# THE ZERO-CURVATURE REPRESENTATION FOR THE NONLINEAR $O(3)$ SIGMA-MODEL 

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Dedicated to L. D. Faddeev on the occasion of his 60th birthday

We consider the $O(3)$ sigma-model as a reduction of the principal chiral field. This approach allows us to introduce currents with ultralocal Poisson brackets and to obtain the zero-curvature equation which admits the fundamental Poisson bracket. Bibliography: 5 titles.

## Introduction

In the present paper we consider the nonlinear $O(3)$ sigma-model, which is one of the field theory models, within the classical framework of the inverse scattering method. An important step in this approach is to find a zero-curvature representation for the investigated model. An attempt to obtain this representation for the $O(3)$ sigma-model was made in [1]. But in [1] some extra conditions on the components of the energy-momentum tensor were imposed. In addition, an essential point in [1] is transition to variables of the sine-Gordon type and application of the Backlund transformation. This transition is classically admissible, but it presents a serious obstacle for further using the quantum version of the inverse scattering method.

On the other hand, one may consider the $O(3)$ sigma-model as a reduction of the model of the principal chiral field, which is more general. The zero-curvature representation for the principal chiral field was obtained in [2]. The quantum inverse scattering method was applied to this model in [3], where an essential trick was the change of the Poisson brackets for the currents. In the present paper we realize this approach for the $O(3)$ sigma-model by means of constructing a new pair of currents. We shall see, though, that it is impossible to obtain directly the XXX-magnetic model in a way similar to that used in [3].

Thus, in the present paper new currents are constructed for the model investigated and a new $U-V$ pair is obtained via these variables. Also, the fundamental Poisson bracket is given and some properties of the new currents are studied.

## 1. The $\vec{n}$-field model

The nonlinear $O(3)$ sigma-model (also known as the $\vec{n}$-field model) describes a three-dimensional vector $\vec{n}=\left(n_{1}, n_{2}, n_{3}\right)$ whose components depend on the coordinates in the ( $1+1$ )-dimensional space-time. The vector $\vec{n}$ is subject to the space-periodicity condition

$$
\vec{n}(x+L, t)=\vec{n}(x, t) .
$$

The Lagrangian of the model coincides with that of the free field,

$$
\begin{equation*}
\mathcal{L}=\frac{1}{2} \int_{0}^{L}\left(\left(\partial_{0} \vec{n}\right)^{2}-\left(\partial_{x} \vec{n}\right)^{2}\right) d x \tag{1}
\end{equation*}
$$

(here we use the notation $\partial_{0} \equiv \frac{\partial}{\partial t}, \partial_{x} \equiv \frac{\partial}{\partial x}$ ). We also impose the extra condition that the length of the vector $\vec{n}$ is fixed,

$$
\begin{equation*}
\vec{n}^{2}=n_{1}^{2}+n_{2}^{2}+n_{3}^{2}=1 . \tag{2}
\end{equation*}
$$

[^0]The Lagrangian (1), combined with constraint (2), implies the following equations of motion:

$$
\begin{equation*}
\partial_{0}^{2} \vec{n}-\partial_{x}^{2} \vec{n}+\left(\left(\partial_{0} \vec{n}\right)^{2}-\left(\partial_{x} \vec{n}\right)^{2}\right) \vec{n}=0 \tag{3}
\end{equation*}
$$

To obtain the Hamiltonian description of the model we introduce the variable of momentum density,

$$
\vec{\pi} \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_{0} \vec{n}\right)}=\partial_{0} \vec{n} .
$$

The Legendre transformation of the Lagrangian (1) yields the Hamiltonian of the model,

$$
H=\int_{0}^{L}\left(\vec{\pi} \cdot \partial_{0} \vec{n}\right) d x-\mathcal{L}=\frac{1}{2} \int_{0}^{L}\left(\vec{\pi}^{2}+\left(\partial_{x} \vec{n}\right)^{2}\right) d x .
$$

The Poisson brackets of the variables $\vec{\pi}$ and $\vec{n}$ can be obtained from the canonical brackets as the PoissonDirac brackets under the constraints

$$
\vec{n}^{2}=1, \quad \vec{\pi} \cdot \vec{n}=0
$$

These Poisson brackets have the following form:

$$
\begin{gather*}
\left\{n^{a}(x), n^{b}(y)\right\}=0 \\
\left\{\pi^{a}(x), n^{b}(y)\right\}=\left(\delta^{a b}-n^{a}(x) n^{b}(x)\right) \delta(x-y), \\
\left\{\pi^{a}(x), \pi^{b}(y)\right\}=\left(\pi^{b}(x) n^{a}(x)-\pi^{a}(x) n^{b}(x)\right) \delta(x-y) . \tag{4}
\end{gather*}
$$

Since our model is $O(3)$-invariant, it is more efficient to describe it in terms of the variables $\vec{l}$ and $\vec{n}$, where $\vec{l}$ (the angular momentum) is defined by

$$
\vec{l}(x)=\vec{\pi}(x) \wedge \vec{n}(x)
$$

Here $\wedge$ denotes the vector product, and so $l^{a}=\epsilon^{a b c} \pi^{b} n^{c}$. Note that $\overrightarrow{l^{2}}(x)=\vec{\pi}^{2}(x)$.
The Poisson brackets

$$
\begin{align*}
&\left\{l^{a}(x), l^{b}(y)\right\}=\epsilon^{a b c} l^{c}(x) \delta(x-y)  \tag{5}\\
&\left\{l^{a}(x), n^{b}(y)\right\}=\epsilon^{a b c} n^{c}(x) \delta(x-y)  \tag{6}\\
&\left\{n^{a}(x), n^{b}(y)\right\}=0 \tag{7}
\end{align*}
$$

of these variables define the current algebra of the group $E(3)$. The phase space of the model is the simplectic orbit

$$
\begin{equation*}
\vec{n}^{2}=1, \quad \vec{l} \cdot \vec{n}=0 . \tag{8}
\end{equation*}
$$

## 2. THE STANDARD ZERO-CURVATURE REPRESENTATION

The inverse scattering method is a basic tool for studying the classical as well as quantum models corresponding to different nonlinear equations (see, e.g., [4]). This method starts from representation of the nonlinear equation in the form of the zero-curvature condition,

$$
\begin{equation*}
\partial_{t} U(x, \lambda)-\partial_{x} V(x, \lambda)+[U(x, \lambda), V(x, \lambda)]=0 \tag{9}
\end{equation*}
$$

Here $U$ and $V$ are square matrices of equal size whose elements depend on the space-time variables $x$ and $t$ and on an extra spectral parameter $\lambda$. Equations (9) should be satisfied for all values of $\lambda$.

