TOPOLOGIE ET CALCUL DIFFÉRENTIEL.
II. CALCUL ET ÉQUATIONS DIFFÉRENTIELLES

DMITRY CHELKAK, DMA ENS 2020

[Last updated: January 13, 2021]

This is the second part of informal lecture notes for the L3 course *Topology and Differential Calculus* at the DMA ENS, Paris, given in the Fall 2020 term.

Contents

November 23, 2020

1. Differentiable functions in Banach spaces: basics

Recall that a function $f:\mathbb{R}\supset U\to\mathbb{R}$ is called differentiable at a point $a\in U$ if there exists $f'(a)\in\mathbb{R}$ such that $f(x)=f(a)+f'(a)(x-a)+o(|x-a|)$ as $x\to a$. To generalize this definition to the context of mappings between Banach spaces we can view the second term as a linear operator $h\mapsto f'(a)h$ acting from $\mathbb{R}$ to $\mathbb{R}$.

Definition 1.1. *Let E,F be Banach spaces and $U\subset E$ is an open set. A mapping $f:U\to F$ is called* differentiable at a point $a\in U$ (in the Fréchet sense) *if there exists a bounded linear operator $(Df)(a)\in\mathcal{L}(E;F)$ such that*

$$f(x)=f(a)+[(Df)(a)](x-a)+o(\|x-a\|)\quad \textit{as}\quad x\to a.$$

One says that f is continuously differentiable on U *(and writes $f\in C^1(U,F)$) if is differentiable at all points of U and the mapping $Df:U\to\mathcal{L}(E;F)$ is continuous.*

Let us briefly discuss this definition.

- Clearly, nothing changes if one replaces the norm in E (or in F) by an equivalent. However, let us emphasize that one needs to require that E is a *normed* space to be able to write the error term $o(\|x-a\|)$, which is uniform in the direction of the increment $x-a$.
- There exists a weaker notion, called the *differentiability in the Gateaux sense*. Namely, one requires that for each $h\in E$ there exists a vector $(Df)(a,h)\in F$ such that $f(a+th)=f(a)+t(Df)(a,h)+o(t)$ as $t\to 0$. Compared to Definition 1.1, there are two important differences: we do not require neither linearity nor continuity of $(Df)(a,h)$ in h. In particular, if we consider the the mapping
$$f:\mathbb{R}^2\to\mathbb{R},\quad f(x_1,x_2):=\frac{x_1^3}{x_1^2+x_2^2},$$
near $a=(0,0)$, then $(Df)(a,h)$ exists for all $h\in\mathbb{R}^2$ but is *not* linear in h. In what follows, all the derivatives are understood in the sense of Definition 1.1 (= Fréchet) and not in this weaker (= Gateaux) sense.

One can now iterate Definition 1.1 in order to define the second (and higher) order derivatives of a continuous mapping. Let us first discuss the types of objects arising along this way. We should have $D^2f=DDf\in\mathcal{L}(E;\mathcal{L}(E;F))$ and similar for higher order derivatives. However, instead of considering bounded linear operators acting to the spaces of bounded linear operators, it is much more transparent to speak about bounded multi-linear mappings.

Definition 1.2. *Let $E_1,\ldots,E_m$ and F be Banach spaces. A multi-linear (i.e., linear in each of its arguments) mapping $L:E_1\times\ldots E_m\to F$ is called bounded if*

$$\|L\|_{\mathcal{L}(E_1,\ldots,E_m;F)}\ :=\sup_{h_1\neq0,\ldots,h_m\neq0}\frac{\|L(h_1,\ldots,h_m)\|_F}{\|h_1\|_{E_1}\cdot\ldots\cdot\|h_m\|_{E_m}}<+\infty.$$

Similarly to bounded linear operators, it is easy to see that the vector-space $\mathcal{L}(E_1,\ldots,E_m)$ of bounded multi-linear mappings is *complete* with respect to the norm introduced above. Also, note that a multi-linear mapping L is bounded if and only if it is continuous at 0. (Indeed, due to the multi-linearity it is enough to consider $\|h_1\|_{E_1}=\ldots=\|h_m\|_{E_m}=1$ in the definition of $\|L\|_{\mathcal{L}(E_1,\ldots,E_m;F)}$.)

Lemma 1.3. *There exists a canonical isomorphism of Banach spaces*

$$\mathcal{L}(E;\mathcal{L}(E;F)) \;\cong\; \mathcal{L}_2(E;F) := \mathcal{L}(E,E;F), \qquad \mathrm{L}(h_1)h_2 = L(h_1,h_2),$$

where $\mathrm{L} \in \mathcal{L}(E;\mathcal{L}(E;F))$ *and* $L \in \mathcal{L}_2(E;F)$. *The same holds true for higher orders:*

$$\mathcal{L}(E;\mathcal{L}(E;\mathcal{L}(E;F))) \cong \mathcal{L}_3(E;F) := \mathcal{L}(E,E,E;F) \quad \textit{etc.}$$

Proof. This is almost a tautology: the linearity is trivial and

$$\|\mathrm{L}\|_{\mathcal{L}(E;\mathcal{L}(E;F))} = \sup_{h_1\neq 0} \frac{\|\mathrm{L}(h_1)\|_{\mathcal{L}(E;F)}}{\|h_1\|_E} = \sup_{h_1\neq 0}\sup_{h_2\neq 0} \frac{\|\mathrm{L}(h_1)h_2\|_F}{\|h_1\|_E\|h_2\|_E} = \|L\|_{\mathcal{L}_2(E;F)}. \quad \square$$

The higher derivatives $D^m f$ of $f : E \to F$ and classes $C^m(U;F)$ of m times continuously differentiable mappings are defined inductively using Definition 1.1.

