BROWNIAN MOTION AND STOCHASTIC CALCULUS

Master class 2015-2016

1. Gaussian vectors

(a) Let ξ be a (real-valued) Gaussian variable with mean μ and variance σ^2. Compute the characteristic function $\varphi(z) = \mathbb{E}[\exp(iz\xi)]$, $z \in \mathbb{R}$.

(b) Let $\xi = (\xi_1, \dots, \xi_d)$ be a Gaussian vector with mean $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{R}^d$ and covariance matrix $G = (G_{jk})_{j,k=1}^d \in \mathbb{R}^{d\times d}$. Prove that the matrix G is positive definite, i.e. $\lambda^\top G\lambda = \sum_{j,k=1}^d \lambda_j G_{jk} \lambda_k > 0$ for all $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$ except $\lambda = 0$.

(c) Let $\xi = (\xi_1, \dots, \xi_d)$ be a Gaussian vector with mean $\mu \in \mathbb{R}^d$ and covariance matrix $G \in \mathbb{R}^{d\times d}$. Compute the characteristic function $\varphi(z) = \mathbb{E}[\exp(iz^\top\xi)]$, $z \in \mathbb{R}^d$. *Hint:* write $G = U^\top \Lambda U$, where U is an orthogonal matrix and Λ is diagonal.

(d) Check that if $\xi_1, \dots, \xi_d$ are independent Gaussian variables, then $\xi = (\xi_1, \dots, \xi_d)$ is a Gaussian vector. For any matrix $U \in \mathbb{R}^{d\times d}$ check that $U\xi$ is also a Gaussian vector. What can be said about their covariance matrices?

(e) Let $\xi = (\xi_1, \dots, \xi_d)$ be a Gaussian vector. Prove that its components $\xi_1, \dots, \xi_d$ are independent if and only if the covariance matrix G is diagonal. Is it true that two Gaussian variables are independent if and only if their covariance is zero?

2. Fourier series

(a) Prove that both families $(\sqrt{2}\cos(\pi nt))_{n\geq 0}$ and $(\sqrt{2}\sin(\pi nt))_{n\geq 1}$ are orthonormal bases in $L^2[0,1]$. *Hint:* Use the fact that $(e^{i\pi nt})_{n\in\mathbb{Z}}$ is an orthogonal basis in $L^2[-1,1]$.

(b) For all $s, t \in [0,1]$, prove the following identity:

$$\sum_{n=1}^{+\infty} \frac{2\sin(\pi ns)\sin(\pi nt)}{\pi^2 n^2} \;=\; \min\{s,t\} - st.$$

(c) (*) Note that the identity given above can be also derived from the identity

$$\sum_{n=1}^{+\infty} \frac{\cos(\pi nt)}{\pi^2 n^2} \;=\; \frac{t^2}{4} - \frac{|t|}{2} + \frac{1}{6}, \quad |t| \leq 1,$$

which follows (by integration) from the *Poisson summation formula*

$$\sum_{n=-\infty}^{+\infty} e^{i\pi nt} \;=\; 2\sum_{m=-\infty}^{+\infty} \delta_{2m}(t)$$

(this should be understood in terms of Schwartz distributions).

3. Green's function of the Laplacian on $[0,1]$.

(a) Prove that the eigenfunctions and eigenvalues of the Dirichlet boundary value problem

$$-f'' = \lambda f, \qquad f(0) = f(1) = 0.$$

are given by $f_n(t) = \sqrt{2}\sin(\pi n t)$ and $\lambda_n = \pi^2 n^2$ with $n \geq 1$. Find eigenfunctions of the similar problem with Neumann boundary conditions $f'(0) = f'(1) = 0$.