Let us consider the model of the principal chiral field. In this model the dynamical variable $g(x, t)$ takes the values in a certain compact group $G$. The equations of motion have the form

$$
\begin{equation*}
\partial_{0}^{2} g-\partial_{x}^{2} g=\partial_{0} g \cdot g^{-1} \cdot \partial_{0} g-\partial_{x} g \cdot g^{-1} \cdot \partial_{x} g \tag{10}
\end{equation*}
$$

Zakharov and Mikhailov [2] found an appropriate the $U-V$ pair for this model,

$$
\begin{align*}
& U(x, \lambda)=\frac{1}{2} \frac{L_{0}(x)+L_{1}(x)}{1-\lambda}-\frac{1}{2} \frac{L_{0}(x)-L_{1}(x)}{1+\lambda}=\frac{\lambda L_{0}(x)+L_{1}(x)}{1-\lambda^{2}},  \tag{11}\\
& V(x, \lambda)=\frac{1}{2} \frac{L_{0}(x)+L_{1}(x)}{1-\lambda}+\frac{1}{2} \frac{L_{0}(x)-L_{1}(x)}{1+\lambda}=\frac{\lambda L_{1}(x)+L_{0}(x)}{1-\lambda^{2}} . \tag{12}
\end{align*}
$$

Here

$$
\begin{equation*}
L_{0}(x, t)=\partial_{0} g \cdot g^{-1}, \quad L_{1}(x, t)=\partial_{x} g \cdot g^{-1} \tag{13}
\end{equation*}
$$

are left currents, taking values in the Lie algebra of the group $G$. The zero-curvature condition (9) for the $U-V$ pair (11), (12) is written via the currents $L_{\mu}$ as follows:

$$
\begin{gather*}
\partial_{0} L_{0}-\partial_{x} L_{1}=0  \tag{14}\\
\partial_{0} L_{1}-\partial_{x} L_{0}+\left[L_{1}, L_{0}\right]=0 . \tag{15}
\end{gather*}
$$

Note that Eq. (15) follows from definitions (13),

$$
\partial_{0}\left(\partial_{x} g \cdot g^{-1}\right)-\partial_{x}\left(\partial_{0} g \cdot g^{-1}\right)+\left[\partial_{x} g \cdot g^{-1}, \partial_{0} g \cdot g^{-1}\right] \equiv 0
$$

while Eq. (14) follows from the equations of motion (10).
The Lagrangian and the Hamiltonian of the model can be expressed in terms of the currents $L_{\mu}$,

$$
\begin{equation*}
\mathcal{L}=\frac{1}{4} \int_{0}^{L} \operatorname{tr}\left(L_{1}^{2}-L_{0}^{2}\right) d x ; \quad H=-\frac{1}{4} \int_{0}^{L} \operatorname{tr}\left(L_{0}^{2}+L_{1}^{2}\right) d x . \tag{16}
\end{equation*}
$$

The Poisson brackets of the components of currents $L_{\mu}^{a}$ are

$$
\begin{align*}
& \left\{L_{0}^{a}(x), L_{0}^{b}(y)\right\}=\epsilon^{a b c} L_{0}^{c}(x) \delta(x-y),  \tag{17}\\
& \left\{L_{0}^{a}(x), L_{1}^{b}(y)\right\}=\epsilon^{a b c} L_{1}^{c}(x) \delta(x-y)-\delta^{a b} \delta^{\prime}(x-y),  \tag{18}\\
& \left\{L_{1}^{a}(x), L_{1}^{b}(y)\right\}=0 \tag{19}
\end{align*}
$$

In the general case of a principal chiral field with values in the group $G$, the equations of motion (10) admit an extra reduction,

$$
\begin{equation*}
g^{2}(x, t)=I \tag{20}
\end{equation*}
$$

This reduction preserves the form of the $U-V$ pair, as well as the zero-curvature equations (14), (15) and relations (16). It also preserves the Poisson brackets (17)-(19) because the brackets of the currents with the constraint (20) vanish.

Applying reduction (20) to the case of the principal chiral field for $G=S U(2)$, we obtain the $O(3)$ sigma-model. Indeed, in this case the variable $g(x, t)$ can be written in the form

$$
g(x, t)=\vec{n} \cdot \vec{\sigma} \equiv n^{a}(x, t) \sigma_{a}, \quad \vec{n}^{2}=1
$$

where $\sigma_{a}$ are Pauli matrices

$$
\sigma_{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad \sigma_{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right), \quad \sigma_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

For the given representation of $g(x, t)$ the equations of motion (10) turn into the equations of motion (3) for the $\vec{n}$-field.