Definition 1.4. *A mapping* $f : E \supset U \to F$ *has an* m*-th derivative at* $a \in U$ *if* $f \in C^{m-1}(U';F)$, *where* $a \in U' \subset U$, *and the mapping* $D^{m-1}f : U' \to \mathcal{L}_{m-1}(E;F)$ *is differentiable at* a. *Note that*

$$(D^m f)(a) \;:=\; (DD^{m-1}f)(a) \;\in\; \mathcal{L}(E;\mathcal{L}_{m-1}(E;F)) \;\cong\; \mathcal{L}_m(E;F).$$

We say that $f \in C^m(U;F)$ *if the mapping* $D^m f : U \to \mathcal{L}_m(E;F)$ *is continuous.*

Let us now give more comments on this definition:

- A crucial property of higher order derivatives is that they are <u>*symmetric*</u> multi-linear mappings: (at least) if $f \in C^m(U;F)$, then

$$\begin{aligned}(D^m f)(a) \;&\in \mathcal{L}_m^{\mathrm{sym}}(E;F)\\ &:= \{L \in \mathcal{L}_m(E;F) : L(h_1,\dots,h_m) = L(h_{\sigma(1)},\dots h_{\sigma(m)})\ \forall \sigma \in S_m\}.\end{aligned}$$

  This is not fully straightforward; the proof is given in Theorem 2.3 below.
- If $L \in \mathcal{L}_2^{\mathrm{sym}}(E;F)$, then

$$L(h_1,h_2) = \tfrac{1}{4}(L(h_1+h_2,h_1+h_2) - L(h_1-h_2,h_1-h_2)).$$

  In other words, a symmetric bi-linear mapping L can be reconstructed from its values $L(h,h)$ on the diagonal; sometimes, one calls such restrictions $E \ni h \mapsto L(h,h) \in F$ *quadratic* mappings. The same holds true for higher orders: if $L \in \mathcal{L}_m^{\mathrm{sym}}(E;F)$, then

$$L(h_1,\dots h_m) = \frac{1}{2^m m!}\sum\nolimits_{\varepsilon_1=\pm1,\dots,\varepsilon_m=\pm1} \varepsilon_1\dots\varepsilon_m L\big(\textstyle\sum_{j=1}^m \varepsilon_j h_j,\dots,\sum_{j=1}^m \varepsilon_j h_j\big).$$

  (For the proof, expand the right-hand side by multi-linearity and note that only the terms $L(h_{\sigma(1)},\dots,h_{\sigma(m)})$ survive under the summation over ε_j.)

Example. Let $\mathcal{E} := \mathcal{L}(E)$ and $\mathcal{E} \supset \mathcal{U} := \{A \in \mathcal{L}(E) : \exists A^{-1} \in \mathcal{L}(E)\}$ be the open set of invertible operators. Consider the mapping $\mathrm{Inv} : \mathcal{U} \to \mathcal{E}$, $A \mapsto A^{-1}$. For each $A \in \mathcal{U}$ we have a 'geometric series' expansion (see part I)

$$\mathrm{Inv}(A+H) = A - A^{-1}HA^{-1} + A^{-1}HA^{-1}HA^{-1} - \dots, \tag{1.1}$$

which converges for $\|H\| < \|A^{-1}\|^{-1}$. In this example,

$$\begin{aligned}&(D\,\mathrm{Inv})(A) : H \mapsto -A^{-1}HA^{-1},\\ &(D^2\,\mathrm{Inv})(A) : (H_1,H_2) \mapsto A^{-1}H_1A^{-1}H_2A^{-1} + A^{-1}H_2A^{-1}H_1A^{-1}\end{aligned}$$

and (1.1) is the Taylor expansion of the mapping Inv at A as we will discuss below. Let us now discuss a several straightforward properties of the operation of taking the derivative of a mapping $F : E \to F$.

- Linearity: $D(\alpha f+\beta g)=\alpha Df+\beta Dg$, where $\alpha,\beta\in\mathbb{R}$ and $f,g:U\to F$.
- Chain rule: $D(g\circ f)(a)=(Dg)(f(a))\circ Df(a)$, where the sign $\circ$ in the right-hand side means the composition of linear operators. Indeed, exactly as in the one-real-variable setup, we see that

$$\begin{aligned}g(f(x)) &= g\big(f(a)+[(Df)(a)](x-a)+o(\|x-a\|)\big)\\ &= g(f(a))+[(Dg)(f(a))]\big([(Df)(a)](x-a)+o(\|x-a\|)\big)+o(\|f(x)-f(a)\|)\\ &= g(f(a))+(Dg)(f(a))(Df)(a)(x-a)+o(\|x-a\|),\end{aligned}$$

  where we use the boundedness of the linear operator $(Dg)(f(a))$ (and that of $(Df)(a)$) to control the error terms via $\|x-a\|$.
- If $L\in\mathcal{L}(F_1,\dots,F_m;F)$ is a bounded multi-linear mapping, $f_j:E\supset U\to F_j$ are differentiable at $a\in U$, and $F=L(f_1,\dots,f_m)$, then $f:U\to F$ is also differentiable at a and

$$[Df(a)]h=\sum_{j=1}^{n}L(f_1(a),\dots,f_{j-1}(a),(Df_j)(a)h,f_{j+1}(a),\dots,f_m(a)).$$

  Again, the proof simply repeats the computation of the derivative of a product of two real-valued functions: one expands the expression

$$f(a+h)=L\big(f_1(a)+[Df_1(a)]h+o(\|h\|),\dots,f_m(a)+[Df_m(a)]h+o(\|h\|)\big)$$

  by multi-linearity of L and collect all the linear (in h) terms, all the others lead to $o(\|h\|)$, $h\to 0$, errors since L is bounded.

  Example: Let $\beta:E\supset U\to\mathbb{R}$ and $f:E\subset U\to F$. Then,

$$D(\beta f)(a)\;=\;D\beta(a)\otimes f(a)+\beta(a)Df(a),$$

  where we use the notation $(e'\otimes f)h:=e'(h)\cdot f$ for $e'\in E'$ and $f\in F$. In other words, the first term acts on vectors $h\in F$ as follows: we should first apply the functional $D\beta(a)\in\mathcal{L}(E;\mathbb{R})=E'$ to h and then multiply the (scalar) result by the vector $f(a)\in F$.

We conclude this lecture by introducing the notion of *partial derivatives*. Assume that $E=E_1\times\dots E_m$; an instructive particular case is $E_1=\dots=E_m=\mathbb{R}$. Given a point $E\supset U\ni a$, denote

$$U^{\widehat{a}_j}:=\{x_j\in E_j:(a_1,\dots,a_{j-1},x_j,a_{j+1},\dots,a_m)\in U\},$$

this set can be also identified with the set of all points $x\in U$ whose all but the j-th coordinates coincide with those of a. Let

$$f^{\widehat{a}_j}:U^{\widehat{a}_j}\to F,\qquad f^{\widehat{a}_j}(x_j):=f(a_1,\dots,a_{j-1},x_j,a_{j+1},\dots,a_m),$$

be the restriction of f onto this set.

