ENS, Stochastic Processes, Fall 2016

§0. Reminders, introduction. [13.09.2016]

- Random variable $(\Omega, \mathcal{A}, \mathbb{P})$, $X: \Omega \to \mathbb{R}$
  $[\mathbb{P}(\Omega) = 1]$ measurable wrt $(\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

  NB: Ω is abstract.
  in principle one can think about $\mu_X := X_*(\mathbb{P})$ (pushforward of $\mathbb{P}$ by X) which is a measure on $\mathbb{R}$.
  But this is not appropriate when working with many r.v. simultaneously

- In fact, one can replace $\mathbb{R}$ by some other (say, complete metric) space.

Terminology:
- random variable $X: \Omega \to \mathbb{R}\ (\mathbb{P})$
- random vector $X: \Omega \to \mathbb{R}^n\ (\mathbb{P}^n)$.
  can be thought of a collection $X_1, \dots, X_n$ of (possibly dependent) defined on the same Ω) random variables indexed by a finite set

- random/stochastic process
  - $\to$ discrete time $X_0, X_1, X_2, \dots, X_n, \dots$ (collection of r.v. indexed by $\mathbb{N}$ or $\mathbb{Z}$)
  - $\to$ cont. time $(X_t)_{t \in [0,T]}$ (or $[0,\infty)$ or $\mathbb{R}$ etc)
    (collection of r.v. indexed by a totally ordered set)

- random/stochastic/statistical field
  (e.g. $(X_u)_{u \in \Omega}$ where $\Omega \subset \mathbb{R}^d$).

Important remark [digression]: Once the index set becomes uncountable, one should be careful with measurability issues

Def: $(X_t)_{t \in [0,T]}$ is a measurable stochastic process if $(t, \omega) \mapsto X_t(\omega)$ is measurable wrt $\mathcal{B}([0,T]) \otimes \mathcal{A} \to \mathcal{B}(\mathbb{R})$

Alternatives: (a) $\forall t \in [0,T]$ $X_t : \Omega \to \mathbb{R}$ is measurable

(b) Fix some functional space, e.g. $C([0,T])$ and ask that $X: \Omega \to C([0,T])$ is measurable

Exercise: Provided we know that $X \in C([0,T])$ $\forall \omega \in \Omega$,
(a) $\Leftrightarrow$ Def $\Leftrightarrow$ (b)

Basic examples of stochastic processes:

- In discrete time: random walks (more generally, Markov chains)
- In continuum time: Brownian motion, Poisson process, etc.

NB: can be thought of as a limit of random walks

[Some informal discussion, more details later]

• Characteristic function

$$X : \Omega \to \mathbb{R} \quad \rightsquigarrow \quad \varphi_X(z) := \mathbb{E}[e^{izX}], \quad z \in \mathbb{R}$$
$$= \int_\Omega e^{izX} \, d\mathbb{P}(\omega)$$

NB: why z ?
If $X \geq 0$, then is well defined for $z \in \overline{\mathbb{H}}$ (where $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$)
In particular, $z = i\lambda$ gives the Laplace transform $\mathcal{L}_X(\lambda) := \mathbb{E}[e^{-\lambda X}]$, $\lambda \geq 0$.

Properties:
- φ - cont. & bdd (by $\varphi(0) = 1$) on $\mathbb{R}$
- If $X \geq 0$, then cont., bdd. on $\overline{\mathbb{H}}$ and analytic in $\mathbb{H}$ (since cont. diff.)
- smoothness of $\mu_X \longleftrightarrow$ decay of $\varphi(z)$ as $z \to \pm\infty$
  [e.g. if there exists a density $d\mu_X = p(x)dx$, $p \in L^1$, then $\varphi(z) \xrightarrow[z \to \pm\infty]{} 0$
  (Riemann - Lebesgue lemma)]
- Smoothness of $\varphi(z)$ (enough at $z=0$) $\longleftrightarrow$ moments of X.

Lemma: (a) $\mathbb{E}[|X|^n] < \infty \Rightarrow \varphi_X \in C^n(\mathbb{R})$ and
$$\varphi_X(z) = 1 + iz\,\mathbb{E}X + \frac{(iz)^2}{2}\mathbb{E}X^2 + \dots + \frac{(iz)^n}{n!}\mathbb{E}X^n + o(z^n), \; z \to 0$$
(and similarly at every other point)

(b) $\exists\, \varphi_X^{(2k)}(0) \Rightarrow \mathbb{E}[X^{2k}] < \infty$ $[\Rightarrow \varphi_X \in C^{2k}(\mathbb{R})]$

Remarks:
- (b) is not true if n is odd
- If $X \geq 0$, then $\mathbb{E}X^n < \infty \Leftrightarrow \exists\, \mathcal{L}_X^{(n)}(0+)$

Proof:
(a) is trivial, for (b) consider $[\Delta_\delta f](x) := \frac{f(x+\delta) - 2f(x) + f(x-\delta)}{\delta^2}$

$$\Delta_\delta^k \varphi_X(z) = \mathbb{E}\left[\left(-4\frac{\sin^2\frac{\delta X}{2}}{\delta^2}\right)^k e^{izX}\right]$$

They remain bounded as $\delta \to 0$ at $z = 0$
$$\Rightarrow \infty > \liminf_{\delta \to 0} \mathbb{E}\left[\left|\frac{\sin \delta X}{\delta}\right|^{2k}\right] \geq \underbrace{\mathbb{E}\left[\liminf_{\delta \to 0}\left|\frac{\sin \delta X}{\delta}\right|^{2k}\right]}_{\text{Fatou's lemma}} = \mathbb{E}[X^{2k}]$$

• Convergence in distribution (aka conv. in law / weak conv.)

Def: $X_n \xrightarrow[n \to \infty]{(d)} X$ iff $\forall h \in C_{bdd}(\mathbb{R})$
$$\mathbb{E}[h(X_n)] \xrightarrow[n \to \infty]{} \mathbb{E}[h(X)]$$

Thm (Lévy):
$$X_n \xrightarrow[n \to \infty]{(d)} X \iff \forall z \in \mathbb{R} \;\; \varphi_{X_n}(z) \xrightarrow[n \to \infty]{} \varphi_X(z)$$

[Moreover, if we do not know a priori that $\varphi_X(z)$ is a characteristic fct of some random variable, than it is enough to know that φ_X is cont. at $z=0$]

NB: Effectively, we need to prove that $\{\mu_{X_n}\}$ is a tight

Pf: Lemma: $\mathbb{P}[|X| \geq 1/a] \leq cst \cdot \frac{1}{a}\int_0^a (1 - \mathrm{Re}\, \varphi_X(z))\,dz$

$\Rightarrow$ trivial

$\Leftarrow$

Pf: $rhs \overset{\text{Fubini}}{=} \mathbb{E}\left[\frac{1}{a}\int_0^a (1-\cos(zX))\,dt\right]$

$= \mathbb{E}\left[1 - \frac{\sin aX}{aX}\right] \geq cst \cdot \mathbb{P}[|X| \geq 1/a]$ ⊠

$\forall$ fixed $a > 0$ $\quad \frac{1}{a}\int_0^a (1 - \mathrm{Re}\,\varphi_{X_n}(z))\,dz \underset{n\to\infty}{\longrightarrow} \frac{1}{a}\int_0^a (1-\mathrm{Re}\,\varphi_X(z))\,dz$

Choose a so that $\left|\frac{1}{a}\int_0^a \ldots\right| < \frac{\varepsilon}{2}$ [here we use that φ_X is cont. at 0]

Then choose n_0: $\left|\frac{1}{a}\int_0^a \ldots\right| < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow \mathbb{P}[|X_n| \geq 1/a] < \varepsilon$

Thus, the family $\{\mu_{X_n}\}$ is tight (Banach-Alaoglu)

$\Rightarrow \exists\ \mu_{X_{n_k}} \xrightarrow{(w)} \mu$ [probability measure]

Clearly, μ is the same for all subsequences since $\varphi_{X_{n_k}}(z) \to \varphi_X(z)$, which characterizes μ ⊠

- Law of large numbers, Central limit theorem, etc.