Using the relation

$$
(\vec{a} \vec{\sigma}) \cdot(\vec{b} \vec{\sigma})=(\vec{a} \vec{b}) \cdot I+i(\vec{a} \wedge \vec{b}) \cdot \vec{\sigma},
$$

we can express the currents $L_{\mu}=\sum_{a=1}^{3} i L_{\mu}^{a} \sigma_{a}$ via the variables of the $\vec{n}$-field,

$$
\begin{align*}
& L_{0}(x, t)=i\left(\partial_{0} \vec{n} \wedge \vec{n}\right) \cdot \vec{\sigma} \equiv i \vec{l} \cdot \vec{\sigma},  \tag{21}\\
& L_{1}(x, t)=i\left(\partial_{x} \vec{n} \wedge \vec{n}\right) \cdot \vec{\sigma} . \tag{22}
\end{align*}
$$

Thus, considering the $O(3)$ sigma-model as a reduction of the principal chiral field for the group $G=$ $S U(2)$, we immediately obtain for it a zero-curvature representation with the $U-V$ pair given by (11), (12) and with the equations for the currents given by (14), (15). As mentioned above, one of these equations is a consequence of the equations of motion (3),

$$
\partial_{0} L_{0}-\partial_{x} L_{1}=i\left(\partial_{0}^{2} \vec{n} \wedge \vec{n}-\partial_{x}^{2} \vec{n} \wedge \vec{n}\right) \cdot \vec{\sigma}=i\left(\left(\partial_{0}^{2} \vec{n}-\partial_{x}^{2} \vec{n}\right) \wedge \vec{n}\right) \cdot \vec{\sigma}=0
$$

To apply the inverse scattering method for solving the $\vec{n}$-field model, one should represent the Poisson brackets of elements of the matrix $U(x, \lambda)$ as a fundamental Poisson bracket,

$$
\begin{equation*}
\{\stackrel{1}{U}(x, \lambda), \stackrel{2}{U}(y, \mu)\}=[r(\lambda, \mu), \stackrel{1}{U}(x, \lambda)+\stackrel{2}{U}(x, \mu)] \delta(x-y) \tag{23}
\end{equation*}
$$

Here $r(\lambda)$ is some classical $r$-matrix and we use the notation

$$
\stackrel{1}{U}=U \otimes I, \quad \stackrel{2}{U}=I \otimes U
$$

Unfortunately, since formula (18) has a term with $\delta^{\prime}(x-y)$ (the so-called nonultralocal term), it is impossible to introduce an appropriate fundamental Poisson bracket.

## 3. Currents $J_{\mu}$ and the new $U-V$ pair

In order to find the new $U-V$ pair, which satisfies the zero-curvature representation and possesses an ultralocal fundamental Poisson bracket, we define the new current v iables

$$
\begin{gather*}
J_{0}(x, t)=\frac{1}{2} i\left(\partial_{0} \vec{n} \wedge \vec{n}-i \partial_{x} \vec{n}\right) \cdot \vec{\sigma} \equiv \frac{1}{2} i\left(\vec{l}-i \partial_{x} \vec{n}\right) \cdot \vec{\sigma},  \tag{24}\\
J_{1}(x, t)=\frac{1}{2} i\left(\partial_{x} \vec{n} \wedge \vec{n}-i \partial_{0} \vec{n}\right) \cdot \vec{\sigma} \equiv \frac{1}{2} i\left(\partial_{x} \vec{n} \wedge \vec{n}-i \vec{\pi}\right) \cdot \vec{\sigma} . \tag{25}
\end{gather*}
$$

Note that in the case of $S U(2)$, adding terms like $\frac{1}{2} \partial_{\mu} g=\frac{1}{2} \partial_{\mu} \vec{n} \cdot \vec{\sigma}$ to the currents $L_{\mu}$ does not bring them out of the Lie algebra $s u(2)$. Consequently, $J_{\mu}$ can be regarded as current variables. But if we apply the above formulas for constructing new currents in the case of an arbitrary Lie group $G$, then the new objects $J_{\mu}$ will not belong to the Lie algebra of this group (but, nevertheless, they will satisfy the zero-curvature equations).

Let us consider an analogue of the pair (11), (12) for the currents $J_{\mu}$,

$$
\begin{equation*}
U^{\prime}(x, \lambda)=\frac{\lambda J_{0}(x)+J_{1}(x)}{1-\lambda^{2}}, \quad V^{\prime}(x, \lambda)=\frac{\lambda J_{1}(x)+J_{0}(x)}{1-\lambda^{2}} . \tag{26}
\end{equation*}
$$

The zero-curvature condition (9) leads to the system of equations

$$
\begin{align*}
& \partial_{0} J_{0}-\partial_{x} J_{1}=0,  \tag{27}\\
& \partial_{0} J_{1}-\partial_{x} J_{0}=\left[J_{0}, J_{1}\right] . \tag{28}
\end{align*}
$$

Formally, system (27), (28) coincides with (14), (15), but it should be noted that now both Eqs. (27) and (28) follow from the equations of motion (3),

$$
\partial_{0} J_{0}-\partial_{x} J_{1}=\frac{1}{2} i\left(\partial_{0}^{2} \vec{n} \wedge \vec{n}-i \partial_{0 x} \vec{n}-\partial_{x}^{2} \vec{n} \wedge \vec{n}+i \partial_{x 0} \vec{n}\right) \cdot \vec{\sigma}=\frac{1}{2} i\left(\left(\partial_{0}^{2} \vec{n}-\partial_{x}^{2} \vec{n}\right) \wedge \vec{n}\right) \cdot \vec{\sigma}=0
$$

In order to check Eq. (28) one should write the new currents $J_{\mu}$ in the form

$$
J_{0}=\frac{1}{2} \partial_{0} g \cdot g^{-1}+\frac{1}{2} \partial_{x} g=\frac{1}{2}\left(L_{0}+L_{1} g\right), \quad J_{1}=\frac{1}{2} \partial_{x} g \cdot g^{-1}+\frac{1}{2} \partial_{0} g=\frac{1}{2}\left(L_{1}+L_{0} g\right)
$$

and use the condition of reduction

$$
\begin{equation*}
g^{2}=(\vec{n} \cdot \vec{\sigma})^{2}=\vec{n}^{2} \cdot I \equiv I, \tag{29}
\end{equation*}
$$

as well as its corollaries

$$
\begin{equation*}
g^{-1}=g \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{\mu} g=-g L_{\mu} . \tag{31}
\end{equation*}
$$

