sl_2)$ as follows:

$$A_{kk'}^{(s,n)} = (-1)^{2s-n}\sqrt{[4s-2k+1]_q[4s-2k'+1]_q}\; \begin{Bmatrix} s & s & 2s-k \\ s & 3s-n & 2s-k' \end{Bmatrix}_q . \tag{13}$$

The statement that the Yang–Baxter equation (5) holds when it is reduced to the subspace $W_n^{(s)}$ is equivalent to the following equality:

$$\big(D_0^{(n)}\, A^{(s,n)}\big)^3 = \big(A^{(s,n)}\, D_0^{(n)}\big)^3. \tag{14}$$

In fact, however, a stronger statement holds: The r.h.s. and l.h.s. of (14) are equal up to a multiplicative constant to the identity operator on $W_n^{(s)}$. This follows from the following statement (which is a q-analogue of Lemma 3 of [4]).

Lemma 1. *For all $n = 0, \dots, \lfloor 3s \rfloor$, the following relation holds:*

$$A^{(s,n)}\, D_0^{(n)}\, A^{(s,n)} = \theta_n\, \big(D_0^{(n)}\big)^{-1} A^{(s,n)}\, \big(D_0^{(n)}\big)^{-1}, \tag{15}$$

where $\theta_n \equiv (-1)^n\, q^{\rho(3s-n)-3\rho(s)}$.

The proof of this and other lemmas is given in the Appendix.

The statement of Lemma 1 can be written in the following form:

$$\big(\mathsf{R}_{12}\, \mathsf{R}_{23}\, \mathsf{R}_{12}\big)\big|_{W_n^{(s)}} = \big(\mathsf{R}_{23}\, \mathsf{R}_{12}\, \mathsf{R}_{23}\big)\big|_{W_n^{(s)}} = \theta_n\, A^{(s,n)}. \tag{16}$$

For $q = 1$, this relation turns into $(\mathbb{P}_{13})\big|_{W_n^{(s)}} = (-1)^n A^{(s,n)}$.

From (16) and (12) it follows that

$$\big((\mathsf{R}_{12}\, \mathsf{R}_{23})^3\big)\big|_{W_n^{(s)}} = \big((\mathsf{R}_{23}\, \mathsf{R}_{12})^3\big)\big|_{W_n^{(s)}} = q^{2\rho(3s-n)-6\rho(s)}. \tag{17}$$

Let us note that

$$\big(\mathsf{R}_{12}\, \mathsf{R}_{23}\, \mathsf{R}_{12}\big)^2 = \big(\mathsf{R}_{23}\, \mathsf{R}_{12}\, \mathsf{R}_{23}\big)^2 = \big(\mathsf{R}_{12}\, \mathsf{R}_{23}\big)^3 = \big(\mathsf{R}_{23}\, \mathsf{R}_{12}\big)^3 \tag{18}$$

$$= \pi_s^{\otimes 3}\Big(\big(R^-_{12}R^-_{13}R^-_{23}\big)^{-1}\big(R^+_{12}R^+_{13}R^+_{23}\big)\Big) = \pi_s^{\otimes 3}(\chi_1\chi_2\chi_3\, \Delta^{(2)}(\chi^{-1})), \tag{19}$$

where the element χ is constructed in the following way: Write the R-matrix (4) as $R^+ = \sum_a r_a^{(1)} \otimes r_a^{(2)}$, and let $\mathcal{S}$ stand for the antipode operation; then $\chi = q^{2H}\big(\sum_a \mathcal{S}(r_a^{(2)})r_a^{(1)}\big)$. It is known [7] that the element χ is central, $\pi_s(\chi) = q^{-2\rho(s)}$, and $\chi_1\chi_2\, \Delta(\chi^{-1}) = \big(R^-\big)^{-1}R^+$. The last relation allows us to derive the last equality in (19) (and its generalization for $\Delta^{(N)}(\chi^{-1})$, see the proof of Lemma 1 in [5]). Thus, relation (16) can be regarded as the definition of a certain square root of the operator given by the r.h.s. of (19).

§3. Yang–Baxter equation on $W_n^{(s)}$

We prove Proposition 1 using the following statement (a q-analogue of Lemma 4 of [4]).