(b) *Green's function* $G(s,t)$ of the Laplacian $f \mapsto -f''$ with Dirichlet boundary conditions is defined to be the kernel of the inverse operator, i.e. the unique function G such that $-f'' = g$ and $f(0) = f(1) = 0$ imply $f(t) = \int_0^1 G(s,t)g(s)ds$. Prove that

$$G(s,t) = \min\{s,t\} - st, \qquad s,t \in [0,1].$$

(c) Prove that

$$G(s,t) = \sum_{n=1}^{+\infty} \frac{2\sin(\pi n s)\sin(\pi n t)}{\pi^2 n^2}, \qquad s,t \in [0,1].$$

4. Poisson process

Recall that we defined the *Poisson process* $(N_t)_{t\in[0,+\infty)}$ of intensity $\lambda > 0$ by

$$N_t := \min\{n : \xi_0 + \cdots + \xi_n \geq t\},$$

where $\xi_0, \xi_1, \ldots$ is a sequence of i.i.d. *exponential variables* with density $\lambda e^{-\lambda x}$, $x \in [0,+\infty)$. (Also recall that $(N_t)_{t\in[0,+\infty)}$ is a process with independent increments due to the memoryless property of the exponential variable.)

(a) Prove that the increments $N_{t+s} - N_t$ are stationary and have Poisson distribution with parameter λs, i.e. $\mathbb{P}[N_{t+s} - N_t = n] = e^{-\lambda s} \cdot (\lambda s)^n / n!$, $n \geq 0$.

(b) Assume that $\lambda^{(1)}, \lambda^{(2)} > 0$ and $\lambda = \lambda^{(1)} + \lambda^{(2)}$. Let $N_t^{(1)}$ and $N_t^{(2)}$ be two independent Poisson processes of intensities $\lambda^{(1)}$ and $\lambda^{(2)}$. Prove that the process $N_t := N_t^{(1)} + N_t^{(2)}$ is a Poisson processes of intensity λ.

(c) Let $(N_t)_{t\geq 0}$ be a Poisson processes of intensity $\lambda > 0$, and let $p \in (0,1)$. Let us color every jump point of N_t white or blue independently with probabilities p and $1-p$, respectively. Prove that the collections of white and blue points define jumps of two Poisson processes of intensities λp and $\lambda(1-p)$, respectively.

(d) **(*)** Prove that a counting process $(N_t)_{t\in[0,+\infty)}$ (i.e., a non-decreasing integer-valued right-continuous process with $N_0 = 0$) is a Poisson process of intensity $\lambda > 0$ if and only if for all $0 < t_1 < \ldots < t_k$ and $0 \leq n_1 \leq \ldots \leq n_k$, one has

$$\mathbb{P}_\lambda(N_{t_k+\delta} - N_{t_k} = 0 \,|\, N_{t_j} = n_j,\, 1 \leq j \leq k) = 1 - \lambda\delta + o(\delta),$$
$$\mathbb{P}_\lambda(N_{t_k+\delta} - N_{t_k} = 1 \,|\, N_{t_j} = n_j,\, 1 \leq j \leq k) = \lambda\delta + o(\delta),$$
$$\mathbb{P}_\lambda(N_{t_k+\delta} - N_{t_k} \geq 2 \,|\, N_{t_j} = n_j,\, 1 \leq j \leq k) = o(\delta), \qquad \text{as } \delta \to 0.$$

5. Miscellaneous

(**a**) Let B_t be the standard Brownian motion on $[0,\infty)$. Check that the process $(1-t)B_{\frac{t}{1-t}}$ is a Brownian bridge on $[0,1]$.

(**b**) Let $\widetilde{B}_t$ be the standard Brownian bridge on $[0,1]$. Check that the process $(1+t)\widetilde{B}_{\frac{t}{1+t}}$ is a standard Brownian motion on $[0,\infty)$.

(**c**) Prove that

$$(a+\tfrac{1}{a})^{-1}\cdot e^{-\frac{a^2}{2}} < \int_a^{+\infty} e^{-\frac{x^2}{2}}\,dx < a^{-1}\cdot e^{-\frac{a^2}{2}}.$$

(**d**) Prove that for any dyadic rationals $s=p2^{-m}$ and $t=q2^{-s}$ one has

$$\sum_{n=1}^{+\infty}\ \sum_{k=1,\ k\,\mathrm{odd}}^{2^n-1} g_{k,n}(s)g_{k,n}(t) \ = \ \min\{s,t\}-st,$$

where the functions $g_{k,n}(x)=\int_0^x f_{k,n}(y)dy$ are the primitives of the *Haar functions*

$$f_{k,n}=2^{-\frac{n+1}{2}}\cdot\big(\chi_{[(k-1)2^{-n},k2^{-n})}-\chi_{[k2^{-n},(k+1)2^{-n})}\big).$$

(**e**) **(*)** Prove that the Haar functions $f_{k,n}(t)$ form a *complete* family in $L^2([0,1])$.