(Identity (31) is obtained by differentiating Eq. (30).)*
Now the left-hand side of Eq. (28) can be written in the form

$$
\partial_{0} J_{1}-\partial_{x} J_{0}=\frac{1}{2}\left(\partial_{0} L_{1}-\partial_{x} L_{0}\right)+\frac{1}{2}\left(\partial_{0} L_{0}-\partial_{x} L_{1}\right) g+\frac{1}{2}\left(L_{0}^{2}-L_{1}^{2}\right) g
$$

and taking Eqs. (14), (15) into account (in fact, this means that we use the equations of motion) we obtain

$$
\partial_{0} J_{1}-\partial_{x} J_{0}=\frac{1}{2}\left[L_{0}, L_{1}\right]+\frac{1}{2}\left(L_{0}^{2}-L_{1}^{2}\right) g .
$$

The right-hand side of Eq. (28) can be given the same form by using conditions (29)-(31),

$$
\begin{aligned}
{\left[J_{0}, J_{1}\right] } & =\frac{1}{4}\left[L_{0}+L_{1} g, L_{1}+L_{0} g\right]=\frac{1}{4}\left[L_{0}, L_{1}\right]+\frac{1}{4}\left[L_{0}, L_{0} g\right]+\frac{1}{4}\left[L_{1} g, L_{1}\right]+\frac{1}{4}\left[L_{1} g, L_{0} g\right] \\
& =\frac{1}{4}\left[L_{0}, L_{1}\right]+\frac{1}{4}\left(L_{0}^{2} g-L_{0} g L_{0}\right)+\frac{1}{4}\left(L_{1} g L_{1}-L_{1}^{2} g\right)+\frac{1}{4}\left(L_{1} g L_{0} g-L_{0} g L_{1} g\right) \\
& =\frac{1}{4}\left[L_{0}, L_{1}\right]+\frac{1}{2} L_{0}^{2} g-\frac{1}{2} L_{1}^{2} g+\frac{1}{4}\left[L_{0}, L_{1}\right] g^{2}=\frac{1}{2}\left[L_{0}, L_{1}\right]+\frac{1}{2}\left(L_{0}^{2}-L_{1}^{2}\right) g .
\end{aligned}
$$

Thus, in contrast to the case of the unreduced principal chiral field, in the case of the $\vec{n}$-field there exists a second $U^{\prime}-V^{\prime}$ pair. The next step in studying the model considered is calculation of the Poisson brackets for the new currents.

## 4. The fundamental Poisson bracket

Now let $t^{a}, a=1,2,3$, be the basis elements of the algebra $s u(2)$, which are normalized with respect to the Killing form,

$$
\operatorname{tr}\left(t^{a} t^{b}\right)=-\frac{1}{2} \delta^{a b} ; \quad\left[t^{a}, t^{b}\right]=f^{a b c} t^{c}
$$

The generators $t^{a}$ are expressed via the Pauli matrices,

$$
t^{a}=\frac{1}{2} i \sigma^{a} ; \quad f^{a b c}=-\epsilon^{a b c} .
$$

[^1]In order to calculate the Poisson brackets of the components of the current $J_{0}(x)=J_{0}^{a}(x) t^{a}$, we use formulas $(5)-(7)^{*}$ and obtain

$$
\begin{align*}
\left\{J_{0}^{a}(x), J_{0}^{b}(y)\right\} & =\left\{l^{a}(x)-i \partial_{x} n^{a}(x), l^{b}(y)-i \partial_{y} n^{b}(y)\right\} \\
& =\epsilon^{a b c} l^{c}(x) \delta(x-y)+i \epsilon^{a b c}\left(n^{c}(x)-n^{c}(y)\right) \delta^{\prime}(x-y) \\
& =\epsilon^{a b c} l^{c}(x) \delta(x-y)+i \epsilon^{a b c}\left((x-y) \partial_{x} n^{c}(x)\right) \delta^{\prime}(x-y)  \tag{32}\\
& =\epsilon^{a b c} l^{c}(x) \delta(x-y)-i \epsilon^{a b c} \partial_{x} n^{c}(x) \delta(x-y)=\epsilon^{a b c} J_{0}^{c}(x) \delta(x-y) .
\end{align*}
$$

We also have

$$
\begin{gather*}
\left\{J_{0}^{a}(x), n^{b}(y)\right\}=\left\{l^{a}(x)-i \partial_{x} n^{a}(x), n^{b}(y)\right\}=\epsilon^{a b c} n^{c}(x) \delta(x-y),  \tag{33}\\
\left\{J_{1}^{a}(x), n^{b}(y)\right\}=\left\{\left(\partial_{x} \vec{n} \wedge \vec{n}-i \vec{\pi}\right)^{a}(x), n^{b}(y)\right\}=i\left(n^{a}(x) n^{b}(x)-\delta^{a b}\right) \delta(x-y) . \tag{34}
\end{gather*}
$$

In order to calculate the other Poisson brackets it is convenient to use the following relations for currents (let us recall that $\vec{l} \cdot \vec{n}=\vec{\pi} \cdot \vec{n}=\partial_{x} \vec{n} \cdot \vec{n}=0$ ):

$$
\begin{gather*}
i\left(\vec{J}_{1} \wedge \vec{n}\right)=i\left(\partial_{x} \vec{n} \wedge \vec{n}-i \vec{\pi}\right) \wedge \vec{n}=\vec{\pi} \wedge \vec{n}-i \partial_{x} \vec{n}=\vec{J}_{0},  \tag{35}\\
i\left(\vec{J}_{0} \wedge \vec{n}\right)=i\left(\vec{l}-i \partial_{x} \vec{n}\right) \wedge \vec{n}=\partial_{x} \vec{n} \wedge \vec{n}-i \vec{\pi}=\vec{J}_{1} . \tag{36}
\end{gather*}
$$