Lemma 2. *Let $0 \le \overline{m} \le n \le 2s$, where $\overline{m} \equiv (2s-m)$. The reductions of the operators P_{12}^m, P_{23}^m, $\mathsf{R}_{12}^{\pm 1}$, and $\mathsf{R}_{23}^{\pm 1}$ to $W_n^{(s)}$ satisfy the following relations:*

$$\begin{aligned} \mathsf{R}_l\, \mathsf{R}_{l'}\, \mathsf{R}_l &= \mathsf{R}_{l'}\, \mathsf{R}_l\, \mathsf{R}_{l'}, \\ \mathsf{P}_l^m\, \mathsf{P}_{l'}^m\, \mathsf{P}_l^m &= \eta_{n,\overline{m}}^2\, \mathsf{P}_l^m, \end{aligned} \tag{20}$$

$$\begin{aligned} \mathsf{P}_l^m\, \mathsf{R}_{l'}^{\pm 1}\, \mathsf{P}_l^m &= (\theta_n \xi_{\overline{m}}^{-2})^{\pm 1} \eta_{n,\overline{m}}\, \mathsf{P}_l^m, \\ \mathsf{R}_l^{\pm 1}\, \mathsf{P}_{l'}^m\, \mathsf{R}_l^{\pm 1} &= (\theta_n \xi_{\overline{m}}^{-1})^{\pm 2}\, \mathsf{R}_{l'}^{\mp 1}\, \mathsf{P}_l^m\, \mathsf{R}_{l'}^{\mp 1}, \end{aligned} \tag{21}$$

$$\begin{aligned} \mathsf{P}_l^m\, \mathsf{P}_{l'}^m\, \mathsf{R}_l^{\pm 1} &= (\theta_n \xi_{\overline{m}}^{-1})^{\pm 1} \eta_{n,\overline{m}}\, \mathsf{P}_l^m\, \mathsf{R}_{l'}^{\mp 1}, \\ \mathsf{R}_l^{\pm 1}\, \mathsf{P}_{l'}^m\, \mathsf{P}_l^m &= (\theta_n \xi_{\overline{m}}^{-1})^{\pm 1} \eta_{n,\overline{m}}\, \mathsf{R}_{l'}^{\mp 1}\, \mathsf{P}_l^m, \end{aligned} \tag{22}$$

where $l = \{12\}$, $l' = \{23\}$ or $l = \{23\}$, $l' = \{12\}$, and $\eta_{n,\overline{m}} = A_{\overline{m},\overline{m}}^{(s,n)}$.

Let us remark that not all the relations in Lemma 2 are independent. For instance, the second relation in (21) follows from (22); the first relation in (21) and the second relation in (20) can be derived from each other with the help of (22).

Let us also remark that, for $q=1$, the operators $\mathsf{R}_l^{\pm 1}$ coincide with the permutation operator $\mathbb{P}_l$, and relations (20)–(22) become the relations of the Brauer algebra [2] (taking into account the additional relation $\mathbb{P}_l^2 = \mathbb{E}$, where $\mathbb{E}$ is the identity operator). For $q \neq 1$, the reductions of the operators $\mathsf{R}_l^{\pm 1}$ to $W_1^{(s)}$ can be represented as linear combinations of P_l^m and the identity operator $\mathbb{E}$. As a consequence, relations (20)–(22) for $n=1$ can be derived from the second relation in (20), which is the defining relation for the Temperley–Lieb algebra [13]. For $n \ge 2$, relations (20)–(22) are the relations that hold in the Birman–Wenzl–Murakami algebra [1, 10]. However, in this algebra an additional relation must also hold, which in our case holds only for $n=2$ (the operator R_l^{-1} being reduced to $W_2^{(s)}$ can be represented as a linear combination of the operators R_l, P_l^m, and $\mathbb{E}$).

Returning to consideration of the spectral resolution (8), let us note that, without loss of generality, we can set $r_0 = \xi_0$. Then R' can be represented in the following form:

$$\mathsf{R}' = \mathsf{R} + g\, \mathsf{P}^{2s-n} + \dots , \tag{23}$$

where $n \ge 1$ and $\dots$ stands for the sum involving projectors whose ranks are smaller than the rank of P^{2s-n}.

Substitute ansatz (23) in the Yang–Baxter equation and consider its reduction to $W_n^{(s)}$ for $n \le 2s$. With the help of relations of Lemma 2, one can verify that the Yang–Baxter equation for $\mathsf{R}'|_{W_n^{(s)}}$ is equivalent to the following matrix equation:

$$g\, \mathsf{J} + (\theta_n \xi_n^{-2} \eta_{n,n}\, g^2 + \eta_{n,n}^2\, g^3)\, \mathsf{G} + (\theta_n \xi_n^{-1} \eta_{n,n}\, g^2)\, \mathsf{H} = 0, \tag{24}$$

