Processus Stochastiques (ENS, 2016/17–2018/19)

**1. Level:** M1. **Prerequisites:** Measure theory, Probability-I.

**2. Description/references.** The course consists of three parts: conditional expectations and martingales in discrete time; Markov chains in discrete time; introduction to the Brownian motion. The standard (at the ENS) reference text is the (unpublished) lecture notes *Intégration, Probabilités et Processus Aléatoire* due to Jean–François Le Gall. We mostly follow these notes though give a different perspective on the Brownian motion, positioning this topic as a generalization of the usual central limit theorem. Traditionally for the ENS, the exposition is complemented by introducing several 'modern' topics on discrete Markov chains at the end of the course.

**2. Contents. Part I. Intro to the Brownian motion.**

- Reminders: random variables, charateristic function, convergence in distribution, law of large numbers, central limit theorem, Gaussian vectors.
- Continuous random processes with independent distributions.
- Gaussian vectors and Lévy-Ciesielski construction of the Brownian motion.
- Discussion of the continuity assumption and Poisson processes.
- Random variables in Polish spaces, Prokhorov's tightness criterion (w/o proof).
- Donsker's invariance principle aka the functional CLT.
- Discussion of regularity properties of the Brownian motion trajectories.
- (Optional: the proof of Prokhorov's tightness criterion.)

**Part II. Conditional expectation and martingales in discrete time.**

- Conditional expectation and conditional distribution.
- Discussion: discrete variables, densities, Guassian vectors.
- Filtrations, stopping times, martingales, sub-/super-martingales.
- Optional stopping theorem.
- Example: Dirichlet problem for discrete harmonic functions; exit probabilities on a grid.
- Doob's upcrossing inequality and the almost sure convergence.
- Example: dyadic filtration, decomposition of measures on $[0, 1]$.
- Doob's maximal inequality and the $L^p$ convergence.
- Uniform integrability and the $L^1$ convergence.
- Galton–Watson process.
- Backward martingales.
- De Finetti's theorem for exchangeable random variables in $\{0, 1\}$.

**Part III. Discrete Markov chains.**

- Definition, examples, spectral perspective for finite spaces of states.
- Recurrent and transient states; recurrent components.
- Invariant measures, uniqueness for irreducible recurrent chains.
- Positive and null recurrent chains, construction of the invariant measure.
- Asymptotic behavior of aperiodic recurrent chains for large times, ergodic theorem.
- Time reversal of a Markov chain. Doob's h-transform.
- Harmonic functions on infinite graphs, the notion of the Martin boundary.
- Monte-Carlo sampling. Example: Glauber dynamics, Metropolis' algorithm.
- Coupling from the past. Example: sampling the Ising model.
- Basics of mixing times. Example: bottom-to-top shuffling.