(Here $\vec{J}_{\mu}=\left(J_{\mu}^{1}, J_{\mu}^{2}, J_{\mu}^{3}\right)$.) Taking into account the relation $\vec{J}_{\mu} \wedge \vec{n}=-\vec{n} \wedge \overrightarrow{J_{\mu}}$, one may write formulas (35), (36) as follows:

$$
\begin{align*}
J_{1} g & =-g J_{1}=J_{0}  \tag{37}\\
J_{0} & =-g J_{0}=J_{1} . \tag{38}
\end{align*}
$$

Now, using (32)-(36) we obtain

$$
\begin{align*}
\left\{J_{0}^{a}(x), J_{1}^{b}(y)\right\} & =i \epsilon^{b c d}\left\{J_{0}^{a}(x), J_{0}^{c}(y) n^{d}(y)\right\}=i \epsilon^{b c d}\left(\epsilon^{a c f} J_{0}^{f}(x) n^{d}(y)+\epsilon^{a d f} J_{0}^{c}(y) n^{f}(x)\right) \delta(x-y) \\
& =i\left(-J_{0}^{b}(x) n^{a}(x)+J_{0}^{a}(x) n^{b}(x)\right) \delta(x-y)=\epsilon^{a b c} J_{1}^{c}(x) \delta(x-y) . \tag{39}
\end{align*}
$$

Similarly one obtains

$$
\begin{equation*}
\left\{J_{1}^{a}(x), J_{1}^{b}(y)\right\}=\epsilon^{a b c} J_{0}^{c}(x) \delta(x-y) . \tag{40}
\end{equation*}
$$

An important property of brackets (32), (39), (40) is their ultralocality (this means that they do not include terms like $\delta^{\prime}(x-y)$ ). This allows us to find the fundamental Poisson bracket for the matrix $U^{\prime}(x, \lambda)$ in the $\vec{n}$-field model.

Indeed, using (32), (39), (40) and following [4] we obtain

$$
\begin{gathered}
\left\{\stackrel{1}{U}^{\prime}(x, \lambda), \stackrel{2}{U^{\prime}}(y, \mu)\right\}=-\frac{1}{4} \frac{\sigma^{a} \otimes \sigma^{b}}{\left(1-\lambda^{2}\right)\left(1-\mu^{2}\right)}\left\{\lambda J_{0}^{a}(x)+J_{1}^{a}(x), \mu J_{0}^{b}(y)+J_{1}^{b}(y)\right\} \\
\quad=-\frac{1}{4} \frac{\epsilon^{a b c} \sigma^{a} \otimes \sigma^{b}}{\left(1-\lambda^{2}\right)\left(1-\mu^{2}\right)}\left((\lambda \mu+1) J_{0}^{c}(x)+(\lambda+\mu) J_{1}^{c}(x)\right) \delta(x-y)
\end{gathered}
$$

Now, introducing the permutation operator

$$
P=\frac{1}{2}\left(I \otimes I+\sigma^{a} \otimes \sigma^{a}\right)
$$

and using its property

$$
\epsilon^{a b c} \sigma^{a} \otimes \sigma^{b}=-i\left[P, \sigma^{c} \otimes I\right]=-2\left[P, t^{c} \otimes I\right]
$$

*We also need a well-known property of generalized functions: $\left(x F(x), \delta^{\prime}(x)\right)=-(F(x), \delta(x))$.

$$
\begin{gathered}
\left\{\stackrel{1}{U}^{\prime}(x, \lambda), \stackrel{2}{U}^{\prime}(y, \mu)\right\}=\frac{\left[P,(\lambda \mu+1) \stackrel{1}{J}_{0}(x)+(\lambda+\mu) \stackrel{1}{J_{1}}(x)\right]}{2\left(1-\lambda^{2}\right)\left(1-\mu^{2}\right)} \delta(x-y) \\
=\frac{1}{\left(1-\lambda^{2}\right)\left(1-\mu^{2}\right)}\left[\frac{P}{2(\lambda-\mu)},\left(\lambda^{2}-1\right) \mu \stackrel{1}{J}_{0}(x)+\left(1-\mu^{2}\right) \lambda^{\frac{1}{J}}(x)+\left(\lambda^{2}-1\right) \stackrel{1}{J_{1}}(x)+\left(1-\mu^{2}\right) \stackrel{1}{J_{1}}(x)\right] \delta(x-y) \\
=\left[\frac{P}{2(\lambda-\mu)}, \stackrel{1}{U^{\prime}}(x, \lambda)-\stackrel{1}{U}^{\prime}(x, \mu)\right] \delta(x-y)=\left[\frac{P}{2(\lambda-\mu)}, \stackrel{1}{U^{\prime}}(x, \lambda)+\stackrel{2}{U}^{\prime}(x, \mu)\right] \delta(x-y) .
\end{gathered}
$$

Thus, we have found the fundamental Poisson bracket for the matrix $U^{\prime}(x, \lambda)$ with $r$-matrix

$$
r(\lambda, \mu)=\frac{P}{2(\lambda-\mu)}
$$

Our success in finding the fundamental Poisson brackets is due to the observation that the transition from the standard currents $L_{0}$ and $L_{1}$ (which correspond by formulas (21), (22) to the pair of variables $i$ and $\left.\partial_{x} \vec{n} \wedge \vec{n}\right)$ to the currents $J_{0}$ and $J_{1}$ leads to the disappearance of the term with $\delta^{\prime}(x-y)$ in the Poisson brackets. At first glance, this disappearance is quite unexpected. It is explained by the fact that the new currents are not deformations of the standard currents.