Definition 1.5. *One says that a mapping $f:E=E_1\times\dots E_m\supset U\to F$ admits* partial derivatives *at a point $a\in U$ if each of the mappings $f^{\widehat{a}_j}:U^{\widehat{a}_j}\to F$ is differentiable at a_j. A general notation is as follows:*

$$(D_{x_j}f)(a)\;:=\;(Df^{\widehat{a}_j})(a_j)\;\in\;\mathcal{L}(E_j;F)$$

but one also often writes $\partial f/\partial x_j$ instead of $D_{x_j}f$, especially if $E_j=\mathbb{R}$.

Trivially, if f is differentiable at $a \in U \subset E$ (in the sense of Definition 1.1), then all partial derivatives exist and $(D_{x_j} f)(a)h = (Df)(a)(0, \dots, 0, h, 0, \dots, 0)$. However, the converse is not true as can be seen from the following example (where $E_1 = E_2 = F = \mathbb{R}$): the partial derivatives $\partial f/\partial x_1$ and $\partial f/\partial x_2$ of a function

$$f(x_1, x_2) := \frac{x_1 x_2}{x_1^2 + x_2^2}, \qquad f(0, 0) := 0,$$

exist at all points, including $x_1 = x_2 = 0$ (since $f(x_1, 0) = f(0, x_2) = 0$ for all $x_1, x_2 \in \mathbb{R}$) but the function is not even *continuous* at $(0, 0)$.

Détour. Though this goes far beyond the scope of this class, it is worth mentioning that the discussion of partial derivatives changes drastically if we consider differentiable (= holomorphic = analytic) functions of several *complex* variables. In this case, the existence (in an open neighborhood of a) of partial derivatives $\partial/\partial z_j$ for all $j = 1, \dots, m$ implies the continuity of f and the existence of the 'full' derivative Df near a. This statement is known under the name *Hartog's theorem* and provides another illustration of the fact that the differentiability with respect to a complex variable (i.e., the existence of a local expansion $f(z) = f(a)+f'(a)(z-a)+o(|z-a|)$ as $z \to a$) is a drastically more rigid assumption than the real-differentiability; see the course *Analyse Complexe* in the spring term.

November 25, 2020

1.1. **'Technical lemmas'.** Let us now briefly discuss standard 'technical' facts on differentiable mappings between Banach spaces, which typically can be easily reduced to similar statements for functions of one real variable. The first lemma is almost trivial and serves as an illustration of how such a reduction works.

Lemma 1.6. *Let $[a, b]$ be a straight segment in $U \subset E$. If $f \in C^1(U; F)$, then*

$$f(b) - f(a) = \Big[\int_0^1 (Df)(a + t(b - a))dt \Big] (b - a), \tag{1.2}$$

where the integral of a continuous mapping $t \mapsto (Df)(a + t(b-a)) \in \mathcal{L}(E; F)$ is understood in the Riemann sense.

Proof. If we set $g(t) := f(a + t(b - a))$, then $f'(t) = [(Df)(a + t(b - a))](b - a)$ by the chain rule. Now we can

- either say that the standard proof for functions of one real variable works in the same way for all target Banach spaces F;
- or to use another reduction to a one-dimensional situation – now for the target space F – based upon the Hahn–Banach theorem: for each bounded linear functionals $A \in F'$, the function $t \mapsto Ag(t)$ is a real-valued continuously differentiable function on $[0, 1]$ and hence $Ag(1)-Ag(0) = \int_0^1 Ag'(t)dt$. This implies that

  $$A(f(b) - f(a)) \ = \ A\Big[\int_0^1 (Df)(a + t(b - a))dt \Big] (b - a).$$

  Since this holds for *all* $A \in F'$, we conclude that (1.2) also holds: indeed, if $Af = 0$ for all $A \in F'$, then $f = 0$ due to Hahn–Banach. □

It is trivial that a linear combination of finitely many C^1 mappings is again a C^1 mapping. The following technical statement extends this property to integrals with respect to a (real) parameter.

Lemma 1.7. *Let $U \subset E$ be an open set, $[\tau_0, \tau_1] \subset \mathbb{R}$ and $f : U \times [\tau_0, \tau_1] \to F$ be such that $f(\cdot, \tau) \in C^1(U; F)$ for all $\tau \in [\tau_0, \tau_1]$ and, moreover, $D_x f$ is continuous as a function of both $x \in E$ and $\tau \in [\tau_0, \tau_1]$, i.e., $D_x f \in C(U \times [\tau_0, \tau_1]; \mathcal{L}(E; F))$. Then, the function*

$$F(x) := \textstyle\int_{\tau_0}^{\tau_1} f(x, \tau) dt$$

is continuously differentiable on U and, for all $x \in U$,

$$[DF](x) = \textstyle\int_{\tau_0}^{\tau_1} D_x f(x, \tau)\, d\tau,$$

where both integrals are understood in the Riemann sense.

Proof. For shortness, denote $\varphi(x, \tau) := D_x f(x, \tau)$ and $\Phi(x) := \int_{\tau_0}^{\tau_1} \varphi(x, \tau) d\tau$. Let us first check that Φ is continuous on U. Given $\varepsilon > 0$ and $a \in U$, for each $\tau \in [\tau_0, \tau_1]$ there exists $\delta(\tau, \varepsilon) > 0$ such that

$$\|\varphi(x, \tau') - \varphi(a, \tau)\| < \varepsilon \text{ provided that } |\tau' - \tau| + \|x - a\| < \delta(\tau, \varepsilon)$$

and hence

$$\|\varphi(x, \tau') - \varphi(a, \tau')\| < 2\varepsilon \text{ provided that } |\tau' - \tau| + \|x - a\| < \delta(\tau, \varepsilon).$$

By compactness, we can find a finite sub-cover of the segment $[\tau_0, \tau_1]$ by intervals $(\tau - \frac{1}{2}\delta(\tau, \varepsilon), \tau + \frac{1}{2}\delta(\tau, \varepsilon))$ and denote by $\delta_0 = \delta_0(\varepsilon)$ the minimum of the corresponding (finitely many) values $\delta(\tau, \varepsilon)$. Then,

$$\|\varphi(x, \tau) - \varphi(a, \tau)\| < 2\varepsilon \text{ for all } \tau \in [\tau_0, \tau_1] \text{ provided that } \|x - a\| < \tfrac{1}{2}\delta_0(\varepsilon)$$

and hence

$$\|\Phi(x) - \Phi(a)\| < 2\varepsilon \cdot |\tau_1 - \tau_0| \text{ provided that } \|x - a\| < \tfrac{1}{2}\delta_0(\varepsilon).$$