where

$$\begin{aligned} \mathsf{G} &= \left(\mathsf{P}_{12}^{2s-n} - \mathsf{P}_{23}^{2s-n}\right)\big|_{W_n^{(s)}} = \pi^{(n)} - A^{(s,n)} \pi^{(n)} A^{(s,n)}, \\ \mathsf{J} &= \left(\mathsf{R}_{12}\, \mathsf{P}_{23}^{2s-n}\, \mathsf{R}_{12} - \mathsf{R}_{23}\, \mathsf{P}_{12}^{2s-n}\, \mathsf{R}_{23}\right)\big|_{W_n^{(s)}} \\ &= D_0^{(n)} A^{(s,n)} \pi^{(n)} A^{(s,n)} D_0^{(n)} - \theta_n^2 \xi_n^{-2} (D_0^{(n)})^{-1} A^{(s,n)} \pi^{(n)} A^{(s,n)} (D_0^{(n)})^{-1}, \end{aligned}$$

and

$$\begin{aligned} \mathsf{H} &= \left(\mathsf{P}_{12}^{2s-n}\, \mathsf{R}_{23}^{-1} + \mathsf{R}_{23}^{-1}\, \mathsf{P}_{12}^{2s-n} - \mathsf{P}_{23}^{2s-n}\, \mathsf{R}_{12}^{-1} - \mathsf{R}_{12}^{-1}\, \mathsf{P}_{23}^{2s-n}\right)\big|_{W_n^{(s)}} \\ &= \theta_n^{-1} \xi_n \left(\pi^{(n)} A^{(s,n)} D_0^{(n)} + D_0^{(n)} A^{(s,n)} \pi^{(n)}\right) \\ &\quad - A^{(s,n)} \pi^{(n)} A^{(s,n)} (D_0^{(n)})^{-1} - (D_0^{(n)})^{-1} A^{(s,n)} \pi^{(n)} A^{(s,n)}. \end{aligned}$$

Here $\pi^{(n)}$ is the matrix such that $(\pi^{(n)})_{kk'} = \delta_{kn} \delta_{k'n}$.

Lemma 3. (i) *For $n=1$, the following relations hold:*

$$\begin{aligned} \mathsf{J} &= (\theta_1^2\xi_0^{-2}\xi_1^{-2} - \xi_0^2)\mathsf{G} = \left(q^{4s(s-1)} - q^{4s^2}\right)\mathsf{G}, \\ \mathsf{H} &= 2\xi_0^{-1}\,\mathsf{G} = 2q^{-2s^2}\,\mathsf{G}. \end{aligned} \tag{25}$$

(ii) *For $n=2$, the matrices J and G are linearly independent, and the following relation holds:*

$$\xi_0\xi_1\,\mathsf{H} = (\xi_0+\xi_1)\,\mathsf{G} + (\xi_0+\xi_1)^{-1}\,\mathsf{J}. \tag{26}$$

(iii) *For $n\geq 3$, the matrices J, G, and H are linearly independent, and $\mathsf{J}\neq 0$.*

Substituting relations (25) in (24), we infer that, for $n=1$, the coefficient g must be a root of the following equation:

$$\eta_{1,1}^2\,g^3 + \eta_{1,1}\theta_1\xi_1^{-1}(\xi_1^{-1}+2\xi_0^{-1})\,g^2 + (\theta_1^2\xi_0^{-2}\xi_1^{-2}-\xi_0^2)\,g = 0.$$

Hence, taking into account that $\eta_{1,1} = -\left(q^{2s}+q^{-2s}\right)^{-1}$, we conclude that, for $n=1$, the coefficient g can take one of the following values: $g=0$, $g=q^{2s(s-2)}(1-q^{8s})$, and $g=q^{2s(s-2)}(1+q^{4s})$. In the first and second cases, the spectral resolution of R' coincides in the two highest orders with that of R and $q^{4s^2}\mathsf{R}^{-1}$, respectively. In the third case, $r_1=r_0$.

For $n=2$, substitute relations (26) in (24) and eliminate H. It is easy to check that the resulting coefficients at J and G vanish if either $g=0$ or

$$\eta_{1,1}\,g = -\theta_2\xi_0^{-1}\xi_1^{-1}\xi_2^{-1}(\xi_0\xi_1\xi_2^{-1}+\xi_0+\xi_1) = -\theta_2^{-1}\xi_0\xi_1\xi_2(\xi_0+\xi_1).$$

However, the last equality cannot hold because $\xi_0^2\xi_1^2\xi_2^2 = \theta_2^2$ (see Eq. (34)).

For $n\geq 3$, the coefficient at J in (24) vanishes only if $g=0$. Thus, the coefficient g in (23) must be zero if $n\geq 2$. Therefore, if R' coincides with R in the two highest orders, then $\mathsf{R}'=\mathsf{R}$. An analogous statement can be established if we consider ansatz (23) with R replaced by R^{-1}. Thus, Proposition 1 is proven.