> Remark: $X \perp\!\!\!\perp Y$
> $\Rightarrow \varphi_{X+Y}(z) = \varphi_X(z)\varphi_Y(z)$

Thm: (weak LLN)

$\xi_1, \ldots, \xi_n$ – i.i.d., $\mathbb{E}|\xi| < \infty \Rightarrow X_n := \frac{\xi_1 + \cdots + \xi_n}{n} \underset{n\to\infty}{\xrightarrow{(d)}} a := \mathbb{E}\xi$

Pf: $\varphi_\xi(z) = 1 + iz\,\mathbb{E}\xi + o(|z|)$ as $z \to 0$

$\Rightarrow \varphi_{X_n} = \left(1 + \frac{iza}{n} + o\left(\frac{|z|}{n}\right)\right)^n \underset{n\to\infty}{\longrightarrow} e^{iza}$ ⊠

Thm: (CLT)

$\xi_1, \ldots, \xi_n$ – i.i.d., $\mathbb{E}\xi^2 < \infty \Rightarrow X_n := \frac{\xi_1 + \cdots + \xi_n - an}{\sigma\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0,1)$

where $\sigma^2 := \mathrm{Var}(\xi)$

standard Gaussian, density $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}dx$

Pf: wlog $a = 0$

$\varphi_\xi(z) = 1 - \frac{\sigma^2 z^2}{2} + o(z^2)$ as $z \to 0$

$\Rightarrow \varphi_{X_n}(z) = \left(1 - \frac{z^2}{2n} + o\left(\frac{z^2}{n}\right)\right)^n \underset{n\to\infty}{\longrightarrow} e^{-z^2/2}$ ⊠

[Optional] (*) → NEXT LECTURE

Thm: $\xi_1^{(n)}, \ldots, \xi_n^{(n)}$ – i.i.d. Bernoulli, with $\mathbb{P}[\xi^{(n)} = 1] = c_n$

If $nc_n \underset{n\to\infty}{\longrightarrow} c < \infty$, then

$X_n := \xi_1^{(n)} + \cdots + \xi_n^{(n)} \underset{n\to\infty}{\xrightarrow{(d)}} \mathrm{Poisson}(c)$

Pf: $\varphi_{X_n}(z) = \left[1 + c_n(e^{iz} - 1)\right]^n \underset{n\to\infty}{\longrightarrow} \exp\left[c(e^{iz}-1)\right]$

$= e^{-c}\sum_{n\geq 0} \frac{c^n}{n!} e^{inz}$ ⊠

Now let us prove a version of the CLT that deals with

§1. Continuous random processes with independent increments

Thm: Let $(X_t)_{t\in[0,T]}$ be s.t. $X_0=0$,

$\forall\ (s_k,t_k)$ – disjoint intervals the increments $X_{t_k}-X_{s_k}$ are indep.

& $\mathbb{P}[X_t \text{ is cont. on } [0,T]]=1$.

Remark: A priori, this event is not necessary measurable. Two ways: (a) outer measure, (b) completion of the σ-algebra

Then $\exists$ two cont. fcts $a, \sigma : [0,T]\to\mathbb{R}$ $\quad (a(0)=\sigma(0)=0,\ \sigma \text{ increasing})$

s.t. $X_t - X_s \sim \mathcal{N}(a(t)-a(s),\ \sigma^2(t)-\sigma^2(s))$

Remark: no assumptions on the existence of second moments (or even first)

Pf: $t_j^{(n)} := s + \frac{j}{n}(t-s)$, $\quad \xi := X_t - X_s$

$$\xi_j^{(n)} := \begin{cases} X_{t_{j+1}^{(n)}} - X_{t_j^{(n)}} & \text{if } |\cdot| \le \varepsilon^{(n)} \\ 0 & \text{if } |\cdot| > \varepsilon^{(n)} \end{cases} \qquad \xi^{(n)} := \xi_0^{(n)} + \dots + \xi_{n-1}^{(n)}$$

How to choose $\varepsilon^{(n)}$? First, let $\varepsilon^{(n)} = \varepsilon > 0$ be fixed.

(*) Note that $\mathbb{P}[\xi \ne \xi^{(n)}] \le \mathbb{P}[\max_{j=0,\dots,n-1} |X_{t_{j+1}^{(n)}} - X_{t_j^{(n)}}| > \varepsilon^{(n)}] \xrightarrow[n\to\infty]{} 0$

due to the condition that $\mathbb{P}[X_t \text{ is cont}]=1$.

$\Rightarrow$ one can find $\varepsilon^{(n)} \downarrow 0$ such that (*) still holds true.

Denote $a_j^{(n)} := \mathbb{E}\,\xi_j^{(n)}$, $\eta_j^{(n)} := \xi_j^{(n)} - a_j^{(n)}$, $(\sigma_j^{(n)})^2 := \operatorname{Var} \xi_j^{(n)} = \operatorname{Var} \eta_j^{(n)}$

Then $\varphi_j^{(n)}(z) := \mathbb{E}[\exp(iz\eta_j^{(n)})]$

$$= 1 - \frac{z^2 (\sigma_j^{(n)})^2}{2} + O(|z|^3\, \mathbb{E}[|\eta_j^{(n)}|^3])$$

$$= 1 - \frac{z^2(\sigma_j^{(n)})^2}{2}\left(1 + O(|z|\varepsilon^{(n)})\right)$$

in any fixed neighborhood of $z=0$.

Since $\sigma_j^{(n)} \le 2\varepsilon_j^{(n)}$, this gives

$$\log \varphi_j^{(n)}(z) = -\frac{z^2(\sigma_j^{(n)})^2}{2}\left(1+O(|z|\varepsilon^{(n)})\right) \text{ and hence}$$

$$\log \varphi^{(n)}(z) = -\frac{z^2(\sigma^{(n)})^2}{2}\left(1+O(|z|\varepsilon^{(n)})\right)$$

where $\varphi^{(n)}(z) := \mathbb{E}[\exp(iz\eta^{(n)})]$ and $(\sigma^{(n)})^2 := \sum_{j=0}^{n-1} (\sigma_j^{(n)})^2$

Recall that $\mathbb{E}[\exp(iz\xi^{(n)})] = e^{ia^{(n)}z}\varphi^{(n)}(z)$

where $a^{(n)} := \sum_{j=0}^{n-1} a_j^{(n)}$

and $\mathbb{P}[\xi^{(n)} \ne \xi] \xrightarrow[n\to\infty]{} 0$

Then one easily concludes that $(\sigma^{(n)})^2$ is bdd

[otherwise one would have (along a subsequence)

$\varphi^{(n)}(z) \xrightarrow[n\to\infty]{} 0\ \ \forall z\ne 0$ which means $\mathbb{E}[\exp(iz\xi)] = 0\ \forall z \ne 0$

and thus is impossible]

Therefore, at least along a subsequence, $(\sigma^{(n)})^2 \xrightarrow[n\to\infty]{} \sigma^2$, which gives $f^{(n)}(z) \to e^{-\frac{z^2\sigma^2}{2}}$

Now, note that $a^{(n)}$ must be bdd too: otherwise $\mathbb{E}\left[\exp\left(i \cdot \frac{\pi}{a^{(n)}} \xi^{(n)}\right)\right] \xrightarrow[n\to\infty]{} -1$ (for $a^{(n)} \to \infty$), which contradicts to the continuity of f_ξ at $z=0$.

$\Rightarrow$ along a subsequence there exists a limit $a^{(n)} \xrightarrow[n\to\infty]{} a$ and $f_\xi(z) = \exp\left(iza - \frac{z^2\sigma^2}{2}\right)$, i.e. $\xi \sim N(a, \sigma^2)$.

Finally, the fact that $a = a(t) - a(s)$ is trivial, $\sigma^2 = (\sigma(t))^2 - (\sigma(s))^2$ follows from the independence and continuity of a & σ can be easily deduced from (a.s.) continuity of X_t which implies $X_{t+\varepsilon} - X_t \xrightarrow[\varepsilon\downarrow 0]{} 0$ a.s.

<u>Discussion</u>: • If the increments are i.i.d., then $a(t) = at$ and $(\sigma(t))^2 = \sigma^2 t$.

• In the general setup, $a(t)$ is not important (just subtract) and one can change the parameterization so that $(\sigma(t))^2 = t$

NB: Careful, if $\sigma(t)$ is constant on $[t_1, t_2]$ (which means that $X_{t_1} = X_{t_2}$ a.s.)

This leads to the following question:

What is a(?) / the(?) random process with $a(t) = 0$, $(\sigma(t))^2 = t$?