6. Measurability

Let $\mathcal{A}$ be a σ-algebra on a space Ω and $(X_t)_{t\in[0,T]}$ be a family of mappings $X_t:\Omega\to\mathbb{R}$. Let $X:\Omega\times[0,T]\to\mathbb{R}$ denote the mapping $(\omega,t)\mapsto X_t$. By $\mathcal{B}(\mathrm{M})$ we will denote the Borel σ-algebra on a metric space M. Prove that the following statements are equivalent:

(**a1**) for each $t\in[0,T]$ the mapping $X_t:(\Omega,\mathcal{A})\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is measurable;

(**a2**) the mapping $X:(\Omega,\mathcal{A})\to(\mathbb{R}^{[0,T]},\mathcal{B}(\mathbb{R})^{\otimes[0,T]})$ is measurable.

Assume that for all $\omega\in\Omega$ *the function* $\mathcal{X}=\mathcal{X}(\omega):t\mapsto X_t$ *is continuous on* $[0,T]$ and let $\mathcal{C}([0,T])$ denote the (Banach) space of real-valued continuous functions on $[0,T]$. Prove that the following statements are equivalent:

(**a**) the mapping $X:(\Omega,\mathcal{A})\to(\mathbb{R}^{[0,T]},\mathcal{B}(\mathbb{R})^{\otimes[0,T]})$ is measurable;

(**b**) the mapping $X:(\Omega\times[0,T],\mathcal{A}\otimes\mathcal{B}([0,T]))\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is measurable;

(**c**) the mapping $\mathcal{X}:(\Omega,\mathcal{A})\to(\mathcal{C}([0,T])\,,\,\mathcal{B}(\mathcal{C}([0,T]))$ is measurable.

Without the continuity assumption, check that (**b**) implies (**a**) but not vice versa.

7. Maximum process

Theorem (Bachelier). *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $M_t := \max_{s\in[0,t]} B_s$. Then for each (fixed) $t \geq 0$ one has*

$$M_t \overset{(d)}{=} M_t - B_t \overset{(d)}{=} |B_t|\,.$$

- **(a)** Could it be true that, say, $(M_t)_{t\in[0,1]} \overset{(d)}{=} (|B_t|)_{t\in[0,1]}$?
- **(b)** Let $x \geq 0$, $y \leq x$ and $\tau = \inf\{t : B_t = x\}$. Using reflection principle show that
$$\mathbb{P}[\, M_1 \geq x, B_1 \leq y\,] = \mathbb{P}[\, B_{\min\{1,\tau\}} - (B_1 - B_{\min\{1,\tau\}}) \geq 2x - y\,]\,.$$
- **(c)** Show that the joint distribution of the pair (M_1, B_1) is given by the measure
$$-2p'(2x-y)dxdy, \quad x \geq 0,\ y \leq x,$$
where $p(t) = (2\pi)^{-1/2}\exp(-t^2/2)$ is the standard normal density.
- **(d)** Deduce the theorem for $t = 1$ and use the scaling invariance to treat the general case.

8. Law of the iterated logarithm

Theorem (Khinchin). *Let $(B_t)_{t\geq 0}$ be a Brownian motion. Then we have a.s.*

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{2t\log\log(1/t)}} = 1.$$

- **(a)** Show that $\limsup_{t\to 0}(2t\log\log(1/t))^{-1/2}B_t \overset{(d)}{=} \limsup_{t\to\infty}(2t\log\log t)^{-1/2}B_t$.
- **(b)** Let $M_t = \sup_{s\in[0,t]} B_s$. Use Bachelier's theorem to show that
$$\mathbb{P}[M_t > ut^{1/2}] \sim (2/\pi)^{1/2}u^{-1}e^{-u^2/2} \quad \text{as} \quad u \to \infty.$$
- **(c)** Show that $\limsup_{t\to\infty}(2t\log\log t)^{-1/2}B_t \leq 1$ almost surely.
- **(d)** Show that for $r > 1$
$$\limsup_{t\to\infty} \frac{B_t - B_{t/r}}{\sqrt{2t\log\log t}} \geq (1 - r^{-1})^{1/2} \quad \text{a.s.}$$
- **(e)** Show that for $r > 1$
$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} \geq (1 - r^{-1})^{1/2} - r^{-1/2} \quad \text{a.s.}$$
- **(f)** Show that $\limsup_{t\to\infty}(2t\log\log t)^{-1/2}B_t \geq 1$ almost surely.