Let us now simplify the consideration and, given $a \in U$, replace $f(x, \tau)$ by the function

$$g(x, \tau) := f(x, \tau) - f(a, \tau) - \varphi(a, \tau)(x - a).$$

Note that $D_x g(a, \tau) = D_x f(a, \tau) - \varphi(a, \tau) = 0$ and our goal is to prove that $DG(a) = 0$, where $G(x) := \int_{\tau_0}^{\tau_1} g(x, \tau) d\tau$. This is a variation of the compactness argument used above to prove the continuity of Φ: since $D_x g(a, \tau) = 0$ (and because of the fact that $D_x g(x, \tau) = D_x f(x, \tau) - \varphi(a, \tau)$ is a continuous function of *both* arguments), for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|D_x g(x, \tau)\| \le \varepsilon \text{ for all } \tau \in [\tau_0, \tau_1] \text{ provided that } \|x - a\| < \delta$$

and hence, using Lemma 1.6 applied to a function $f(\cdot, \tau)$ and $b = x$,

$$\|g(x, \tau)\| \le \varepsilon \|x - a\| \text{ for all } \tau \in [\tau_0, \tau_1] \text{ provided that } \|x - a\| < \delta.$$

Integrating this in τ, we get the estimate $\|G(x)\| \le \varepsilon |\tau_1 - \tau_0| \cdot \|x - a\|$ for all x such that $\|x - a\| < \delta = \delta(\varepsilon)$. This means that $\|G(x)\| = o(\|x - a\|)$ as $x \to a$. □

In the proof given above we relied upon Lemma 1.6 when saying that a uniform estimate on the derivative of a mapping implies the natural uniform estimate on the increments of this mapping. Similarly to the one-real-variable context, for such a claim there is no need to assume that f is *continuously* differentiable:

Lemma 1.8. *Let $[a,b]$ be a straight segment in $U \subset E$ and a mapping $f : U \to F$ be differentiable at all points on $[a,b]$. Then,*

$$\|f(b) - f(a)\|_F \;\le\; \sup_{x\in[a,b]} \|Df(x)\|_{\mathcal{L}(E;F)} \cdot \|b-a\|_E.$$

Proof. As in Lemma 1.6, the claim can be easily reduced to the one-real-variable context by considering a functional $A \in F'$ and a function $g(t) := Af(a+t(b-a))$, $g : [0,1] \to \mathbb{R}$. Since $g'(t) = A(Df)(a+t(b-a))(b-a)$, we have

$$\|A(f(b)-f(a))\| \le \sup_{x\in[a,b]} \|A(Df)(x)(b-a)\| \le \|A\| \cdot \sup_{x\in[a,b]} \|Df(x)\| \cdot \|b-a\|.$$

Due to Hahn–Banach, one can choose a functional $A \in F'$ so that $\|A\| = 1$ and $\|A(f(b)-f(a))\| = \|f(b)-f(a)\|$, which implies the desired claim. □

Remark. Let us also briefly recall/discuss the proof of the one-real-variable result:

- Given a function $g : [0,1] \to \mathbb{R}$, the most standard way to estimate the increment $g(1) - g(0)$ by $\sup_{t\in[0,1]} |g'(t)|$ is to find an extremum of the function $g(t) - t(g(1)-g(0))$ (which has the same values $g(0)$ at both $t = 0$ and $t = 1$) and to say that $g'(t) = g(1) - g(0)$ at this extremal point.

  This proof does *not* directly apply to the multi-dimensional setup: even for smooth curves $g : [0,1] \to \mathbb{R}^2$ there is no guarantee that there exists $t \in [0,1]$ such that $g'(t) = g(1) - g(0)$: e.g., consider $g(t) := \cos(2\pi t, \sin 2\pi t)$.
- However, there is another standard one-dimensional proof which can be directly generalized to the setup of Lemma 1.8 in order to avoid using the axiom of choice: denote $M := \sup_{x\in[a,b]} \|Df(x)\|$ and consider the set

  $$\{x \in [a,b] : \|f(x) - f(a)\| \le (M+\varepsilon) \cdot \|x-a\|\}.$$

  For each $\varepsilon > 0$ this set is simultaneously closed (trivially by continuity) and open (if $x \in [a,b]$ belongs to this set, then a certain open neighborhood of x does since $\|f(x') - f(x)\| \le \|Df(x)\| \cdot \|x'-x\| + o(\|x'-x\|)$), thus $\|f(b) - f(a)\| \le (M+\varepsilon)\|b-a\|$ for all $\varepsilon > 0$ and we can send $\varepsilon \to 0$.

The last 'technical' lemma concerns limits of differentiable functions.

Lemma 1.9. *Let $f_n : E \supset U \to F$ be everywhere differentiable in U. Assume that $f_n \to f$ (pointwise) and $Df_n =: \varphi_n \rightrightarrows \varphi$ uniformly on U. Then, f is everywhere differentiable in U and $Df = \varphi$. Moreover, if $f \in C^1(U;F)$, then $f \in C^1(U;F)$.*

Proof. The proof mimics the one-real-variable case. Given $a \in U$ and $\varepsilon > 0$, we can find $N = N(\varepsilon)$ such that $\|Df_n - \varphi\| \le \varepsilon$ and hence $\|Df_n - Df_N\| \le 2\varepsilon$ for all $n \ge N = N(\varepsilon)$, uniformly in U. Applying Lemma 1.8 in a vicinity of a, we see that

$$\|(f_n(x) - f_N(x)) - (f_n(a) - f_N(a))\| \;\le\; 2\varepsilon \cdot \|x-a\|.$$

Since the function f_N is differentiable at a, we know that

$$\|f_N(x) - f_N(a) - Df_N(a)(x-a)\| \;\le\; \varepsilon \cdot \|x-a\| \;\text{ if }\; \|x-a\| \le \delta = \delta(\varepsilon, N).$$

Finally, $\|Df_N(a) - \varphi(a)\| \le \varepsilon$ provided that $N(\varepsilon)$ is chosen large enough. All together, we have

$$\|f_n(x) - f_n(a) - \varphi(a)(x-a)\| \le 4\varepsilon\|x-a\| \;\text{ if }\; \|x-a\| \le \delta(\varepsilon, N(\varepsilon))$$

for all $n \ge N(\varepsilon)$ and hence the same for the limit f of functions f_n. This means that f is differentiable at a and $Df(a) = \varphi(a)$. If, in addition, $f_n \in C^1(U;F)$, then $Df = \varphi \in C(U;F)$ as the uniform limit of continuous mappings $\varphi_n \in C(U;F)$. □

2. The symmetry of partial derivatives and the Taylor formula

Let us now come back to the setup when $E = E_1 \times \ldots E_m$. As discussed in the previous lecture, the existence of all partial derivatives $D_{x_j} f : U \to \mathcal{L}(E_j; F)$ is not enough even to guarantee the continuity of a mapping $f : E \supset U \to F$, letting alone the differentiability. However, if we require that all these partial derivatives are continuous, the situation becomes much nicer.