This is called a Brownian motion or a Wiener process [Robert Brown 1827: pollen in water] [Wiener 1920s]

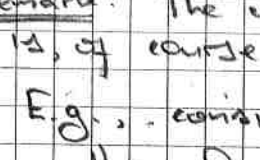
(*) REMINDER ON POISSON VARIABLES (SEE ABOVE)

Remark: The continuity assumption is, of course, very important

E.g., consider the Poisson process of intensity λ:

$\xi_1, \xi_2, \ldots$ i.i.d. $\sim \lambda e^{-\lambda x} \mathbb{1}_{\{x>0\}} dx$ (exponential)

$X_t := \max\{N : \xi_1 + \ldots + \xi_N \le t\}$



Exercise:

(a) $X_t - X_s \sim \text{Poisson}(\lambda(t-s))$

(b) Increments are independent

ALL THIS → NEXT LECTURE

<u>Discussion</u>: LÉVY PROCESSES

$X_0 = 0$, indep. stationary increments $(X_t - X_s \overset{(d)}{=} X_{t-s})$

(+) continuity in probability: $\forall t \ge 0 \ \forall \varepsilon > 0 \ \lim_{h\downarrow 0} \mathbb{P}[|X_{t+h} - X_t| \ge \varepsilon] = 0$

NB: $\Rightarrow \exists$ a cadlag version of X_t [$\forall t \ge 0 \ \mathbb{P}[X_t = \tilde{X}_t] = 1$]

Fact (Lévy–Khin(t)chin(e)), w/o proof

$\forall$ Lévy process is a linear comb. (more rigorously, an integral over the size of jumps) of $\text{cst} \times t$, $\text{cst} \times BM$ and Poisson processes w/ different jumps & intensities

§2. Gaussian vectors and (Lévy - Ciesielski) construction of a Brownian motion.

Motivating discussion:

We want to construct a process X_t s.t. (say, $t\in[0,1]$)

$X_0=0$, increments are independent and s.t. $X_t - X_s \overset{(d)}{=} X_{t-s} \sim N(0,t-s)$

NB: $\Leftrightarrow$ $\mathrm{Cov}(X_t, X_s) = s\wedge t$

(since, for $t\geq s$, $X_t - X_s \perp\!\!\!\perp X_s$)

MORE DETAILS IN EXERCISES

Reminder:

$X:\Omega\to\mathbb{R}^d$ is called a Gaussian vector iff $\forall c_1,\dots,c_d\in\mathbb{R}$ $c_1X_1+\dots+c_dX_d$ is Gaussian

$\Leftrightarrow$ density (on a subspace)

$$p(x)=\frac{\exp\left[-\frac{1}{2}(x-a)^T\Sigma^{-1}(x-a)\right]}{\sqrt{(2\pi)^d\,|\det\Sigma|}}$$

where $\Sigma=\Sigma^T=\mathbb{E}[X^TX]>0$ is a covariance matrix

Important:

$\exists$ basis in which Σ is diagonal $\Rightarrow$ components of X in this basis are independent

Let us think of X_t as a "Gaussian vector in some infinite-dimensional Hilbert space" (e.g., $L^2([0,1])$) and fix some orthonormal basis $f_n(t)$ there

Since $\mathbb{E}[X_sX_t]=s\wedge t$, we can formally (!) write

$$\mathbb{E}[\dot{X}_s X_t] = \begin{cases}1, & s<t\\ 0, & s>t\end{cases}$$

and $\mathbb{E}[\dot{X}_s\dot{X}_t]=\delta_s(t)$ (as a function of t)

NB: THIS IS JUST A FORMAL COMPUTATION

Now expand $\dot{X}_t$ in this basis:

$$\dot{X}_t=\sum_{n=0}^{\infty}\xi_n f_n(t)$$

note that $\xi_n=\langle \dot{X}, f_n\rangle$ should be (centered) Gaussian as linear functionals of Gaussian

Moreover,

$$\mathbb{E}\,\xi_n\xi_m = \mathbb{E}\left[\int_0^1 \dot{X}_s f_n(s)\,ds \int_0^1 \dot{X}_t f_m(t)\,dt\right]$$

$$\underset{\text{formal Fubini}}{=} \int_0^1\int_0^1 \mathbb{E}[\dot{X}_s\dot{X}_t]\, f_n(s) f_m(t)\,ds\,dt$$

$$=\int_0^1 f_n(s) f_m(s)\,ds = \begin{cases}1 & n=m\\ 0 & n\neq m\end{cases}$$

MORAL: WE EXPECT THAT ξ_n ARE i.i.d. STANDARD GAUSSIAN

AND $X_t=\sum_{n=0}^{\infty}\xi_n g_n(t)$

WHERE $g_n(t)=\int_0^t f_n(s)\,ds$

Remark:

This should be true for any orthonormal basis of L^2 (think about $\Sigma=\mathrm{Id}$ for Gaussian vectors)

(1940s) Lévy: Fourier (?) basis

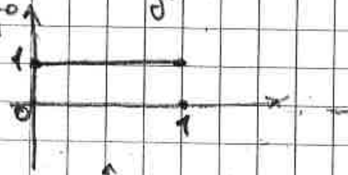
(1961) Ciesielski: Haar basis

In fact, it is easier to use the Haar basis:

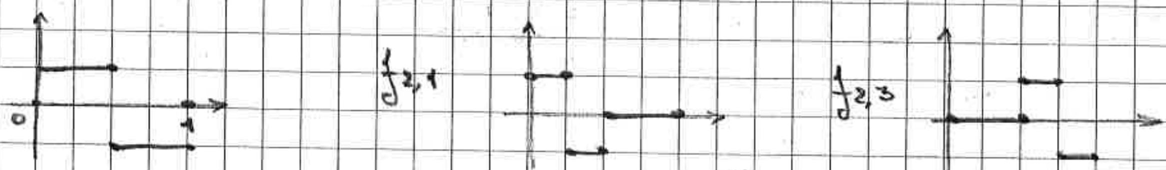
$f_0(t) = 1$, $f_{k,n}(t) = 2^{\frac{n-1}{2}} \times \begin{cases} 1, & {}^{k-1}/_{2^n} \le t < {}^{k}/_{2^n} \\ -1, & {}^{k}/_{2^n} \le t < {}^{k+1}/_{2^n} \\ 0, & \text{otherwise} \end{cases}$

$\begin{cases} n = 1, 2, 3, \dots \\ k = 1, 3, \dots, 2^n - 1 \end{cases}$

f_0

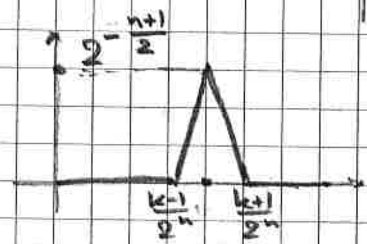


$f_{1,1}$ $f_{2,1}$ $f_{2,3}$



etc

$\rightsquigarrow$ $g_0(t) = t$ $g_{k,n}(t)$:



FOR $t \in [0,1]$, LET

$$\underline{\underline{B_t := \xi_0 t + \sum_{n \ge 1}}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{2^n - 1} \underline{\underline{\xi_{k,n}\, g_{k,n}(t)}}, \qquad (*)$$

where ξ_0, $\xi_{k,n}$ are i.i.d. $\mathcal{N}(0,1)$.

Question 1: Why is it true that this series defines a continuous function ?!

Answer: Proposition: A.s., $(*)$ converges in $\mathcal{C}([0,1])$

Proof: Let $G_n(t) := \sum_{\substack{k=1 \\ k\text{-odd}}}^{2^n-1} \xi_k\, g_{k,n}(t)$.

One has

$$\mathbb{P}\left[\|G_n\|_\infty \ge c\sqrt{n}\, 2^{-\frac{n+1}{2}} \right]$$

big enough constant (in fact, just $> \sqrt{\log 2}$)

$$= \mathbb{P}\left[\max_{k=1,3,\dots,2^n-1} |\xi_{k,n}| \ge c\sqrt{n} \right] \le 2^n \cdot \frac{1}{\sqrt{2\pi}} \int_{c\sqrt{n}}^{\infty} e^{-\frac{x^2}{2}}\, dx$$

$$\boxed{\int_a^{+\infty} e^{-x^2/2}\, dx < \frac{1}{a}\int_a^{+\infty} x e^{-x^2/2}\, dx = \frac{1}{a} e^{-a^2/2}}$$

$$\le 2^n \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{c\sqrt{n}}\, e^{-c^2 n}$$

$$\le cst \cdot \exp\left[(\log 2 - c^2) n \right]$$

Now we can apply Borel-Cantelli and conclude that only a finite number of these events happen ☒

Question 2: Why do we have a proper correlation structure?

Answer: Effectively, this is the computation performed along the motivating discussion, see next page

Exercise: Compute by brute-force $\mathbb{E}[B_{p2^{-m}} B_{q2^{-k}}]$

Remark: Note that $B_{p2^{-m}}$ is fully determined by the values of $\xi_{k,n}$ with $n \le m$.