9. Identities for random walks. Last return to 0 and running maximum.

Theorem (last return to 0, Feller). *Let $(S_m)_{m\geq 0}$ be a simple symmetric random walk in $\mathbb{Z}$ and $\sigma_n := \max\{k \leq n : S_{2k} = 0\}$. Then*

$$\mathbb{P}(\sigma_n = k) \;=\; u_k u_{n-k}, \quad k = 0, \dots, n,$$

where $u_k = \mathbb{P}(S_{2k} = 0) = 2^{-2k}\binom{2k}{k}$.

- **(a)** Formulate a discrete version of the reflection principle for the Brownian motion.
- **(b)** Show that $\mathbb{P}(\sigma_n = k) \;=\; u_k \cdot \mathbb{P}(M_{2(n-k)-1} = 0)$, where $M_m = \max_{k\leq m} S_k$.
- **(c)** Note that $1 - \mathbb{P}(M_{2m-1} = 0) \;=\; \mathbb{P}(M_{2m-1} \geq 1, S_{2m-1} \geq 1) + \mathbb{P}(M_{2m-1} \geq 1, S_{2m-1} \leq -1)$.
- **(d)** Using the reflection principle for the simple random walk, prove that

$$\mathbb{P}(M_{2m-1} = 0) = \mathbb{P}(S_{2m-1} = 1) = u_m.$$

Theorem (running maximum times, Sparre-Andersen). *Let $(S_m)_{m\geq 0}$ be a random walk in $\mathbb{R}$ based on a symmetric diffuse (i.e. absolutely continuous w.r.t. Lebesgue measure) distribution, put $M_n := \max_{k\leq n} S_k$, and write $\tau_n := \min\{k \geq 0 : S_k = M_n\}$. Then*

$$\mathbb{P}(\tau_n = k) \;=\; u_k u_{n-k}, \quad k = 0, \dots, n,$$

where u_k are the same as in the previous theorem.

- **(a)** Show that $\mathbb{P}[\tau_k = 0] = \mathbb{P}[\tau_k = k]$ for all $k \geq 0$.
- **(b)** Show that $\mathbb{P}[\tau_n = k] = v_k v_{n-k}$ for all $0 \leq k \leq n$, where $v_k = \mathbb{P}[\tau_k = k]$.
- **(c)** By induction show that $v_k = u_k$.
- **(d)** **(*)** Is it true that the processes $(\tau_n)_{n\geq 0}$ and $(\sigma_n)_{n\geq 0}$ are identically distributed?

10. Identities for random walks. Sojourns and maxima.

Theorem (sojourns and maxima, Sparre-Andersen). *Let $\xi_1, \dots, \xi_n$ be i.i.d. (more generally, exchangeable) random variables and $S_m = \sum_{k=1}^m \xi_k$ for $0 \leq m \leq n$. Then,*

$$\#\{1 \leq m \leq n : S_m > 0\} \;\overset{(d)}{=}\; \min\{k : S_k \;=\; \max\nolimits_{0\leq m\leq n} S_m\}.$$

- **(a)** This is a deterministic statement. Given some values $\xi_1, \dots, \xi_n \in \mathbb{R}$, let us construct a permutation $\beta : \{1, \dots, n\} \to \{1, \dots, n\}$ by the following algorithm applied consecutively to $k = n,\ k = n-1, \dots,\ k = 1$:
  - if $S_k \leq 0$, denote by $\beta(k)$ the maximal available index in $\{1, \dots, n\}$;
  - if $S_k > 0$, denote by $\beta(k)$ the minimal available index in $\{1, \dots, n\}$.