Theorem 2.1. *Let $a \in U$ and all the partial derivatives $D_{x_j} f$, $j = 1, \ldots, m$, of a mapping $f : U \to F$ exist in an open neighborhood of the point a and are continuous at this point. Then, the mapping f is differentiable at a and*

$$[Df(a)]h = \textstyle\sum_{j=1}^m [D_{x_j} f(a)]h_j, \quad \textit{where} \quad h = (h_1, \ldots, h_m) \in E = E_1 \times \ldots \times E_m.$$

In particular, if $D_{x_j} f \in C(U; \mathcal{L}(E_j; F))$ for all $j = 1, \ldots, m$, then $f \in C^1(U; F)$.

Proof. This is a simple corollary of Lemma 1.8. Let

$$g(x) := f(x) - \textstyle\sum_{j=1}^m [D_{x_j} f(a)](x_j - a_j).$$

Note that $D_{x_j} g = 0$ for all $j = 1, \ldots, m$ and that we aim to prove that $Dg = 0$. For x close enough to a, denote a sequence of points $x^{(j)} \in U$, $j = 0, \ldots, m$, as follows:

$$x^{(j)} := (x_1, \ldots, x_j, a_{j+1}, \ldots, a_m);$$

note that $x^{(0)} = a$ and $x^{(m)} = x$. Applying Lemma 1.8 on each of the segments $[x^{(j-1)}, x^{(j)}] \subset U$ we see that

$$\|g(x^{(j)}) - g(x^{(j-1)})\| \le \sup\nolimits_{[x^{(j-1)}, x^{(j)}]} \|D_{x_j} g\| \cdot \|x_j - a_j\|.$$

Since $D_{x_j} g$ is continuous at the point a and $(D_{x_j} g)(a) = 0$, for each ε there exists $\delta > 0$ such that all $\|D_{x_j} g\| \le \varepsilon$ provided that $\|x - a\| = \sum_{j=1}^m \|x_j - a_j\| \le \delta$. Therefore,

$$\|g(x) - g(a)\| \le \textstyle\sum_{j=1}^m \|g(x^{(j)} - g(x^{(j-1)}))\| \le m\varepsilon \cdot \|x - a\|$$

provided that $\|x - a\| \le \delta = \delta(\varepsilon)$, i.e., $g(x) = g(a) + o(\|x - a\|)$ as $x \to a$. □

The next important fact (which is also a corollary of Lemma 1.8) to discuss is the symmetry of partial derivatives under the assumption of their continuity. It is convenient to start the consideration with a particular case $E = \mathbb{R}^2$.

Proposition 2.2. *Let $(0, 0) \in U \subset \mathbb{R}^2$ and $f \in C^1(U; F)$. Assume that the partial derivative of the function $\partial f / \partial x_1$ with respect to x_2 exists in an open neighborhood of the point $(0, 0)$ and is continuous at this point. Then, the partial derivative of the function $\partial f / \partial x_2$ with respect to x_1 at the point $(0, 0)$ also exists and*

$$\frac{\partial}{\partial x_1} \frac{\partial f}{\partial x_2}(0, 0) = \frac{\partial}{\partial x_2} \frac{\partial f}{\partial x_1}(0, 0).$$

Proof. Note that we can assume that $\partial / \partial x_2 (\partial f / \partial x_1)(0, 0) = 0$ without loss of generality: indeed, replacing $f(x_1, x_2)$ by $f(x_1, x_2) - \partial / \partial x_2 (\partial f / \partial x_1)(0, 0) \cdot x_1 x_2$ neither change the differentiability assumptions nor the claim to be proved.

For (x_1, x_2) close enough to $(0, 0)$, let

$$\begin{aligned} g(x_1, x_2) :=& f(x_1, x_2) - f(x_1, 0); \\ h(x_1, x_2) :=& g(x_1, x_2) - g(0, x_2) \\ =& f(x_1, x_2) - f(x_1, 0) - f(0, x_2) + f(0, 0); \end{aligned}$$

note that the latter expression is symmetric in x_1, x_2 and that we aim to prove that

$$\lim_{x_1\to 0} \frac{(\partial f/\partial x_2)(x_1,0) - (\partial f/\partial x_2)(0,0)}{x_1} = \lim_{x_1\to 0}\lim_{x_2\to 0} \frac{h(x_1,x_2)}{x_1x_2} \overset{[?]}{=} 0\,. \tag{2.1}$$

On the other hand, it directly follows from Lemma 1.8 that

$$\|h(x_1,x_2)\| \ \le\ \sup\nolimits_{t_1\in[0,x_1]} \|(\partial g/\partial x_1)(t_1,x_2)\| \cdot |x_1|$$

and

$$\begin{aligned}\|(\partial g/\partial x_1)(t_1,x_2)\| \ &= \|(\partial f/\partial x_1)(t_1,x_2) - (\partial f/\partial x_1)(t_1,0)\| \\ &\le\ \sup\nolimits_{t_2\in[0,x_2]} \|\partial/\partial x_2(\partial f/\partial x_1)(t_1,t_2)\| \cdot |x_2|.\end{aligned}$$

As the latter second partial derivative is assumed to be continuous at the point $(0,0)$ and vanishes at this point, the proof is in fact complete: for each $\varepsilon > 0$ one can find $\delta > 0$ such that

$$\|\partial/\partial x_2(\partial f/\partial x_1)(t_1,t_2)\| \le \varepsilon \ \text{ and hence } \ \|h(x_1,x_2)\| \le \varepsilon\cdot|x_1||x_2|.$$

for all $(t_1,t_2)\in[0,x_1]\times[0,x_2]$, provided that $|x_1|+|x_2| < \delta = \delta(\varepsilon)$. In particular, this uniform bound implies that $|\lim_{x_2\to 0} h(x_1,x_2)/x_2| \le \varepsilon\cdot|x_1|$ if $|x_1| < \delta(\varepsilon)$. Thus, the limit as $x_1 \to 0$ in (2.1) exists and equals to 0. □

We will start the next lecture by discussing why Proposition 2.2 implies that the m-th derivative of a mapping $f\in C^m(E;F)$ is a *symmetric* multi-linear mapping.