Indeed, if we take the initial Hamiltonian with an extra parameter $\gamma$,

$$
H=\frac{1}{2} \int_{0}^{L}\left(\vec{\pi}^{2}+\gamma^{2}\left(\partial_{x} \vec{n}\right)^{2}\right) d x
$$

then we obtain the following standard and new currents:

$$
\begin{gathered}
\vec{L}_{0}(x)=\vec{l}(x), \quad \vec{L}_{1}(x)=\gamma \partial_{x} \vec{n} \wedge \vec{n} ; \\
\vec{J}_{0}(x)=\vec{l}(x)-\gamma \partial_{x} \vec{n}, \quad \vec{J}_{1}(x)=\gamma \partial_{x} \vec{n} \wedge \vec{n}-i \vec{\pi}=-\left(\vec{\pi}+i \gamma \partial_{x} \vec{n} \wedge \vec{n}\right) .
\end{gathered}
$$

These formulas imply that one can consider the currents $J_{0}$ and $J_{1}$ as deformations with parameter $\gamma$ of the variables $\vec{l}$ and $-i \vec{\pi}$. The same conclusion follows from simply comparing relations (32), (39), (40) for $J_{0}$ and $J_{1}$ with formulas (4), (5) for $\vec{l}$ and $\vec{\pi}$.

## 5. Lagrangian, Hamiltonian, and currents $\widehat{J}_{\mu}$

As mentioned above, the Lagrangian and the Hamiltonian of the $\vec{n}$-field can be written via the currents $L_{\mu}$,

$$
\begin{align*}
\mathcal{L} & =\frac{1}{4} \int_{0}^{L} \operatorname{tr}\left(L_{1}^{2}-L_{0}^{2}\right) d x=\frac{1}{2} \int_{0}^{L}\left(\vec{l}^{2}-\left(\partial_{x} \vec{n} \wedge \vec{n}\right)^{2}\right) d x  \tag{41}\\
H & =-\frac{1}{4} \int_{0}^{L} \operatorname{tr}\left(L_{0}^{2}+L_{1}^{2}\right) d x=\frac{1}{2} \int_{0}^{L}\left(\vec{l}^{2}+\left(\partial_{x} \vec{n} \wedge \vec{n}\right)^{2}\right) d x . \tag{42}
\end{align*}
$$

It turns out that similar formulas can be obtained in terms of the new currents $J_{\mu}(x)$ as well. Indeed, consider the Lagrangian

$$
\begin{align*}
\widehat{\mathcal{L}}^{\prime} & =\frac{1}{2} \int_{0}^{L} \operatorname{tr}\left(J_{1}^{2}-J_{0}^{2}\right) d x=\int_{0}^{L} \operatorname{tr} J_{1}^{2} d x=-\int_{0}^{L} \operatorname{tr} J_{0}^{2} d x=\frac{1}{2} \int_{0}^{L}\left(\vec{l}-i \partial_{x} \vec{n}\right)^{2} d x \\
& =\frac{1}{2} \int_{0}^{L}\left(\vec{l}^{2}-\left(\partial_{x} \vec{n}\right)^{2}\right) d x-i \int_{0}^{L}\left(\vec{l} \cdot \partial_{x} \vec{n}\right) d x \tag{43}
\end{align*}
$$

which differs from the initial Lagrangian (1). But the difference

$$
\Theta=\mathcal{L}^{\prime}-\mathcal{L}=i \int_{0}^{L}\left(\vec{l} \cdot \partial_{x} \vec{n}\right) d x=i \int_{0}^{L}\left(\partial_{x} \vec{n} \wedge \partial_{0} \vec{n}\right) \cdot \vec{n} d x
$$

between these Lagrangians is the so-called topological term, which gives no contribution to the variation of the action (see, e.g., [5]). Indeed, we have

$$
S_{\Theta}=\int_{t_{1}}^{t_{2}} \Theta d t=i \int_{t_{1}}^{t_{2}}(d \vec{n} \wedge d \vec{n}) \vec{n} \equiv i \int_{t_{1}}^{t_{2}} \Omega
$$

and since the form $\Omega$ is closed, $d \Omega=0$, it follows that $\delta S=i \int_{t_{1}}^{t_{2}} d \Omega=0$. Thus, the Lagrangians $\mathcal{L}^{\prime}$ and $\mathcal{L}$ are equivalent.

The Hamiltonian of the model can be written in the form

$$
H=-\int_{0}^{L} \operatorname{tr} J_{0} \widehat{J}_{0} d x=\int_{0}^{L} \operatorname{tr} J_{1} \widehat{J}_{1} d x=\frac{1}{2} \int_{0}^{L}\left(\vec{l}^{2}+\left(\partial_{x} \vec{n}\right)^{2}\right) d x .
$$

Here we have introduced one more pair of currents, which up to sign are Hermitian conjugates of the currents $J_{\mu}$,

$$
\begin{gather*}
\widehat{J}_{0}(x, t) \equiv-J_{0}^{*}(x, t)=\frac{1}{2} i\left(\vec{l}+i \partial_{x} \vec{n}\right) \cdot \vec{\sigma}  \tag{44}\\
\widehat{J}_{1}(x, t) \equiv-J_{1}^{*}(x, t)=\frac{1}{2} i\left(\partial_{x} \vec{n} \wedge \vec{n}+i \vec{\pi}\right) \cdot \vec{\sigma} . \tag{45}
\end{gather*}
$$

It is easy to check that $\widehat{J}_{\mu}$ satisfy zero-curvature Eqs. (27), (28), while Eqs. (37), (38) take the form

$$
\begin{align*}
& g \widehat{J}_{1}=-\widehat{J}_{1} g=\widehat{J}_{0}  \tag{46}\\
& g \widehat{J}_{0}=-\widehat{J}_{0} g=\widehat{J}_{1} \tag{47}
\end{align*}
$$