Lecture #3 (23.09.2016)

- To finish Lévy-Ciesielski construction (see previous page)

(*) $B_t := \xi_0 t + \sum_{n\geq 1} \sum_{k=1,3,\dots,2^n-1} \xi_{n,k}\, g_{n,k}(t)$

$= \int_0^t f_{n,k}(s)\,ds$

Lemma 1: A.s. $B_t \in C([0,1])$

↑ on a set of full measure (or assume $\mathcal{A}$ to be complete w.r.t. $\mathbb{P}$)

Pf: see previous page ⊠

Lemma 2: ⓐ $\forall t_1,\dots,t_n \in [0,1]$ $\quad c_1 B_{t_1} + \dots + c_n B_{t_n}$ is Gaussian

ⓑ $\mathbb{E} B_t = 0$, $\quad \mathbb{E} B_s B_t = s \wedge t$

Pf: ⓐ limit of Gaussian variables is Gaussian (char. fcts) w/ convergent variances.

ⓑ $\mathbb{E} B_s B_t = \sum_{n\geq 0} \sum_k g_{n,k}(s)\, g_{n,k}(t)$

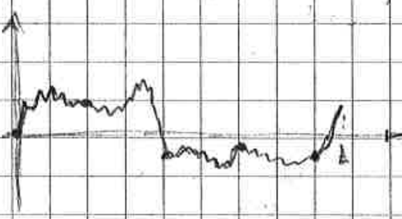
$= \sum_{n\geq 0} \sum_k \underbrace{\int_0^s f_{n,k}(r)\,dr \int_0^t f_{n,k}(r)\,dr}_{= \langle \mathbb{1}_{[0,s]}, f_{n,k} \rangle \langle \mathbb{1}_{[0,t]}, f_{n,k} \rangle} = \langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \rangle = s \wedge t.$ ⊠

Thus, we have constructed a continuous Gaussian process on $[0,1]$ with the desired covariance structure.

To extend it to $\mathbb{R}_+$, one can proceed as follows:

Take $B_t^{(k)}$, $t \in [0,1]$ – i.i.d. copies $(k \geq 0)$ and define

$$B_t := \sum_{k=0}^{\lfloor t \rfloor - 1} B_1^{(k)} + B_{t-\lfloor t \rfloor}^{(\lfloor t \rfloor)}$$



Trivial exercise: to check lemma 2 (for all $t \in \mathbb{R}_+$).

IMPORTANT REMARK:

To know the value $B_{p 2^{-m}}$ (in this construction) it is enough to know the variables $\xi_{k,n}$ with $n \leq m$.

In fact, partial sums of (*) look like (& converge to B_t in $C([0,1])$)

etc.

Def: Multidimensional Brownian motion

$B_t \in C([0,\infty); \mathbb{R}^d)$ is a cont. random process

$B_t = (B_t^{(1)}, \dots, B_t^{(d)})$ where $B_t^{(k)}$ are independent copies of the 1D BM

Simple but Important Exercise: (independence of the choice of the basis) check that $\forall\, 0 < t_1 < t_2 < \dots < t_n$ $\quad (B_{t_1}, \dots, B_{t_n}) \in \mathbb{R}^{dn}$ is a Gaussian vector whose distribution does not depend on the choice of the basis.

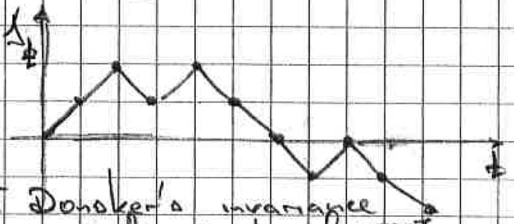
§3. Discussion of the continuity assumption & Poisson processes.

NB: this is something similar to what Brown observed

REMARK: Another (more standard) approach to the Brownian motion is to consider (scaling) limits of random walks:

Let $\xi_1, \xi_2, \ldots$ be i.i.d. and $S_0 = 0$, $S_n := \xi_1 + \ldots + \xi_n$

E.g. Simple random walk on $\mathbb{Z}$: $\xi = \pm 1$ with proba $1/2$



IN THE NEXT PARAGRAPH, we will prove the following theorem [Donsker's invariance principle, functional CLT]

Let $S_t := S_{[t]} + \{t\} (S_{[t+1]} - S_{[t]})$.

If $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$, then $\left(N^{-\frac{1}{2}} S_{Nt}\right)_{t \geq 0} \xrightarrow{(d)} (B_t)_{t \geq 0}$

Remark: There is no surprise: the scaling is given by the usual CLT, the increments should be independent, so the only question is the continuity.

Remark: $N = 2^n$ and $\xi \sim N(0,1)$ gives exactly the Lévy-Ciesielski construction

But, what would happen if we drop the assumption $\mathbb{E}\xi^2 < \infty$? (and find some other way to scale and to pass to the limit as $n \to \infty$)?

Reminder: Poisson random variables. [see page 3]

Fact: $\xi_k^{(n)} = 0, 1$ w/ proba c_n

If $n c_n \xrightarrow[n\to\infty]{} c$, then $S_n := \xi_1^{(n)} + \ldots + \xi_n^{(n)} \to \text{Poisson}(c)$

Def: Let $\tau_0, \tau_1, \ldots \sim \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}\, dx$ be i.i.d. exp. variables

and $X_t := \min\{N : \tau_0 + \ldots + \tau_N \geq t\}$

This is called a Poisson process of intensity λ

EXERCISES:

$$\boxed{\begin{array}{l} \text{(a) } X_t - X_s \sim \text{Poisson}(\lambda(t-s)) \\ \text{(b) Increments are independent} \end{array}}$$

NB: describes decay of atoms.

REAL PHYSICAL RANDOMNESS (AT LEAST, BELIEVED)

So, this is another example of a random process with stationary indep. increments

i.e. $X_t - X_s \overset{(d)}{=} X_{t-s} - X_0$

⇝ INFORMAL DISCUSSION OF LÉVY PROCESSES, SEE PAGE 5 (BOTTOM).

LECTURE #4 (26.09.2016)

4. Donsker's invariance principle (functional CLT)

Goal for today : prove Donsker's invariance principle :

Examples :

① Simple random walk on $\mathbb{Z}$

$\xi = \pm 1$ with proba $\frac{1}{2}$

Thm : $\xi_1, \xi_2, \dots, \xi_n, \dots$ — i.i.d. $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$

$S_n := \xi_1 + \dots + \xi_n$

$S_x := S_n + (x-n)(S_{n+1} - S_n)$ for $n \le x \le n+1$

Then $\left(N^{-\frac{1}{2}} S_{Nt}\right)_{t\in[0,1]} \xrightarrow{(d)} (B_t)_{t\in[0,1]}$

where the weak convergence is understood in the space $C([0,1])$

② $N = 2^n$ $\xi \sim N(0,1)$ gives the laws of successive approximations in the Lévy–Ciesielski construction.

[this follows both from the previous lecture or exercises]

But some preliminary discussion is needed ... DEFINITION:

- We deal with probability measures on $\mathcal{X} := C([0,1])$
  $\mathcal{B}(M)$ – Borel σ-algebra (generated by open sets) – separable complete metric space
  [= generated by evaluation maps $\pi_t : f \mapsto f(t)$]
- In particular, if X, Y : $\forall\ 0 \le t_1 < \dots < t_n \le 1$
  $(X_{t_1}, \dots X_{t_n}) \overset{(d)}{=} (Y_1, \dots, Y_n)$ $\Rightarrow$ $X \overset{(d)}{=} Y$ [see exercises]
- $(B_t)_{t\in[0,1]}$ defines some measure on $\mathcal{B}(\mathcal{X})$ (which is called the Wiener measure W)
  It has Gaussian marginals
  $$W(\{X \in C([0,1]) : X_{t_1} \in A_1, X_{t_2} \in A_2, \dots, X_{t_n} \in A_n\})$$
  $$= \frac{1}{\sqrt{(2\pi)^n |\det \Sigma|}} \int\limits_{A_1\times\dots\times A_n} e^{-\frac{1}{2} y \Sigma^{-1} y^T} dy, \quad \Sigma = \begin{pmatrix} t_1 & \dots & t_1 \\ \vdots & t_2 \dots & t_2 \\ t_1 & t_2 \dots & t_n \end{pmatrix}$$
  [just a Gaussian vector with covariances $\mathbb{E}[X_{t_1} X_{t_2}] = t_1 \wedge t_2$] $[\Sigma - t_1(t_2 - t_1)\dots(t_n - t_{n-1})]$
- What is the weak convergence of measures on $\mathcal{X}$?

Def : $X^{(N)} \xrightarrow{(d)} X$ iff $\forall h \in C_{bdd}(\mathcal{X})$ $\mathbb{E}[h(X^{(N)})] \to \mathbb{E}[h(X)]$

[$\mu_N \xrightarrow{(w)} \mu$ iff $\forall h \in C_{bdd}(\mathcal{X})$ $\int_{\mathcal{X}} h(x)\, d\mu_N \xrightarrow[N\to\infty]{} \int_{\mathcal{X}} h(x)\, d\mu$]

Theorem : $\forall \mathcal{X}$ – complete separable TFAE (for finite measures on $\mathcal{X}$)

(i) $\mu_N \xrightarrow{(w)} \mu$

(ii) $\forall G$ – open $\liminf\limits_{N\to\infty} \mu_N(G) \ge \mu(G)$

(iii) $\forall F$ – closed $\limsup\limits_{N\to\infty} \mu_N(F) \le \mu(F)$.