November 30, 2020

In the previous lecture we proved Proposition 2.2, which says – under a certain continuity assumption – that the second partial derivatives of a function $f:\mathbb{R}^2\to F$ are symmetric with respect to the order of the derivations. The next theorem is a straightforward corollary of this proposition.

Theorem 2.3. *Let E, F be Banach spaces and $f\in C^m(U;F)$ be a m times continuously differentiable function defined on an open set $U\subset E$. Then, its m-th derivative is a symmetric multi-linear mapping: $D^m f\in C(U;\mathcal{L}_m^{\mathrm{sym}}(E;F))$.*

Proof. Let $a\in U$ and $h_1,\dots,h_m\in E$. Consider a function $g:\mathbb{R}^m\supset V\to F$ defined by

$$g(t_1,\dots,t_m)\ :=\ f(a+t_1h_1+\ldots+t_mh_m),$$

where $V := \{(t_1,\dots,t_m)\in\mathbb{R}^m : a+t_1h_1+\ldots+t_mh_m\in U\}$. It is easy to see (e.g., by induction in m) that

$$[D^m f(a+t_1h_1+\ldots+t_mh_m)](h_1,\dots,h_m)\ =\ \frac{\partial}{\partial t_1}\cdots\frac{\partial g}{\partial t_m}(t_1,\dots,t_m).$$

As we assume the continuity of $(D^m f)(a)$ in a, Proposition 2.2 yields that the right-hand side is symmetric with respect to the order of derivations. Therefore, the multi-linear mapping $[(D^m f)(a)](h_1,\dots,h_m)$ is symmetric in $h_1,\dots,h_m$. □

Before going further to the Taylor formula, let us discuss two more exercises on how usual formulas for second derivatives read in the multi-dimensional situation.

(1) What happens with the standard formula $(f_1f_2)'' = f_1''f_2 + 2f_1'f_2' + f_1f_2''$?

Let $L \in \mathcal{L}(F_1, F_2; F)$ and $f_{1,2} : E \supset U \to F_{1,2}$ be twice differentiable mappings. Recall that, if $f = L(f_1, f_2)$, then

$$[(Df)(a)]h = L((Df_1)(a)h, f_2(a)) + L(f_1(a), (Df_2)(a)h).$$

Differentiating this one more time in the direction k, we get

$$\begin{aligned}[(Df)(a)](k,h) &= L((D^2f_1)(a)(k,h), f_2(a)) + L((Df_1)(a)h, (Df_2)(a)k) \\ &+ L((Df_1)(a)k, (Df_2)(a)h) + L(f_1(a), (D^2f_2)(a)(k,h)).\end{aligned}$$

(2) What happens with the standard formula $(g\circ f)'' = (g''\circ f)\cdot(f')^2 + (g'\circ f)\cdot f''$?

Recall that

$$[D(g\circ f)(a)]h = [Dg(f(a))](Df)(a)h.$$

Differentiating this once more in the direction k, we get

$$\begin{aligned}[D^2(g\circ f)(a)](k,h) &= [D^2g(f(a))]((Df)(a)k, (Df)(a)h) \\ &+ (Dg)(f(a))\,[(D^2f)(a)](k,h).\end{aligned}$$

Let us now discuss the Taylor formula for mappings between Banach spaces.

Theorem 2.4 (Taylor's formula). *Let $f \in C^{m-1}(U;F)$ and, moreover, there exists the m-th derivative $(D^mf)(a)$ of f at a point $a \in U$. Then,*

$$f(x) = \sum_{k=0}^{m} \frac{1}{k!}[(D^kf)(a)](x-a) + o(\|x-a\|^m) \quad \text{as} \quad x \to a.$$

Moreover, if $f \in C^m(U;F)$ and $D^{m+1}f$ exists at all points of the segment $[a,x]$, then the remainder is bounded by $\frac{1}{(m+1)!}\sup_{y\in[a,x]}\|(D^{m+1}f)(y)\|\cdot\|x-a\|^{m+1}$.

Proof. Denote $g(x) := f(x) - \sum_{k=0}^{m}\frac{1}{k!}[(D^kf)(a)](x-a)$. It is easy to see that $(D^kg)(a) = 0$ for all $k = 0,\ldots,m$. Indeed, if $L \in \mathcal{L}_k(E;F)$ is a multi-linear mapping and $\ell(x) := L(x-a) = L(x-a,\ldots,x-a)$, then

- $(D^s\ell)(a) = 0$ if $s < k$ since at least one of $x-a$ survive in $D^s\ell$;
- $[(D^s\ell)(x)]h = k!L(h)$ if $s = k$ for all x, the factor $k!$ appears since each time – when differentiating – we should replace $x-a$ by h and there are $k!$ ways to obtain all arguments h from all arguments $x-a$.
- $(D^s\ell)(x) = 0$ for all x if $s > k$.

We need to prove that $\|g(x)\| = o(\|x-a\|^m)$. This can be easily done by induction: $(D^mg)(a) = 0$ means that $\|(D^{m-1}g)(x)\| = o(\|x-a\|)$ as $x \to a$; then it follows from Lemma 1.8 and $(D^{m-1}g)(a) = 0$ that $\|(D^{m-2}g)(x)\| = o(\|x-a\|^2)$ etc.

The quantitative control of the remainder term through $\sup_{y\in[a,x]}\|(D^{m+1}f)(y)\|$ can be obtained in the same way (i.e., by inductively applying Lemma 1.8). □

Let us now discuss how the Taylor formula reads in terms of the partial derivatives when $E = \mathbb{R}^n$. It is easy to see by induction that

$$[(D^kf)(a)](h^{(1)},\ldots,h^{(k)}) = \sum_{j_1,\ldots,j_k=1}^{n} \frac{\partial}{\partial x_{j_1}}\cdots\frac{\partial f}{\partial x_{j_k}}(a)\cdot h^{(1)}_{j_1}\ldots h^{(k)}_{j_k}.$$