Appendix

Proof of Lemma 1. The 6–j symbols of the algebra $U_q(sl_2)$ satisfy the following q-analogue of the Racah identity [8, 11]:

$$\begin{aligned} &\sum_p \left((-1)^p\,[2p+1]_q \begin{Bmatrix} r_1 & r_3 & l \\ r_2 & r_4 & p \end{Bmatrix}_q q^{\rho(p)-\rho(r_1)-\rho(r_4)} \begin{Bmatrix} r_1 & r_2 & l' \\ r_3 & r_4 & p \end{Bmatrix}_q \right) \\ &\quad = (-1)^{l+l'}\,q^{\rho(r_2)-\rho(l)} \begin{Bmatrix} r_3 & r_1 & l \\ r_2 & r_4 & l' \end{Bmatrix}_q q^{\rho(r_3)-\rho(l')}. \end{aligned} \tag{27}$$

(Note that the identity remains true if we set $\rho(t) = -t(t+1)$ since the 6–j symbols are self-dual with respect to the replacement $q\to q^{-1}$.)

Consider the matrix entry (kk') of equality (15). Using formula (10) and taking into account that $A^{(s,n)}$ is a symmetric matrix, we conclude that

$$\sum_m (-1)^m\,A^{(s,n)}_{km}\,q^{\rho(2s-m)-2\rho(s)}\,A^{(s,n)}_{k'm} = (-1)^{n+k+k'}\,q^{\rho(3s-n)+\rho(s)-\rho(2s-k)-\rho(2s-k')}\,A^{(s,n)}_{kk'}. \tag{28}$$

Now, taking into account formula (13), it is easy to see that relation (28) follows from identity (27) if we set $r_1=r_2=r_3=s$, $r_4=3s-n$, $l=2s-k$, $l'=2s-k'$, and $p=2s-m$.

Proof of Lemma 2. We prove those relations of Lemma 2 that contain R^{+1} on the l.h.s. Their counterparts with R^{-1} on the l.h.s. can be proven similarly.

The second relation in (20):

$$\pi^{(\overline{m})}\hat{\pi}^{(\overline{m})}\pi^{(\overline{m})} = \pi^{(\overline{m})}A^{(s,n)}\pi^{(\overline{m})}A^{(s,n)}\pi^{(\overline{m})} = (A^{(s,n)}_{\overline{m}\overline{m}})^2\pi^{(\overline{m})}.$$

Here and below, we denote $\hat{\pi}^{(\overline{m})} \equiv A^{(s,n)}\pi^{(\overline{m})}A^{(s,n)}$.

Relations (21):

$$
\begin{aligned}
\pi^{(\overline{m})}\hat{D}_0^{(n)}\pi^{(\overline{m})} &\overset{(11)}{=} \pi^{(\overline{m})}A^{(s,n)}D_0^{(n)}A^{(s,n)}\pi^{(\overline{m})}\\
&\overset{(15)}{=} \theta_n\pi^{(\overline{m})}(D_0^{(n)})^{-1}A^{(s,n)}(D_0^{(n)})^{-1}\pi^{(\overline{m})}\\
&\overset{(10)}{=} \theta_n\xi_{\overline{m}}^{-2}\pi^{(\overline{m})}A^{(s,n)}\pi^{(\overline{m})} = \theta_n\xi_{\overline{m}}^{-2}A^{(s,n)}_{\overline{mm}}\pi^{(\overline{m})},\\
D_0^{(n)}\hat{\pi}^{(\overline{m})}D_0^{(n)} &= D_0^{(n)}A^{(s,n)}\pi^{(\overline{m})}A^{(s,n)}D_0^{(n)}\\
&\overset{(10)}{=} \xi_{\overline{m}}^{-2}D_0^{(n)}A^{(s,n)}D_0^{(n)}\pi^{(\overline{m})}D_0^{(n)}A^{(s,n)}D_0^{(n)}\\
&\overset{(15)}{=} \theta_n^2\xi_{\overline{m}}^{-2}A^{(s,n)}(D_0^{(n)})^{-1}A^{(s,n)}\pi^{(\overline{m})}A^{(s,n)}(D_0^{(n)})^{-1}A^{(s,n)}\\
&\overset{(11)}{=} \theta_n^2\xi_{\overline{m}}^{-2}(\hat{D}_0^{(n)})^{-1}\pi^{(\overline{m})}(\hat{D}_0^{(n)})^{-1}.
\end{aligned}
$$