The Poisson brackets of the currents $\widehat{J}_{\mu}$ exactly repeat (32), (39), and (40),

$$
\begin{align*}
& \left\{\widehat{J}_{0}^{a}(x), \widehat{J}_{0}^{b}(y)\right\}=\epsilon^{a b c} \widehat{J}_{0}^{c}(x) \delta(x-y), \\
& \left\{\widehat{J}_{0}^{a}(x), \widehat{J}_{1}^{b}(y)\right\}=\epsilon^{a b c} \widehat{J}_{1}^{c}(x) \delta(x-y),  \tag{48}\\
& \left\{\widehat{J}_{1}^{a}(x), \widehat{J}_{1}^{b}(y)\right\}=\epsilon^{a b c} \widehat{J}_{0}^{c}(x) \delta(x-y)
\end{align*}
$$

However, as one should have expected, the Poisson brackets of $J_{\mu}$ and $\widehat{J}_{\mu}$ contain nonultralocal terms,

$$
\begin{aligned}
\left\{J_{0}^{a}(x), \widehat{J}_{0}^{b}(y)\right\} & =\epsilon^{a b c} l^{c}(x) \delta(x-y)-i \epsilon^{a b c}\left(n^{c}(x)+n^{c}(y)\right) \delta^{\prime}(x-y)= \\
& =\epsilon^{a b c} J_{0}^{c}(x) \delta(x-y)-2 i \epsilon^{a b c} n^{c}(x) \delta^{\prime}(x-y), \\
\left\{J_{0}^{a}(x), \widehat{J}_{1}^{b}(y)\right\} & =\epsilon^{a b c} \widehat{J}_{1}^{c}(x) \delta(x-y)-2 \delta^{a b} \delta^{\prime}(x-y), \\
\left\{\widehat{J}_{0}^{a}(x), J_{1}^{b}(y)\right\} & =\epsilon^{a b c} J_{1}^{c}(x) \delta(x-y)-2 \delta^{a b} \delta^{\prime}(x-y) .
\end{aligned}
$$

Note that the simultaneous presence of currents $\widehat{J}_{\mu}$ and $J_{\mu}$ in the Hamiltonian $H$ is quite natural. Indeed, from the standard currents $L_{\mu}$, one can obtain four new variables:

$$
\begin{array}{ll}
J_{0}=\frac{1}{2}\left(L_{0}+L_{1} g\right), & J_{1}=\frac{1}{2}\left(L_{1}+L_{0} g\right), \\
\widehat{J}_{0}=\frac{1}{2}\left(L_{0}-L_{1} g\right), & \widehat{J}_{1}=\frac{1}{2}\left(L_{1}-L_{0} g\right) .
\end{array}
$$

It follows from relations (37), (38) and (46), (47) that neither the pair $J_{0}, J_{1}$ nor the pair $\widehat{J}_{0}, \widehat{J}_{1}$ are actually independent variables. Therefore, in the general case, in order to preserve the number of independent variables one should use, for example, the pair of currents $J_{0}$ and $\widehat{J}_{0}$. From this point of view, it is rather unexpected that the formulas for the $U-V$ pair and the expression for the Lagrangian contain, in fact, only one independent current.

By way of example we give the following expressions for the $U-V$ pair:

$$
\begin{gathered}
U^{\prime}(\lambda)=\frac{\lambda J_{0}(x)+J_{1}(x)}{1-\lambda^{2}}=J_{0}(g+\lambda)[(g-\lambda)(g+\lambda)]^{-1}=J_{0}(g-\lambda)^{-1}=-(g+\lambda)^{-1} J_{0}, \\
V^{\prime}(\lambda)=\frac{\lambda J_{1}(x)+J_{0}(x)}{1-\lambda^{2}}=J_{1}(g-\lambda)^{-1}=-(g+\lambda)^{-1} J_{1} .
\end{gathered}
$$

Finally, note that replacing the currents $J_{\mu}$ by $\widehat{J}_{\mu}$ in formulas (26) gives us another pair of matrices $U^{\prime \prime}(x, \lambda), V^{\prime \prime}(x, \lambda)$ satisfying the zero-curvature equation. It follows from relation (48) that the matrix $U^{\prime \prime}(x, \lambda)$ has the same fundamental Poisson bracket and $r$-matrix as the matrix $U^{\prime}(x, \lambda)$.

## 6. The light-CONE COORDinates

Since the Lagrangian (1) is relativistic invariant, it is natural to use the light-cone coordinates

$$
\xi=\frac{t+x}{2}, \quad \eta=\frac{t-x}{2} .
$$

It is also convenient to introduce the new variables

$$
\begin{aligned}
& J_{+}(x, t)=\sum_{a=1}^{3} J_{+}^{a} t^{a}=\frac{1}{2}\left(J_{0}(x, t)+J_{1}(x, t)\right)=\frac{1}{4} \partial_{\xi} g \cdot g^{-1}+\frac{1}{4} \partial_{\xi} g, \\
& J_{-}(x, t)=\sum_{a=1}^{3} J_{-}^{a} t^{a}=\frac{1}{2}\left(J_{0}(x, t)-J_{1}(x, t)\right)=\frac{1}{4} \partial_{\eta} g \cdot g^{-1}-\frac{1}{4} \partial_{\eta} g .
\end{aligned}
$$

It follows from (37), (38) that

$$
\begin{align*}
& J_{+} g=-g J_{+}=J_{+}  \tag{49}\\
& g J_{-}=-J_{-} g=J_{-} \tag{50}
\end{align*}
$$

Using (32), (39), (40) we get

$$
\begin{gather*}
\left\{J_{+}^{a}(x), J_{+}^{b}(y)\right\}=\epsilon^{a b c} J_{+}^{c}(x) \delta(x-y)  \tag{51}\\
\left\{J_{-}^{a}(x), J_{-}^{b}(y)\right\}=\epsilon^{a b c} J_{-}^{c}(x) \delta(x-y),  \tag{52}\\
\left\{J_{+}^{a}(x), J_{-}^{b}(y)\right\}=0 . \tag{53}
\end{gather*}
$$

Now, following [3], we can consider the variables $S \equiv J_{+}(x)$ and $T \equiv J_{-}(x)$ as a pair of independent spin variables. In [3] this was used for transition from the principal chiral field model to the XXX-magnetic model, and the well-developed approach via the Bethe ansatz was used for further investigation.