(iv) $\forall A \in \mathcal{B}(\mathcal{X})$: $\mu(\partial A) = 0$, $\exists \lim\limits_{N\to\infty} \mu_N(A) = \mu(A)$

IS IT KNOWN ? PROOF ? NEXT TIME ?

OK, AT LEAST WE NOW UNDERSTAND WHAT THIS THEOREM MEANS...

• Consequences of the weak conv.: ① In particular, if $F : \mathcal{X} \to \mathbb{R}$ is cont., and $X^{(n)} \xrightarrow{(d)} X$ then $F(X^{(n)}) \xrightarrow{(d)} F(X)$

Pf: e.g., consider $h(X) := \exp[izF(X)]$

Examples: particular values X_t, maximum $M_t := \max_{s\in[0,t]} X_t$

modulus of continuity $\omega(X_t, \delta) := \max_{|s-t|\le\delta} |X_t - X_s|$

② More generally, $F : \mathcal{X} \to \mathcal{Y}$ - cont., $X^{(n)} \xrightarrow{(d)} X$ (in $\mathcal{X}$)

$\Rightarrow F(X^{(n)}) \xrightarrow{(d)} F(X)$ (in $\mathcal{Y}$)

Pf: we need to check that

$$(F_* \mu_n)(A) \to (F_* \mu)(A) \quad \forall A : (F_*\mu)(\partial A) = 0$$

$$\mu_n(F^{-1}(A)) \qquad \mu(F^{-1}(A)) \qquad \mu(\partial(F^{-1}(A))) = 0 \Leftarrow \mu(\underbrace{F^{-1}(\partial A)}_{\supset \partial F^{-1}(A)})$$

Example: $F : (X_t)_{t\in[0,1]} \longmapsto (M_t)_{t\in[0,1]}$

is called the maximum process

• (Pre) compactness wrt the weak convergence

• Imagine that $\mathcal{X}$ was compact.

Then $\underbrace{\mathcal{C}(\mathcal{X})^*}_{\text{here } h \text{ lives}} = \mathcal{M}(\mathcal{X}) = \{$signed Borel measures on $\mathcal{X}\}$

[RIESZ REPRESENTATION THEOREM]

and positive measures are those s.t. $\{h \ge 0\} \mapsto \mathbb{R}_+$

BANACH - ALAOGLU THM:

$\Longrightarrow \forall \mu_n$ - proba measure on $\mathcal{X}$

$\exists \mu_{n_k} \xrightarrow{(w)} \mu$. Clearly, μ is a positive measure and $\mu(\mathcal{X}) = 1$: just take $h = 1$.

BUT $\mathcal{X}$ IS FAR FROM BEING COMPACT...

• How does it work for $\mathcal{X} = \mathbb{R}^d$?

Vague convergence of (locally) finite measures: $\mu_n \xrightarrow{(v)} \mu$ if

$\forall h \in C_0(\mathbb{R}^d)$ (compact support) $: h \ge 0 \quad \int_{\mathbb{R}^d} h\, d\mu_n \to \int_{\mathbb{R}^d} h\, d\mu$

Thm: (Helly) $\forall \mu_n$ - proba measures on $\mathbb{R}^d$ $\exists \mu_{n_k} \xrightarrow{(v)} \mu$

[but μ is not necessarily probabilistic]

Tightness condition: $\sup_{K \Subset \mathbb{R}^d} \liminf_{n\to\infty} \mu_n(K) = 1$

Thm: $\mu_n \xrightarrow{(v)} \mu$. Then $\mu(\mathbb{R}^d) = 1 \iff \{\mu_n\}$ is tight

In this case $\mu_n \xrightarrow{(w)} \mu$.

UNFORTUNATELY, THE PROOF THIS THEOREM USES THE GEOMETRY OF $\mathbb{R}^d$ IN AN ESSENTIAL WAY (DISTRIBUTION FUNCTIONS...)

IS IT KNOWN? PROOF? NEXT TIME?

BUT ...

- PROKHOROV'S THEOREM: μ_N – proba measures on $\mathcal{X}$ – separable complete

$[\nabla_0]$ $\{\mu_N\}$ is tight (i.e. $\sup_K \liminf_{N\to\infty} \mu_N(K) = 1$) NB: maybe not σ-comp.

$\Rightarrow \exists\ \mu_{N_k} \xrightarrow{(w)} \mu$ – proba measure

PROOF: THIS IS NOT EASY! DO WE WANT IT? NEXT TIME?

- Now the last reminder

Arzelà–Ascoli theorem: $\{f_n\} \subset C([0,1])$ SO IT DESCRIBES A COMPACT IN $C([0,1])$

$\exists M: \|f_n\| \le M$ (uniform boundedness)

and $\forall \varepsilon>0\ \exists \delta>0: |x-y|\le\delta \Rightarrow |f_n(x)-f_n(y)|\le\varepsilon\ \forall n$ (equicontinuity)

$\Rightarrow \exists\ f_{n_k} \underset{n_k\to\infty}{\rightrightarrows} f$ ($\Longleftrightarrow$ convergence in $C([0,1])$)

PROOF OF DONSKER'S THEOREM:

There are two parts: ⓐ tightness of the family $(N^{-\frac12} S_{Nt})$ and ⓑ characterization of the limit.

Note that ⓑ is almost trivial due to CLT:

Let $0=t_0<t_1<\ldots<t_n\le 1$ $\quad Y_j^{(N)} := N^{-\frac12}(S_{Nt_j} - S_{Nt_{j-1}})$ and

Note that $\tilde{Y}_j^{(N)}$ are independent $\quad \tilde{Y}_j^{(N)} := N^{-\frac12}(S_{\lfloor Nt_j\rfloor} - S_{\lfloor Nt_{j-1}\rfloor})$

and converge to $\mathcal{N}(0, t_j - t_{j-1})$ by CLT [in distribution]

since $$\tilde{Y}_j^{(N)} = \underbrace{\frac{\sqrt{[Nt_j]-[Nt_{j-1}]}}{\sqrt{N(t_j-t_{j-1})}}}_{\text{deterministic}\ \to 1} \cdot \underbrace{\frac{S_{[Nt_j]} - S_{[Nt_{j-1}]}}{\sqrt{[Nt_j]-[Nt_{j-1}]}}}_{\to\ \mathcal{N}(0,1)} \cdot \sqrt{t_j - t_{j-1}}$$

[Lemma: $\xi^{(N)} \xrightarrow{(d)} \xi$, $\alpha_N \to 1 \Rightarrow \alpha_N \xi^{(N)} \xrightarrow{(d)} \xi$

Pf: $$\mathbb{E}\left[\left|e^{i\alpha_N z\xi^{(N)}} - e^{i\alpha z\xi^{(N)}}\right|\right] \le \underbrace{M|\alpha_N-\alpha||z|}_{\xrightarrow{N\to\infty} 0} + \underbrace{2\mathbb{P}[|\xi^{(N)}|\ge M]}_{\text{uniformly in } N\ (!)\ \xrightarrow{M\to\infty} 0}\ ]$$

Thus, $(\tilde{Y}_1^{(N)}, \ldots, \tilde{Y}_n^{(N)}) \xrightarrow{(d)} (B_{t_1}, B_{t_2}-B_{t_1}, \ldots, B_{t_n}-B_{t_{n-1}})$.

- Finally, $\forall \varepsilon>0\ \mathbb{P}[|Y_j^{(N)} - \tilde{Y}_j^{(N)}| \ge \varepsilon] \xrightarrow[N\to\infty]{} 0$

and hence $\forall z_1,\ldots,z_n \in \mathbb{R}$

$$\mathbb{E}[e^{i(z_1 Y_1^{(N)}+\ldots+z_n Y_n^{(N)})}] - \mathbb{E}[e^{i(z_1\tilde{Y}_1^{(N)}+\ldots+z_n\tilde{Y}_n^{(N)})}] \xrightarrow[N\to\infty]{} 0$$

$$\Rightarrow (Y_1^{(N)}, \ldots, Y_n^{(N)}) \xrightarrow{(d)} (B_{t_1}, B_{t_2}-B_{t_1}, \ldots, B_{t_n}-B_{t_{n-1}})$$

THUS, THE MAIN DIFFICULTY IS TO PROVE ⓐ, I.E. TIGHTNESS.