If $h^{(1)} = \ldots = h^{(k)} = h$, then we can use the symmetry of partial derivatives and collect similar terms. Let $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$ be a *mutli-index*, where s_j denotes the number of instances of $\partial/\partial x_j$ in the k-th partial derivative of f. In particular,

$$|s| := s_1 + \ldots + s_n \ = \ k.$$

Then, $[(D^k f)(a)](h) = [(D^k f)(a)](h, \ldots, h)$ contains

$$\frac{k!}{s!} \ := \ \frac{k!}{s_1! \ldots s_n!} \ = \ \begin{pmatrix} & k & \\ s_1 \ s_2 & \ldots & s_n \end{pmatrix} \ =: \ \begin{pmatrix} k \\ s \end{pmatrix}$$

terms

$$\frac{\partial^k f}{\partial x^s}(a) \cdot h^s \ := \ \frac{\partial^k f}{\partial x_1^{s_1} \ldots \partial x_n^{s_n}}(a) \cdot h_1^{s_1} \ldots h_n^{s_n} \,.$$

To summarize,

$$[(D^k f)(a)](h) \ = \ \sum_{s \in \mathbb{N}^n :\, |s| = k} \begin{pmatrix} k \\ s \end{pmatrix} \frac{\partial^k f}{\partial x^s}(a) \cdot h^s$$

and the Taylor formula can be rewritten as

$$f(x) \ = \sum_{s \in \mathbb{N}^n :\, |s| \le m} \frac{1}{s!} \frac{\partial^{|s|} f}{\partial x^s}(a) \cdot (x-a)^s + o(\|x-a\|^m) \ \text{ as } \ x \to a,$$

where we use the same notation $(x-a)^s := (x_1 - a_1)^{s_1} \cdot \ldots \cdot (x_n - a_n)^{s_n}$ as above.

Let us now briefly discuss a traditional terminology used in finite-dimensional situations.

- Let $E = \mathbb{R}^n$ and $F = \mathbb{R}$. Then, the vector $\nabla f := (\partial f/\partial x_1 \ \ldots \ \partial f/\partial x_n)$ is called the *gradient* of f. In what follows we view $\mathbb{R}^n$ as the space of <u>column</u> vectors so that the real number $[(Df)(a)](h) = (\nabla f)(a) \cdot h$ can be simply viewed as a product of a $1 \times n$ and $n \times 1$ matrices; in other words, we view the (row) vector $\nabla f(a)$ as an element of the dual space. However, in many situations it is convenient to use the self-duality of $\mathbb{R}^n$ and to define the gradient as a column vector too so that $[(Df)(a)](h) = \langle (\nabla f)(a), h \rangle$.
- Let $E = F = \mathbb{R}^n$. The *Jacobian (determinant)* of $f : \mathbb{R}^n \supset U \to \mathbb{R}^n$ is
$$\det \mathrm{J}(f), \quad \mathrm{J}(f) := \left[\frac{\partial f_p}{\partial x_q} \right]_{p,q=1}^n ;$$
the $n \times n$ matrix $\mathrm{J}(f)$ of partial derivatives (which is nothing but the matrix representation of $(Df)(a) \in \mathcal{L}(\mathbb{R}^n)$) is called a *Jacobian matrix.*
- Let $E = \mathbb{R}^n$ and $F = \mathbb{R}$. The *Hessian (matrix)* of $f : \mathbb{R}^n \supset U \to \mathbb{R}$ is
$$\mathrm{H}(f) := \left[\frac{\partial^2 f}{\partial x_p \partial x_q} \right]_{p,q=1}^n$$
The symmetric(!) matrix $\mathrm{H}(f)$ represents the second derivative of f as follows: $[(D^2 f)(a)](k, h) = k^\top \cdot \mathrm{H}(f)(a) \cdot h$.

Finally, let us formulate the usual criterion for extremal points of a mapping $f : E \supset U \to R$ at a point $a \in U$.

Proposition 2.5. *Let $f \in C^1(U; \mathbb{R})$ and there exists $(D^2 f)(a) \in \mathcal{L}_2^{\mathrm{sym}}(E; \mathbb{R})$. Then,*

- ◦ *if a is an extremal point of f, then $(Df)(a) = 0$ and $(D^2 f)(a) \geq 0$ (minimum) or $(D^2 f)(a) \leq 0$ (maximum), in the sense of quadratic forms (i.e., $[(D^2 f)(a)](h) \geq 0$ for all $h \in E$ at a minimum and similarly for maxima);*
- ◦ *vice versa, if $(Df)(a) = 0$ and $[(D^2 f)(a)](h) \geq c \cdot \|h\|^2$ for all $h \in E$ and some $c > 0$, then a is a local minimum of f; and similarly for maxima.*

Proof. This is a trivial corollary of the Taylor formula with $m = 2$. □

We conclude this lecture by a few simple remarks on this standard 'second derivative' criterion of extremal points.

- The sufficient condition $[(D^2 f)(a)](h) \geq c\|h\|^2$ actually makes sense only when we work in Hilbert spaces or, more precisely, with norms that are equivalent to Hilbert ones: since we also have $[(D^2 f)(a)](h) \leq C\|h\|^2$, the quadratic form $[(D^2 f)(a)](h)$ can be used to introduce the scalar product structure, which gives rise to a norm $([(D^2 f)(a)](h))^{1/2} \asymp \|h\|$.
- In the one-dimensional situation, the roots of the derivative $f'(a) = 0$ are typically extrema of f, unless the second derivative at a degenerates. This is <u>not</u> the case in the multi-dimensional situation: if $(Df)(a) = 0$ and the second derivative $(D^2 f)(a)$ is non-degenerate, it is typically neither positive nor negative definite. (Indeed, the Hessian matrix $(\mathrm{H}f)(a)$ typically has eigenvalues of both signs.) Such points a are called *saddle points* of f.
- A possible way to check whether a $n \times n$ matrix $(\mathrm{H}f)(a)$ is strictly(!) positive definite is to consider its minors $\det[\partial^2/\partial_{x_p}\partial_{x_q}]_{p,q=1}^{k}$ for $k = 1, \ldots, n$. The *Sylvester criterion* says that the quadratic form $(\mathrm{H}f)(a)$ is strictly positive definite if and only if all these n determinants are positive.

December 02, 2020

3. Inverse and implicit function theorems

Today we discuss two important 'technical' statements on smooth functions, which can be loosely formulated as follows:

- a local inverse f^{-1} to a smooth mapping f exists and is smooth provided that the derivative of f is non-degenerate ('inverse function theorem');
- the zero set $\{(x, y) : f(x, y) = 0\}$ of a smooth mapping f can be locally viewed as a graph $\{(x, g(x))\}$ of a smooth mapping g provided that the partial derivative $D_y f$ is non-degenerate ('implicit function theorem').