  Let $S_m^{(\beta)} = \sum_{k=1}^m \xi_{\beta(k)}$ for $0 \leq m \leq n$. Prove that

$$\#\{1 \leq m \leq n : S_m > 0\} \;=\; \min\{k : S_k^{(\beta)} = \max\nolimits_{0\leq m\leq n} S_m^{(\beta)}\}.$$

- **(b)** Prove that $(\xi_{\beta(1)}, \dots, \xi_{\beta(n)}) \;\overset{(d)}{=}\; (\xi_1, \dots, \xi_n)$ and deduce the theorem.

  Hint: re-write the condition $S_k \leq 0$ as $\xi_{k+1} + \dots + \xi_n \geq S_n = S_n^{(\beta)}$.

11. Reflected Brownian motion. The processes $(M_t - B_t)_{t\geq 0}$ and $(|B_t|)_{t\geq 0}$.

(a) Let $(S_m)_{m\geq 0}$, $(S'_m)_{m\geq 0}$ be two independent simple symmetric random walks in $\mathbb{Z}$ started at the origin and $\widetilde{S}'_m := S'_m + \frac{1}{2}$ for $m \geq 0$. Let $M_m := \max_{k\leq m} S_k$ and

$$L'_m := \#\left\{1 \leq k \leq m : \widetilde{S}'_k = -\widetilde{S}'_{k-1} \in \{\pm\tfrac{1}{2}\}\right\}$$

denote the number of steps before time m when the trajectory $(\widetilde{S}'_m)_{m\geq 0}$ crosses the horizontal line. Prove that the processes

$$(M_m - S_m\,, M_m)_{m\geq 0} \quad \text{and} \quad (|S'_m| - \tfrac{1}{2}\,, L'_m)_{m\geq 0}$$

are identically distributed. *Hint:* note that both processes can be described as the (identically distributed) random walks in $\mathbb{Z}_+ \times \mathbb{Z}_+$.

(b) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $M_t := \max_{s\in[0,t]} B_s$. Using Donsker's theorem, prove that

$$(M_{t_1} - B_{t_1}, \ldots, M_{t_p} - B_{t_p}) \overset{(d)}{=} (|B_{t_1}|, \ldots, |B_{t_p}|)$$

for all $0 \leq t_1 \leq \ldots \leq t_p$.

(c) Conclude that the processes $(M_t - B_t)_{t\geq 0}$ and $(|B_t|)_{t\geq 0}$ are identically distributed. *Hint:* The mapping $f(t) \mapsto \max_{s\in[0,t]} f(s)$ is continuous in $C([0,T])$ for each $T > 0$.

(d) **(*)** Denote by $(\mathcal{F}_t^{(1)})_{t\geq 0}$ and $(\mathcal{F}_t^{(2)})_{t\geq 0}$ the filtrations generated by these two processes. Is it true that $\mathcal{F}_t^{(1)} = \mathcal{F}_t$, where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by $(B_t)_{t\geq 0}$ itself? (In other words, can one reconstruct B_t from $M_t - B_t$?) Is it true that $\mathcal{F}_t^{(2)} = \mathcal{F}_t$?

12. Uniform laws for the Brownian bridge

Theorem. *Let $(B_t)_{t\in[0,1]}$ be a Brownian bridge on $[0,1]$ and $M := \max_{s\in[0,1]} B_s$. Then the following random variables are both $U(0,1)$, i.e. uniformly distributed on $[0,1]$:*

$$\tau_1 = \lambda\{t \in [0,1] : B_t > 0\}, \quad \tau_2 = \inf\{t : B_t = M\}.$$

(a) Given $u \in [0,1]$, define $B_t^u := B_{(u+t)} - B_u$, where $(x) := x - \lfloor x \rfloor$. Show that the process $(B_t^u)_{t\in[0,1]}$ is distributed as a Brownian bridge.

(b) Let $\tau_2^u := \inf\{t : B_t^u = M - B_u\}$ and let u be uniformly distributed over $[0,1]$. Apply Fubini's theorem to show that $\mathbb{P}(\tau_2 \leq t) = \int_0^1 \mathbb{P}(\tau_2^u \leq t)du$ is also $U(0,1)$.

(c) Using Donsker's theorem and Exercise 10, show that τ_1 and τ_2 have the same law.