FOR THIS, WE NEED A SIMPLE LEMMA.

Lemma: $\xi_1, \dots, \xi_N$ – independent with $\mathbb{E}\xi_i = 0$. Then

$$\forall \alpha>0 \quad \mathbb{P}\Big[\max_{k=1,\dots,N} |S_k| \geq 2\alpha\Big] \leq \Big(1 - \frac{\mathbb{E}S_N^2}{\alpha^2}\Big)^{-1} \mathbb{P}\big[|S_N| \geq \alpha\big]$$

Pf: $\Omega_k := \{\omega\in\Omega : |S_1|, \dots, |S_{k-1}| < 2\alpha \ \& \ |S_k| \geq 2\alpha\}$

Then
$$\mathbb{P}[|S_N|\geq\alpha] \geq \sum_{k=1}^N \mathbb{P}(\Omega_k)\cdot\mathbb{P}(|S_N - S_k| \leq \alpha)$$
$$\geq \sum_{k=1}^N \mathbb{P}(\Omega_k)\Big(1 - \frac{\mathbb{E}(S_N-S_k)^2}{\alpha^2}\Big)$$
$$\geq \Big(1 - \frac{\mathbb{E}S_N^2}{\alpha^2}\Big)\sum_{k=1}^N \mathbb{P}(\Omega_k) \quad \boxtimes$$

BACK TO THE THEOREM:

Uniform boundedness:

$$\mathbb{P}\Big[\max_{t\in[0,1]} |N^{-\frac12} S_{tN}| \geq 2M\Big] \leq \Big(1 - \frac{1}{M^2}\Big)^{-1} \mathbb{P}\big[\underbrace{|N^{-\frac12} S_N|}_{\xrightarrow[N\to\infty]{(d)} N(0,1)} \geq M\big]$$

$$\xrightarrow[N\to\infty]{} \Big(1-\frac{1}{M^2}\Big)^{-1}\cdot\sqrt{\frac{2}{\pi}}\int_M^{+\infty} e^{-x^2/2}\,dx$$

Thus, $\forall \eta>0 \ \exists M>0$:
$$\liminf_{N\to\infty} \mathbb{P}\Big[\max_{t\in[0,1]} |N^{-\frac12} S_{tN}| \leq 2M\Big] \geq 1-\eta.$$

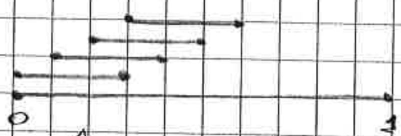
Equicontinuity:

Given $\varepsilon>0$, we need to find $\delta>0$:
$$\mathbb{P}\Big[\max_{|s-t|\leq\delta/3} N^{-\frac12}|S_{Nt} - S_{Ns}| \leq 4\varepsilon\Big] \geq 1-\eta \qquad ⊛$$

Note that this is guaranteed if

$\forall m = 0, \lfloor \frac{\delta}{3}N\rfloor, \lfloor\frac{2\delta}{3}N\rfloor, \lfloor\delta N\rfloor, \dots,$ ($\approx \frac{3}{\delta}$ choices)

one has
$$\boxed{\max_{k=0,\dots,\lfloor\delta N\rfloor} N^{-\frac12}|S_{m+k} - S_m| \leq 2\varepsilon}$$



If $|s-t|\leq\delta/3$, they fall into one of these segments and one uses
$|S_{Nt} - S_{Ns}| \leq |S_{Nt} - S_m| + |S_{Ns} - S_m|$

Now estimate the probability of bad events using the same lemma:

$$\mathbb{P}\Big[\max_{|s-t|\leq\delta/3} N^{-\frac12}|S_{Nt} - S_{Ns}| \geq 4\varepsilon\Big]$$
$$\leq \mathbb{P}\Big[\exists m : \max_{k=0,\dots,\lfloor\delta N\rfloor} N^{-\frac12}|S_{m+k} - S_m| \geq 2\varepsilon\Big]$$
$$\leq \frac{3}{\delta}\Big(1 - \frac{\delta N}{\varepsilon^2 N}\Big)^{-1}\mathbb{P}\big[\underbrace{|N^{-\frac12} S_{\lfloor\delta N\rfloor+1}|}_{\xrightarrow[N\to\infty]{(d)} N(0,\delta)} \geq \varepsilon\big]$$
$$\xrightarrow[N\to\infty]{} \frac{3}{\delta}\Big(1-\frac{\delta}{\varepsilon^2}\Big)^{-1}\cdot\underbrace{\sqrt{\frac{2}{\pi}}\int_{\varepsilon/\sqrt{\delta}}^{+\infty} e^{-x^2/2}\,dx}_{\leq \frac{\sqrt{\delta}}{\varepsilon} e^{-\varepsilon^2/2\delta}}$$

THIS CAN BE MADE $\leq \eta$ BY CHOOSING δ SMALL ENOUGH (E.G. $\eta := \varepsilon^{2+d}$ FOR $\varepsilon\to 0$)

THUS, $\forall \eta>0$ WE HAVE FOUND (ARZELÀ-ASCOLI) A COMPACT SET $K = K(\eta) \subset C([0,1])$ S.T.
$$\liminf_{N\to\infty} \mathbb{P}\Big[(N^{-\frac12} S_{Nt})_{t\in[0,1]} \in K\Big] \geq 1 - 2\eta$$

! NOW APPLY PROKHOROV'S THEOREM ! $\boxtimes$

30.09.2016

Reminder: Donsker's thm: $(N^{-\frac{1}{2}} S_{Nt})_{t\in[0,1]} \xrightarrow[N\to\infty]{(d)} (B_t)_{t\in[0,1]}$ [functional CLT] in distribution in $\mathcal{C}([0,1])$

Pf: two steps:
(a) tightness
(b) identification of the limit (CLT)

(a1) Prokhorov's theorem [tightness in metric spaces]

(a2) Arzela-Ascoli thm [characterization of compacts in $\mathcal{C}([0,1])$]

(a3) $\forall \eta>0 \ \exists M>0$: $\liminf_{N\to\infty} \mathbb{P}[\max_{t\in[0,1]} |N^{-\frac{1}{2}} S_{Nt}| \geq M] \leq \eta$

$\exists \delta>0$: $\liminf_{N\to\infty} \mathbb{P}[\max_{s,t:|s-t|\leq\delta} N^{-\frac{1}{2}} |S_{Nt}-S_{Ns}| \geq \varepsilon] \leq \eta$

§5. Some insights on the (modulus of) continuity of the Brownian Motion

It was proven that

$$\liminf_{N\to\infty} \mathbb{P}[\max_{s,t:|s-t|\leq\delta/3} N^{-\frac{1}{2}}|S_{Nt}-S_{Ns}| \geq 4\varepsilon] \leq \frac{3}{\delta}\left(1-\frac{\delta}{\varepsilon^2}\right)^{-1} \cdot \frac{\sqrt{\delta}}{\varepsilon} e^{-\varepsilon^2/2\delta}$$

In particular, one can take $\delta := \varepsilon^{2+\alpha}$ provided that ε is small enough

THIS IMPLIES THAT

$\forall \alpha>0 \quad \mathbb{P}[(B_t)_{t\in[0,1]}$ is $(\frac{1}{2}-\alpha)$-Hölder on $[0,1]] = 1$.

Remark: There is no surprise in the approximate exponent $\frac{1}{2}$ since we have $\mathbb{E}|B_t-B_s|^2 = t-s$.

Theorem: $\mathbb{P}[B_t$ is nowhere differentiable on $[0,1]] = 1$.

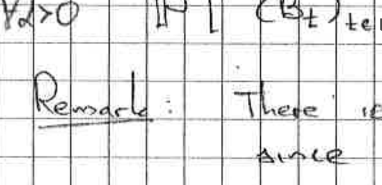
Paley, Wiener, Zigmund

Moreover, $\forall \alpha>0 \quad \mathbb{P}[\exists t\in[0,1] : \limsup_{h\downarrow 0} \frac{|B_{t+h}-B_t|}{h^{1/2+\alpha}} < +\infty] = 0$

Remark: In particular, $(B_t)_{t\in[0,1]}$ is nowhere $(\frac{1}{2}+\alpha)$-Hölder; see also the list of facts after the proof.