But in our case this method cannot be applied directly. Indeed, (35)-(38) imply the relations

$$
\begin{gather*}
J_{0}^{2}=-J_{1}^{2}  \tag{54}\\
J_{0} J_{1}=-J_{0} g J_{0}=-J_{1} J_{0} \tag{55}
\end{gather*}
$$

whence it follows that

$$
\begin{equation*}
S^{2} \equiv J_{+}^{2}=\frac{1}{4}\left(J_{0}+J_{1}\right)^{2}=0, \quad T^{2} \equiv J_{-}^{2}=\frac{1}{4}\left(J_{0}-J_{1}\right)^{2}=0 . \tag{56}
\end{equation*}
$$

Thus, the Casimir operators $S^{2}$ and $T^{2}$ corresponding to the spin variables $\vec{S}$ and $\vec{T}$ vanish in the quantum case and we cannot use the Bethe ansatz approach as was done in [3].

Note that relation (56) gives an extra condition on solutions of an auxiliary system of differential equations. In our case, taking into account the concrete form of matrices $U^{\prime}(x, \lambda)$ and $V^{\prime}(x, \lambda)(26)$ and using light-cone coordinates, we can write the auxiliary problem as follows:

$$
\begin{equation*}
\partial_{\xi} \vec{F}=\frac{2}{1-\lambda} J_{+} \vec{F}, \quad \partial_{\eta} \vec{F}=\frac{2}{1+\lambda} J_{-} \vec{F} . \tag{57}
\end{equation*}
$$

Relations (56) imply the extra conditions

$$
\begin{equation*}
J_{+} \partial_{\xi} \vec{F}=0, \quad J_{-} \partial_{\eta} \vec{F}=0 \tag{58}
\end{equation*}
$$

Using relations (49), (50) we can write these conditions in the form

$$
(g+1) \partial_{\xi} \vec{F}=0, \quad(g-1) \partial_{\eta} \vec{F}=0
$$

Finally, note that relations (57), (58) yield the following second-order differential equations for the vector $\vec{F}$ :

$$
\begin{equation*}
\partial_{\xi}^{2} \vec{F}=\frac{2}{1-\lambda}\left(\partial_{\xi} J_{+}\right) \vec{F}, \quad \partial_{\eta}^{2} \vec{F}=\frac{2}{1+\lambda}\left(\partial_{\eta} J_{-}\right) \vec{F} . \tag{59}
\end{equation*}
$$

## 7. Conclusion

As shown above, the fact that the currents $\vec{J}_{+}$and $\vec{J}_{-}$are of zero length does not allow us to apply the standard method of quantization. But the same fact points out another possible approach to investigation of the model-transition to fermion-type variables. Indeed, if we write the currents $\vec{J}_{+}$and $\vec{J}_{-}$in the form

$$
\vec{J}_{+}=\frac{1}{2}\left(\begin{array}{c}
i\left(z_{1}^{2}-\bar{z}_{1}^{2}\right) \\
z_{1}^{2}+\bar{z}_{1}^{2} \\
2 i z_{1} \bar{z}_{1}
\end{array}\right), \quad \vec{J}_{-}=\frac{1}{2}\left(\begin{array}{c}
i\left(z_{2}^{2}-\bar{z}_{2}^{2}\right) \\
z_{2}^{2}+\bar{z}_{2}^{2} \\
2 i z_{2} \bar{z}_{2}
\end{array}\right)
$$

where $z_{1}(x)$ and $z_{2}(x)$ are the components of the spinor $\Psi(x)$ with the Poisson brackets

$$
\left\{z_{i}(x), \bar{z}_{j}(y)\right\}=\delta_{i j} \delta(x-y)
$$

then we easily check relations (51)-(53) as well as the relation

$$
\left(\vec{J}_{+}\right)^{2}=\left(\vec{J}_{-}\right)^{2}=0 .
$$

Thus, in principle, the $\vec{n}$-field model can be quantized by means of fermionization of the current variables. We are going to perform a more detailed investigation in a forthcoming paper.

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## Literature Cited

1. K. Pohlmeyer, "Integrable hamiltonian systems and interactions through quadratic constraints," Commun. Math. Phys., 46, 207-221 (1976).
2. V. E. Zakharov and A. V. Mikhailov, "Two-dimensional relativistic invariant models integrable by means of the inverse scattering method," Zh. Eksp. Teor. Fiz., 74, 1953-1973 (1978).
3. L. D. Faddeev and N. Yu. Reshetikhin, "Integrability of the principal chiral field model in the $1+1$ dimension," Ann. Phys., 167, 227-256 (1986).
4. L. D. Faddeev and L. A. Takhtadzhyan (Takhtajan), Hamiltonian Methods in the Theory of Solitons, Springer-Verlag (1986).
5. L. D. Faddeev, "In quest of multidimensional solitons," in: Nonlocal, Nonlinear, and Nonrenormalizable Field Theories (IV Conference on Nonlocal Field Theories, USSR, Alushta (1976) [in Russian], Dubna (1976), pp. 207-223.

[^0]:    Translated from Zapiski Nauchnykh Seminarov POMI, Vol. 215, 1994, pp. 100-114. Original article submitted January 27, 1993.

[^1]:    *It is interesting to note that relation (31) leads to coincidence of the right and left currents in this model, $R_{\mu} \equiv-g^{-1} \cdot \partial_{\mu} g=$ $\partial_{\mu} g \cdot g^{-1} \equiv L_{\mu}$.