(d) **(*)** Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, $(\widetilde{B_t})_{t\in[0,1]}$ be a Brownian bridge. For each $T < 1$ prove that the distributions of the processes $(B_t)_{t\in[0,T]}$ and $(\widetilde{B}_t)_{t\in[0,T]}$ are mutually absolutely continuous. *Hint:* prove that the processes $(B_t - tT^{-1}B_T)_{t\in[0,T]}$ and $(\widetilde{B}_t - tT^{-1}\widetilde{B}_T)_{t\in[0,T]}$ are identically distributed (as a re-scaled Brownian bridge).

13. Exit time from $[-a, a]$ and time spent in $[0, -a]$ during a downcrossing

(a) Let $a > 0$ and $\tau_{\pm a} := \inf\{t \geq 0 : |B_t| = a\}$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion (started from 0). Show that

$$\mathbb{E}[\exp(-\mu\tau_{\pm a})] = 1/\cosh(a\sqrt{2\mu}), \quad \mu \geq 0.$$

Compute the expectations $\mathbb{E}[\tau_a]$ and $\mathbb{E}[\tau_a^2]$. Is is true that $\mathbb{E}[\exp(\theta\tau_{\pm a})] < +\infty$ for some $\theta > 0$? What is the optimal upper bound for such θ's?

Proposition (removing of negative excursions). *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, $s(t) := \lambda(\{s' \in [0,t] : B_{s'} \geq 0\})$ and $t(s) := \inf\{t \geq 0 : s(t) \geq s\}$. Then*

$$(B_{t(s)})_{s\geq 0} \overset{(d)}{=} (|B_s|)_{s\geq 0}\,.$$

(b) Check a similar statement for simple random walks (this is trivial).

(c) Prove that for each $T > 0$ one has $\lim_{\varepsilon\downarrow 0} \lambda(t \in [0,T] : |B_t| \leq \varepsilon) = 0$ almost surely.

(d) Let $s_f(t)$ and $t_f(s)$ be defined as in the proposition via a continuous function $f = f(t)$ instead of B_t. Check that the mapping $(f(t))_{t\in[0,T]} \mapsto (f(t(s \wedge s(T))))_{s\in[0,T]}$ is continuous (in the $C([0,T])$ metric) at almost every Brownian motion trajectory $(B_t)_{t\in[0,T]}$.

(e) Prove the proposition using (b) and Donsker's invariance principle.

(f) Let $\tau_{-a} := \inf\{t \geq 0 : B_t = -a\}$. Prove that $\lambda(s \in [0, \tau_{-a}] : 0 \geq B_s \geq -a) \overset{(d)}{=} \tau_{\pm a}$.

14. Recurrence/transience of the d-dimensional Brownian motion

Let $d \geq 2$ and $A(r,R) := \{x \in \mathbb{R}^d : r < \|x\| < R\}$. Let $(B_t^x)_{t\geq 0}$ denote a standard d-dimensional Brownian motion started from x and $\tau_{r,R} = \tau_{r,R}^x := \inf\{t > 0 : B_t^x \notin A(r,R)\}$.

(a) For $x \in A(r,R)$, prove that

$$\mathbb{P}^x[|B^x_{\tau_{r,R}}| = r] \;=\; \begin{cases} (\log R/\|x\|)\cdot(\log R/r)^{-1} & \text{if } d = 2,\\ \left(\|x\|^{2-d} - R^{2-d}\right)\cdot\left(\|r\|^{2-d} - R^{2-d}\right)^{-1} & \text{if } d \geq 3.\end{cases}$$

(b) Prove that $\mathbb{P}[\,\exists t > 0 : B_t^x = 0\,] = 0$ for all $x \in \mathbb{R}^d$.

(c) Let $d = 2$ and $\|x\| > r > 0$. Prove that $\mathbb{P}[\,\exists t \geq 0 : |B_t^x| = r\,] = 1$ and deduce that

$$\mathbb{P}[\,\exists 0 < t_1 < t_2 < \ldots : t_k \to \infty \text{ and } |B^x_{t_k}| = r\,] \;=\; 1$$

for any $x \in \mathbb{R}^d$ and $r > 0$ (in other words, the Brownian motion in 2D is *recurrent*).