Pf: IT IS ENOUGH TO SHOW THAT, $\forall M>0$,

$$\mathbb{P}[\exists t\in[0,1] \ \& \ h_0>0 : |B_{t+h}-B_t| \leq M h^{1/2+\alpha} \ \forall h\leq h_0] = 0$$



Let this happen for $t\in[\frac{k-1}{2^n}, \frac{k}{2^n}]$ and n : $\frac{l+1}{2^n} < h_0$,

WHERE l IS CHOSEN SO THAT $l\alpha > 1$, E.G. $l := \lfloor 1/\alpha \rfloor + 1$

Then, for each $j=1,\dots,l$, $|B_{(k+j)2^{-n}} - B_{(k+j-1)2^{-n}}| \leq 2M\left(\frac{l+1}{2^n}\right)^{1/2+\alpha}$ (*)

& all these increments are i.i.d. $\mathcal{N}(0,2^{-n})$

$\Rightarrow \mathbb{P}[$(*) happens for fixed $k,j] \leq cst(M,\alpha) \cdot 2^{-n\alpha}$

and hence, due to independence

$$\mathbb{P}[\circledast \text{ holds for } \underline{\text{fixed}}\ k \text{ and } \underline{\text{all}}\ j=1,\dots,\ell] \leq cst(M,\alpha)\cdot 2^{-n\alpha\ell}$$

Finally, the union bound gives

$$\mathbb{P}[\circledast \text{ holds for } \underline{\text{some}}\ k \text{ and all } j=1,\dots,\ell] \leq cst(M,\alpha)\cdot 2^{-n(\underbrace{\alpha\ell-1}_{>0 \text{ by our choice of } \ell})}.$$

Borel-Cantelli

$\Rightarrow$ this can happen for ~~finite~~ number of n's only

$$\Rightarrow \mathbb{P}[\exists t\in[0,1]\ \&\ h_0>0 : |B_{t+h}-B_t| \leq M h^{1/2+\alpha}\ \forall h\leq h_0] = 0$$ ☒

SOME (DEEPER AND NON-CONTRADICTORY) FACTS ON THE MODULUS OF CONTINUITY [W/O PROOF, AT LEAST NOW]

(1) KHINCHIN / KHINTCHINE : LAW OF THE ITERATED LOG

[1933] $$\forall t\in[0,1]\quad \mathbb{P}\left[\limsup_{h\downarrow 0}\frac{|B_{t+h}-B_t|}{\sqrt{2h\log\log 1/h}}=1\right]=1.$$

(2) LÉVY : EXACT MODULUS OF CONTINUITY

[1937] $$\mathbb{P}\left[\limsup_{h\downarrow 0}\ \max_{t\in[0,1-h]}\frac{|B_{t+h}-B_t|}{\sqrt{2h\log 1/h}}=1\right]=1.$$

Rem: note the difference! This does not contradict to each other because of non-countability of the set of times $[0,1]$: $\exists$ exceptional ones

MOREOVER, THERE EXIST EXCEPTIONAL TIMES:

(3) $$\mathbb{P}\left[\sup_{t\in[0,1]}\ \limsup_{h\downarrow 0}\frac{|B_{t+h}-B_t|}{\sqrt{2h\log 1/h}}=1\right]=1$$

Rem: this is stronger than (2) as we now claim that one can find the same t for $h\downarrow 0$.

(4) $$\mathbb{P}\left[\inf_{t\in[0,1]}\ \limsup_{h\downarrow 0}\frac{|B_{t+h}-B_t|}{\sqrt{h}}=1\right]=1$$

Rem: $\Rightarrow$ there exist exceptional times at which B_t is $\frac{1}{2}$-Hölder

[thm of 1983]

- SEE MÖRTERS - PERES "BROWNIAN MOTION" (CHAPTER 10) FOR MANY OTHER AMAZING RESULTS ON THE FRACTAL DIMENSIONS OF EXCEPTIONAL SETS
- QUESTIONS OF THAT KIND, ESPECIALLY IN DIMENSIONS $D\geq 3$ ARE SUBJECT OF VERY RECENT (AND ONGOING) RESEARCH !

["GEOMETRY" OF EXCEPTIONAL SETS]

§6. Tightness: the proof of Prokhorov's criterion.

REMARK: It is convenient for the proof to work with NOT NECESSARILY COMPLETE metric spaces. Also there is no need to assume separability.

• Reminder: $\mathcal{X}$ - metric space. $\mathcal{B}(\mathcal{X})$ - Borel σ-algebra.
$\{\mu_n\}$ - collection of proba measures is called tight if $\sup_K \liminf_{n\to\infty} \mu_n(K) = 1$.
(K ← compact subsets of $\mathcal{X}$)

GOAL:
$\{\mu_n\}$ - tight $\Rightarrow$ $\exists\, n_k \to \infty$: $\mu_{n_k} \xrightarrow[n_k\to\infty]{(w)} \mu$ - proba measure

i.e. $\forall h \in C_{bdd}(\mathcal{X})$ $\int_{\mathcal{X}} h\, d\mu_n \to \int_{\mathcal{X}} h\, d\mu$

• Portmanteau thm: [TO SKIP]

TFAE
(i) $\mu_n \xrightarrow[n\to\infty]{(w)} \mu$
(ii) $\forall G$ - open $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$
(iii) $\forall F$ - closed $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$
(iv) $\forall A : \mu(\partial A) = 0$ $\lim_{n\to\infty} \mu_n(A) = \mu(A)$

(i) $\Rightarrow$ (ii)
$\forall G$ $\forall h \in C(\mathcal{X}) : 0 \leq h \leq 1_G$
$\liminf_{n\to\infty} \mu_n(G) \geq \liminf_{n\to\infty} \int_{\mathcal{X}} h\, d\mu_n = \int_{\mathcal{X}} h\, d\mu$
now take $h \uparrow 1_G$

Pf: (ii) $\Leftrightarrow$ (iii) : trivial
$\Downarrow$
(iv) : $\mu(\mathring{A}) \leq \liminf_{n\to\infty} \mu_n(\mathring{A}) \leq \liminf_{n\to\infty} (A)$
$\leq \limsup_{n\to\infty} \mu(A) \leq \limsup_{n\to\infty} (\bar{A}) \leq \mu(\bar{A})$

(iv) $\Rightarrow$ (iii) :
let $F^\varepsilon := \{x \in \mathcal{X} : \operatorname{dist}(x, F) \leq \varepsilon\}$, choose $\varepsilon \to 0$: $\mu(\partial F^\varepsilon) = 0$
(exist as ∂F^ε are disjoint)
for such ε one has $\mu_n(F) \leq \mu_n(F^\varepsilon) \xrightarrow[n\to\infty]{} \mu(F^\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{} \mu(F)$

(ii) $\Rightarrow$ (i) : let $h \geq 0$, then
$$\int_{\mathcal{X}} h\, d\mu \underset{\text{Fubini}}{=} \int_0^{+\infty} \mu(\{x \in \mathcal{X} : h(x) > t\})\, dt \overset{(ii)}{\leq} \int_0^{+\infty} \liminf_{n\to\infty} \mu_n(\{x \in \mathcal{X} : h(x) > t\})$$
$$\underset{\text{Fatou}}{\leq} \liminf_{n\to\infty} \int_0^{+\infty} \mu_n(\{x \in \mathcal{X} : h(x) > t\}) = \liminf_{n\to\infty} \int_{\mathcal{X}} h\, d\mu_n$$
For $|h| \leq M$, apply this to $M \pm h$ to get $M \pm \int_{\mathcal{X}} h\, d\mu \leq M \pm \begin{matrix}\liminf \\ \limsup\end{matrix} \int_{\mathcal{X}} h\, d\mu_n$ ∎

Remark: ⓐ $\left.\begin{matrix} F : \mathcal{X} \to \mathcal{Y} \text{ - cont.} \\ \mu_n \xrightarrow[n\to\infty]{(w)} \mu \text{ in } \mathcal{X} \end{matrix}\right\} \Rightarrow F_* \mu_n \xrightarrow[n\to\infty]{(w)} F_* \mu$ in $\mathcal{Y}$.