The first relation in (22) (the second one can be proven similarly):

$$
\begin{aligned}
\pi^{(\overline{m})}\hat{\pi}^{(\overline{m})}D_0^{(n)} &= \pi^{(\overline{m})}A^{(s,n)}\pi^{(\overline{m})}A^{(s,n)}D_0^{(n)} = A^{(s,n)}_{\overline{mm}}\pi^{(\overline{m})}A^{(s,n)}D_0^{(n)}(A^{(s,n)})^2\\
&\overset{(15)}{=} \theta_n A^{(s,n)}_{\overline{mm}}\pi^{(\overline{m})}(D_0^{(n)})^{-1}A^{(s,n)}(D_0^{(n)})^{-1}A^{(s,n)} \overset{(10)}{=} \theta_n\xi_{\overline{m}}^{-1}A^{(s,n)}_{\overline{mm}}\pi^{(\overline{m})}(\hat{D}_0^{(n)})^{-1}.
\end{aligned}
$$

Proof of Lemma 3. For $n=1$, the matrices G, H, and J are of size 2×2, and relations (25) can be verified straightforwardly using the explicit form of the matrix $A^{(s,1)}$ (see Eq. (73) of [3]).

In order to examine the case $n\geq 2$, let us write down explicitly the matrix entries of G, H, and J:

$$
\mathsf{G}_{kk'} = \delta_{kn}\,\delta_{k'n} - A^{(s,n)}_{nk}A^{(s,n)}_{nk'}, \tag{29}
$$

$$
\mathsf{H}_{kk'} = \theta_n^{-1}\xi_n(\delta_{kn}\,\xi_{k'}A^{(s,n)}_{nk'} + \delta_{k'n}\,\xi_k A^{(s,n)}_{nk}) - (\xi_k^{-1}+\xi_{k'}^{-1})\,A^{(s,n)}_{nk}A^{(s,n)}_{nk'}, \tag{30}
$$

and

$$
\mathsf{J}_{kk'} = (\xi_k\xi_{k'} - \theta_n^2\xi_n^{-2}\xi_k^{-1}\xi_{k'}^{-1})\,A^{(s,n)}_{nk}A^{(s,n)}_{nk'}. \tag{31}
$$

Recall that $k,k'=0,1,\ldots,n$.

Considering (31) for $k=0$ and $k'=0,1$, it is easy to infer that $\mathsf{J}\neq 0$ (since $\xi_0^2\neq\xi_1^2$).

Assume that the following relation holds:

$$
\alpha\mathsf{G} + \beta\mathsf{J} - \gamma\mathsf{H} = 0, \tag{32}
$$

where $\alpha\beta\gamma\neq 0$. Using formulas (29)–(31), write down the matrix entries of (32) for $(k,k')=(0,0)$, $(0,1)$, and $(1,1)$ dividing them by $A^{(s,n)}_{nk}A^{(s,n)}_{nk'}$ (note that $A^{(s,n)}_{nk}\neq 0$ for all k, see Eq. (97) of [3]):

$$
\begin{aligned}
&-\alpha + (\xi_0^2 - \theta_n^2\xi_n^{-2}\xi_0^{-2})\beta + 2\xi_0^{-1}\gamma = 0,\\
&-\alpha + (\xi_0\xi_1 - \theta_n^2\xi_n^{-2}\xi_0^{-1}\xi_1^{-1})\beta + (\xi_0^{-1}+\xi_1^{-1})\gamma = 0,\\
&-\alpha + (\xi_1^2 - \theta_n^2\xi_n^{-2}\xi_1^{-2})\beta + 2\xi_1^{-1}\gamma = 0.
\end{aligned} \tag{33}
$$

The determinant of this system of equations is $d=(\xi_0^{-1}-\xi_1^{-1})^3(\theta_n^2\xi_n^{-2}-\xi_0^2\xi_1^2)$. Since $\xi_0\neq\xi_1$, the equality $d=0$ can be satisfied only if

$$
\theta_n^2 = \xi_0^2\xi_1^2\xi_n^2, \tag{34}
$$

which is equivalent to the following condition: $\rho(3s-n)+3\rho(s)-\rho(2s)-\rho(2s-1)-\rho(2s-n)=2s(2-n)=0$. Thus, relation (32) cannot hold for $n\geq 3$.

For $n=2$, a solution of system (33) is as follows: $\alpha=\beta^{-1}=\xi_0+\xi_1$ and $\gamma=\xi_0\xi_1$. A direct check using the explicit form of the matrix $A^{(s,2)}$ (see Eq. (74) of [3]) shows that relation (32) with such coefficients indeed holds. Since system (33) has no solution for $\gamma=0$, we conclude that G and J are linearly independent.

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