(d) Let $d \geq 3$. Prove that $B_t \to \infty$ as $t \to \infty$ almost surely (in other words, the Brownian motion in 3+D is *transient*). *Hint:* consider events $A_n := \{\exists s < t : B_s \geq n^3, B_t \leq n\}$.

(e) **(*)** Let $d \geq 2$, $U \in \mathbb{R}^d$ is a bounded open set, $x \in \partial U$ and there exist an open cone C (with the vertex at x) such that $B(x,r) \cap C \subset \mathbb{R}^d \setminus U$ for some $r > 0$. Prove that x is a *regular boundary point of* U, i.e. $\inf\{t > 0 : B_t^x \notin U\} = 0$ almost surely.

15. N-dimensional Brownian motion: Itô's calculus, local martingales and Bessel processes

(**a**) Let $\alpha \in \mathbb{R}$ and B_t be a N-dimensional Brownian motion started at $x \neq 0$. Consider (local) semi-martingale $X_t := |B_t|^\alpha$. Compute dX_t using Itô's calculus.

(**b**) Note that X_t is a local martingale if $\alpha = 2-N$, which agrees with the fact that the function $H(x) = |x|^{2-N}$ is harmonic in $\mathbb{R}^N$ (one should consider $\log |B_t|$ for $N = 2$).

Let $N = 3$, $\alpha = -1$ and $X_t = |B_t|^{-1}$ so that X_t is a local martingale. Note that, almost surely, the process $(X_t)_{t\geq 0}$ is well-defined for all $t \geq 0$ and $\lim_{t\to\infty} X_t = 0$.

(**c**) Give an example of a sequence $(\tau_n)_{n\in\mathbb{N}}$ of localizing stopping times for $(X_t)_{t\geq 0}$.

(**d**) Check that $\mathbb{E}[X_t^2] \leq \text{const} < +\infty$ for some constant independent of $t \geq 0$.

(**e**) What is the law of the random variable $\max_{t\in[0,+\infty)} X_t$? Does it belong to $L^1(\Omega)$?

(**f**) Using the explicit formula for the Gaussian density, prove that $\mathbb{E}[X_T] < X_0 = x^{-1}$.
Hint: Use the identity $|S_r|^{-1} \int_{S_r} |y+x|^{-1} d\lambda_{S_r}(x) = \min\{x^{-1}, r^{-1}\}$, where λ_{S_r} denotes the Lebesgue measure on the sphere $S_r = \{y \in \mathbb{R}^3 : |y|=r\}$ and $|S_r|$ is the area of S_r.

Remark. *Thus, X_t is a local martingale but not a (true) martingale.*

Let $(B_t)_{t\geq 0}$ be a N-dimensional Brownian motion started at $x \neq 0$. Denote

$$\beta_t \;:=\; \sum_{k=1}^{N} \int_0^t \frac{B_t^k}{|B_t|}\, dB_t^k,$$

(we set $B_t^k/|B_t| = 0$ if $|B_t| = 0$). Check that this stochastic integral is well-defined.

(**g**) Prove that the process $(\beta_t)_{t\geq 0}$ is a 1-dimensional Brownian motion started at 0.
Hint: Check that $(\beta_t)_{t\geq 0}$ and $(\beta_t^2 - t)_{t\geq 0}$ are local martingales, and use Lévy's theorem.

(**h**) Using (a), show that

$$|B_t| \;=\; |x| + \beta_t + \frac{N-1}{2} \int_0^t \frac{ds}{|B_s|}\,,$$
$$|B_t|^2 \;=\; |x|^2 + 2\int_0^t |B_s| d\beta_s + Nt\,.$$

Definition. *Let $(\beta_t)_{t\geq 0}$ be a standard 1-dimensional Brownian motion started at 0 and $m \geq 0$. A process $(X_t)_{t\geq 0}$ satisfying the stochastic differential equation*

$$dX_t \;=\; 2\sqrt{X_t}\, d\beta_t + m dt$$

is called a squared Bessel process of dimension m. The process $Y_t := \sqrt{X_t}$ (if $m < 2$, there is an issue with signs of Y_t) is called an m-dimensional Bessel process.