Pf: trivial using (i) [SEE PREVIOUS LECTURE]

ⓑ If $\mathcal{X} \subset \mathcal{Y}$, then $\mu_n \xrightarrow[n\to\infty]{(w)} \mu$ in $\mathcal{X}$
iff $\mu_n \xrightarrow[n\to\infty]{(w)} \mu$ in $\mathcal{Y}$

Pf: trivial using (ii)

• The case $\mathcal{X} = \mathbb{R}^d$ [TO SKIP]

Write $F_n(x) := \mu_n\{t \in \mathbb{R}^d : t \leq x\}$ ← coordinate-wise
and take a subsequence : $F_{n_k}(r) \xrightarrow[n_k\to\infty]{} F(r)$ $\forall r \in \mathbb{Q}^d$

Define $F(x) := \inf_{x < r} F(r)$ ← this is monotone and right-continuous

$\Rightarrow \exists\, \mu$ on $\mathbb{R}^d$: $\mu(t \in \mathbb{R}^d : x < t \leq y) = F(y) - F(x)$

Note that F is cont at all $x \in (\mathbb{R} \setminus \underbrace{\mathcal{C}}_{\text{countable set of all jumps of all coordinate projections}})^d$

and $F_{n_k}(x) \xrightarrow[n_k\to\infty]{} F(x)$ $\forall x \in (\mathbb{R}\setminus\mathcal{C})^d$

$\Rightarrow$ $\mu_{n_k}(B) \xrightarrow[n_k\to\infty]{} \mu(B)$ $\forall B =$ finite union of rectangular boxes with corners in $\mathbb{R}\setminus\mathcal{C}$

$\Rightarrow$ $\mu(\mathring{B}) \le \liminf_{n_k\to\infty} \mu_{n_k}(\mathring{B})$

$\le \limsup_{n_k\to\infty} \mu_{n_k}(\bar{B}) \le \mu(\bar{B})$ $\forall$ bounded B $\Rightarrow$ (iv)

Also, it is clear (as compacts in $\mathbb{R}^d$ = bounded closed sets) that $\sup_K \liminf_{n\to\infty} \mu_n(K) = 1$ $\Rightarrow$ $\mu(\mathbb{R}^d) = 1$. ⊠

- **The case $\mathcal{X} = \mathbb{R}^\infty$.**

Remark: $\mathbb{R}^\infty = \{(x_1, \dots, x_n, \dots), \ x_n \in \mathbb{R}\}$

metric: $d(x,y) := \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1+|x_k-y_k|}$

topology: $x^{(n)} \to y^{(n)} \iff \forall k \ x_k^{(n)} \to y_k^{(n)}$

base of topology: $B(x_1,r_1)\times \dots \times B(x_n,r_n)\times\mathbb{R}\times\mathbb{R}\times\dots$

$\mathcal{B}(\mathbb{R}^\infty)$ is generated by cylindric sets

Lemma: Let $\pi^{(N)}: \mathbb{R}^\infty \to \mathbb{R}^N$
$x \mapsto (x_1, \dots, x_N)$

Then $\mu_n \xrightarrow[n\to\infty]{(w)} \mu$ in $\mathbb{R}^\infty$ if and only if $\forall N$ $\pi^{(N)}_* \mu_n \xrightarrow[n\to\infty]{(w)} \pi^{(N)}_*\mu$

Pf: ($\Rightarrow$): trivial as $\pi^{(N)}$ is continuous

($\Leftarrow$): Let $\tilde{\pi}^{(N)}: x \mapsto (x_1, \dots, x_N, 0, 0, \dots)$
and note that $\forall x$ $d(x, \tilde{\pi}^{(N)}(x)) \le 2^{-N}$

Then, $\forall F \subset \mathbb{R}^\infty$ - closed one can define

$$F^{(N)} := (\tilde{\pi}^{(N)})^{-1}\left[\{x\in\mathbb{R}^\infty : d(x,F) \le 2^{-N}\} \cap \tilde{\pi}^{(N)}(\mathbb{R}^\infty)\right]$$

Clearly, $F^{(N)}$
- is closed
- contains F
- $F^{(N)} \downarrow F$ as $N\to\infty$

and write

$$\limsup_{n\to\infty} \mu_n(F) \le \limsup_{n\to\infty}\mu_n(F^{(N)}) \le \mu(F^{(N)}) \xrightarrow[N\to\infty]{} \mu(F)$$

since $\pi^{(N)}_* \mu_n \xrightarrow{(w)} \pi^{(N)}_*\mu$ ⊠

To prove Prokhorov's criterion for $\mathcal{X} = \mathbb{R}^\infty$, consider $\{\mu_n\}$ and use diagonal process to find n_k: $\pi^{(N)}_* \mu_{n_k} \xrightarrow[n\to\infty]{(w)} \mu^{(N)}$ - some proba measures $\forall N$

It is easy to check that $\pi^{(N_1)}_* \mu^{(N_2)} = \mu^{(N_1)}$ if $N_1 < N_2$

and hence $\exists \mu$ - proba measure on $\mathcal{B}(\mathbb{R}^\infty)$: $\mu^{(N)} = \pi^{(N)}_*\mu$ ⊠

- **The case $\mathcal{X} \subset \mathbb{R}^\infty$** [**Remark:** not necessarily complete!]

$\{\mu_n\}$ - tight in $\mathcal{X}$ $\Rightarrow$ tight in $\mathbb{R}^\infty$

$\Rightarrow$ $\exists n_k$: $\mu_{n_k} \xrightarrow[n_k\to\infty]{(w)} \mu$ - proba measure in $\mathbb{R}^\infty$

To check that μ is supported on $\mathcal{X}$, note that tightness implies

$\exists \underbrace{K_m}_{\text{compact in } \mathcal{X}} \subset \mathcal{X}$: $\liminf_{n\to\infty} \mu_n(K_m) \ge 1 - 2^{-m}$

$\Rightarrow$ compact in $\mathbb{R}^\infty$ $\Rightarrow$ $\mu(K_m) \ge \limsup_{n\to\infty}\mu_n(K_m) \ge 1 - 2^{-m}$

• **The case $\mathcal{X}$ is separable**

Consider $\{x^{(s)}\}_{s=1}^{\infty}$ - dense subset and write

$$g : \mathcal{X} \to \mathbb{R}^{\infty}$$
$$x \mapsto (d(x, x^{(s)}))_{s=1}^{\infty}$$

This is a homeomorphism (exercise)

$x_n \xrightarrow{\mathcal{X}} x \Rightarrow \forall s \ d(x_n, x^{(s)}) \to d(x, x^{(s)})$ trivial

$\Leftarrow$: since $\{x^{(s)}\}$ is dense, there exist arbitrary small $d(x, x^{(s)})$ ⊥

[BUT NOTE THAT $g(\mathcal{X})$ CAN BE NOT COMPLETE AS A SUBSPACE OF A METRIC SPACE $\mathbb{R}^{\infty}$ EVEN IF $\mathcal{X}$ IS A COMPLETE METRIC SPACE !]

$\{\mu_n\}$ - tight in $\mathcal{X}$ $\Rightarrow$ $\{g_* \mu_n\}$ - tight in $g(\mathcal{X})$

$\Rightarrow \exists n_k : g_* \mu_{n_k} \xrightarrow[n_k\to\infty]{(w)} \mu$ in $g(\mathcal{X})$ $\Rightarrow$ $\mu_{n_k} \xrightarrow[n_k\to\infty]{(w)} g_*^{-1}\mu$ in $\mathcal{X}$

• Note that if $\mathcal{X}$ is σ-compact [i.e. a countable union of compacts], then it is separable as each compact admits finite 2^{-m} - nets $\forall (\forall m)$.

• **General case** [Exercise: Use tightness to approximate by σ-compact ones]

Remark: e.g. $\mathcal{X} = \ell^{\infty}$ is not separable

$\forall m \ \exists K_m : \liminf_{n\to\infty} \mu_n(K_m) \geq 1 - 2^{-m}$,

take $\tilde{\mathcal{X}} := \bigcup_{m=1}^{\infty} K_m$ - it is σ-compact by construction

and $\tilde{\mu}_n := (\mu_n(\tilde{\mathcal{X}}))^{-1} \cdot \mu_n|_{\tilde{\mathcal{X}}}$ - still tight in $\tilde{\mathcal{X}}$ (clear)

Now find $n_k \to \infty$: $\tilde{\mu}_{n_k} \xrightarrow[n_k\to\infty]{(w)} \mu$ in $\tilde{\mathcal{X}}$ $\Rightarrow$ in $\mathcal{X}$

and note that $\forall F \subset \mathcal{X}$ - closed

$$\mu(F) = \mu(\underbrace{F \cap \tilde{\mathcal{X}}}_{\text{closed in } \tilde{\mathcal{X}}}) \geq \limsup_{n_k\to\infty} \underbrace{\tilde{\mu}_{n_k}(F \cap \tilde{\mathcal{X}})}_{\geq \mu_{n_k}(F \cap \tilde{\mathcal{X}}) \geq \mu_{n_k}(F) - \underbrace{\mu_{n_k}(\mathcal{X} \setminus \tilde{\mathcal{X}})}_{\xrightarrow[n_k\to\infty]{} 0}} \geq \limsup_{n_k\to\infty} \mu_{n_k}(F)$$

☒

COMMENT ON THE STATEMENT OF PROKHOROV'S THEOREM:

In fact, if $\mathcal{X}$ is separable and complete, then

$\{\mu_n\}$ - tight $\underset{!?}{\Longleftrightarrow}$ $\forall n_k \to \infty \ \exists n_{k_s} \to \infty$:

$\mu_{n_{k_s}} \xrightarrow[n_{k_s}\to\infty]{(w)} \mu$ - some proba measure in $\mathcal{X}$.