Remark. *If $m = N \geq 2$ is integer, then $(Y_t)_{t\geq 0} \overset{(d)}{=} (|B_t|)_{t\geq 0}$. Note that, almost surely, $|B_t| \neq 0$ for all $t \geq 0$. Thus in this case there exists a unique continuous choice of the square root of the squared process $(X_t)_{t\geq 0} \overset{(d)}{=} (|B_t|^2)_{t\geq 0}$.*

16. Local time at zero for 1D Brownian motion

Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion. For every $\varepsilon > 0$, we define a function

$$g_\varepsilon : \mathbb{R} \to \mathbb{R}_+ , \qquad g_\varepsilon(x) := \sqrt{x^2 + \varepsilon^2}.$$

Note that $g'_\varepsilon(x) \to \operatorname{sign}(x)$ as $\varepsilon \to 0$ uniformly on each of the sets $\{x \in \mathbb{R} : |x| \geq \delta\}$, $\delta > 0$.

(a) Apply Itô's formula to compute a decomposition $g_\varepsilon(B_t) = g_\varepsilon(B_0) + M^\varepsilon_t + A^\varepsilon_t$, where M^ε_t is a local martingale and A^ε_t is a bounded variation process. Observe that M^ε_t is a square integrable martingale and A^ε_t is increasing.

(b) Show that, for each $T > 0$,

$$(M^\varepsilon_t)_{t\in[0,T]} \underset{\varepsilon\to 0}{\to} (\beta_t)_{t\in[0,T]} \quad \text{in} \quad L^2(\Omega), \quad \text{where} \quad \beta_t := \int_0^t \operatorname{sign}(B_s) dB_s$$

and the convergence is understood in the metric of $C([0, T])$.

Remark. *Recall that Lévy's theorem implies that the process $(\beta_t)_{t\geq 0}$ is another one-dimensional Brownian motion defined on the same filtration as $(B_t)_{t\geq 0}$.*

(c) Infer that there exists an increasing process $(L^0_t)_{t\geq 0}$ such that

$$|B_t| = \int_0^t \operatorname{sign}(B_s) dB_s + L^0_t \quad \text{for all } t \geq 0.$$

Observing that $(A^\varepsilon_t)_{t\in[0,T]} \to (L^0_t)_{t\in[0,T]}$ as $\varepsilon \to 0$ show that, for every choice of the segment $[u, v] \subset \mathbb{R}_+$ and $\delta > 0$, the following is true:

$$|B_t| \geq \delta \ \text{ for all } t \in [u, v] \quad \Rightarrow \quad L^0_u = L^0_v .$$

Infer that L^0_t is constant on each of the intervals $(u, v) \subset \mathbb{R}_+$ such that $B_t \neq 0$ on (u, v).

Definition. *The process L^0_t is called a local time of the Brownian motion B_t at 0.*

(d) Given $\varepsilon > 0$, define two sequences of stopping times $(\sigma^\varepsilon_n)_{n\geq 1}$ and $(\tau^\varepsilon_n)_{n\geq 1}$ inductively by setting $\sigma^\varepsilon_1 := 0$,

$$\tau^\varepsilon_n := \inf\{t > \sigma^\varepsilon_n : \ |B_t| = \varepsilon\} \quad \text{and} \quad \sigma^\varepsilon_{n+1} := \inf\{t > \tau^\varepsilon_n : \ B_t = 0\} .$$

Further, let $N^\varepsilon_t := \max\{n \geq 1 : \tau^\varepsilon_n \leq t\}$ with the convention $\max \emptyset := 0$. Show that

$$\varepsilon N^\varepsilon_t \underset{\varepsilon\to 0}{\to} L^0_t \quad \text{in} \quad L^2(\Omega).$$

Hint: Observe that

$$\left| L^0_t + \int_0^t \left(\textstyle\sum_{n\geq 1} \mathbf{1}_{[\sigma^\varepsilon_n, \tau^\varepsilon_n]}(s)\right) \operatorname{sign}(B_s) dB_s - \varepsilon N^\varepsilon_t \right| \leq \varepsilon.$$

Remark. *Note that L^0_t can be reconstructed from the* absolute value $|B_t|$ *of the Brownian motion B_t. In particular, this means that the filtration generated by the Brownian motion $\beta_t = |B_t| - L^0_t$ is* strictly smaller *than the one generated by B_t itself.*