Theorem 3.1. *Let $f \in C^m(U; F)$, $m \geq 1$, and $a \in U$. Assume that the linear operator $(Df)(a) \in \mathcal{L}(E; F)$ has a bounded inverse $[(Df)(a)]^{-1} \in \mathcal{L}(F; E)$. Then, there exists an open neighborhood $a \in V \subset U$ such that f is a homeomorphism of V onto an <u>open</u> set $W \subset F$ and, moreover, a C^m-diffeomorphism (i.e., $f^{-1} \in C^m(W; E)$). In particular, f^{-1} is differentiable in W and $(Df^{-1})(f(x)) = [(Df)(x)]^{-1}$, $x \in V$.*

Let us mention from the very beginning that – without loss of generality – one can assume that $F = E$ and $[(Df)(a)] = \mathrm{Id}_E$ if we consider the composition

$$[(Df)(a)]^{-1} \circ f : E \to E$$

instead of the mapping $f : E \to F$ itself; note that the boundedness of operators $(Df)(a)$ and $[(Df)(a)]^{-1}$ essentially says that E and F are isomorphic: more precisely, E and F are isomorphic up to a change of the norms by equivalent ones.

Proposition 3.2. *Let $g : E \subset V \to E$ be a q-Lipschitz mapping, where $q < 1$. Then, the mapping $f : x \mapsto x + g(x)$ is a homeomorphism between V and an open set $f(V) \subset E$. Moreover, the inverse mapping f^{-1} is $(1-q)^{-1}$-Lipschitz.*

Proof. It directly follows from the q-Lipschitzness of the mapping g that

$$\|f(x') - f(x)\| \geq (1-q)\|x' - x\|, \quad x, x' \in V.$$

Thus f is a bi-Lipschitz bijection of V and $f(V)$, so essentially we only need to prove that $f(V)$ is an open set in E. This follows from the fixed point theorem for q-Lipschitz mappings, as follows.

Let $b = f(a) \in f(V)$ and $r > 0$ be such that $\overline{B}(a,r) \subset V$. We aim to prove that $\overline{B}(b,(1-q)r) \subset f(V)$. To this end, given a point $y \in \overline{B}(b,(1-q)r)$, consider a mapping $x \mapsto y - g(x)$. This mapping (a) is q-Lipschitz (since so is g) and (b) maps the closed ball $\overline{B}(a,r)$ into itself: if $\|x - a\| \leq r$, then

$$\begin{aligned} \|(y - g(x)) - a\| &\leq \|y - b\| + \|g(x) + (a - b)\| \\ &= \|y - b\| + \|g(x) - g(a)\| \leq (1-q)r + q\|x - a\| \leq r. \end{aligned}$$

Since $\overline{B}(a,r)$ is a complete metric space, there exists a point $x \in \overline{B}(a,r)$ such that $x = y - g(x)$, i.e., $y = f(x)$. Thus, $\overline{B}(f(a),(1-qr)) \subset f(\overline{B}(a,r))$. □

Proof of Theorem 3.1. As discussed above, for simplicity let us assume (without loss of generality) that $E = F$ and $(Df)(a) = \mathrm{Id}_E$. Let $\rho = \rho_{1/2} > 0$ be such that

$$V = V_{1/2} := B(a, \rho_{1/2}) \subset \{x \in U : \|(Df)(x) - \mathrm{Id}\| < \tfrac{1}{2}\}.$$

If we denote $g(x) := f(x) - x$, then the mapping g is $\frac{1}{2}$-Lipschitz in V due to Lemma 1.8. Therefore, it follows from Proposition 3.2 that $W := f(V)$ is an open set in E and that $f : V \to W$ is a homeomorphism.

Let us now prove that the inverse mapping $f^{-1} : W \to V$ is differentiable at the point $b := f(a)$ and that $(Df^{-1})(b) = \mathrm{Id}$. To this end, define open balls $V_\varepsilon = B(a, \rho_\varepsilon)$ similarly to $V_{1/2}$ and let $W_\varepsilon := f(V_\varepsilon)$. Then, for all $y \in W_\varepsilon$ we have

$$\begin{aligned} \|(f^{-1}(y) - a) - (y - b)\| &= \|g(f^{-1}(y)) - g(f^{-1}(b))\| \\ &\leq \varepsilon \cdot \|f^{-1}(y) - f^{-1}(b)\| \leq \varepsilon(1-\varepsilon)^{-1} \cdot \|y - b\|, \end{aligned}$$

where we consecutively used the Lipshitzness of g and the Lipshitzness of f^{-1}. Thus,

$$f^{-1}(y) = a + (y - b) + o(\|y - b\|) \text{ as } y \to b,$$

i.e., $(Df^{-1})(b) = \mathrm{Id} = [(Df)(a)]^{-1}$.

We can apply the same argument for all points $x \in V_{1/2}$ since $(Df)(x)$ is invertible for all $x \in V_{1/2}$. Therefore, the derivative $(Df^{-1})(y) = [(Df)(x)]^{-1}$, where $y = f(x)$, is defined *pointwise* in $W = W_{1/2}$ and it only remains to prove that this derivative depends on y continuously. Note that

$$Df^{-1} = \mathrm{Inv} \circ Df \circ f^{-1}, \quad W \overset{f^{-1}}{\to} V \overset{Df}{\to} \mathcal{L}(E;F) \overset{\mathrm{Inv}}{\to} \mathcal{L}(F;E),$$

is a composition of continuous mappings, i.e., $Df^{-1} \in C(W; \mathcal{L}(F;E))$.

Finally, for $f \in C^m(U;F)$ with $m \geq 2$ one can use an inductive argument: if we already know that $f^{-1} \in C^{m-1}(W;E)$, then the explicit formula for Df^{-1} implies that Df^{-1} is $m-1$ times continuously differentiable, i.e., $f^{-1} \in C^m(W;E)$. □

Theorem 3.3. *Let $(x_0,y_0)\in U\subset E\times F$ and $f\in C^m(U;F)$, where $m\geq 1$. Assume that $f(x_0,y_0)=0$ and that the linear operator $(D_yf)(x_0,y_0)$ is invertible in $\mathcal{L}(F)$. Then, there exist open neighborhoods $x_0\in V\subset E$ and $y_0\in W\subset F$ and a C^m-smooth function $g:V\to W$ such that $V\times W\subset U$ and*

$$f(x,y)=0 \;\Leftrightarrow\; y=g(x) \;\text{ for }\; (x,y)\in V\times W.$$

Proof. Let

$$\psi(x,y):=(x,f(x,y)),\quad \psi:E\times F\supset U\to E\times F,$$

and note that

$$[(D\psi)(x_0,y_0)](h_x,h_y)=(h_x,[(D_xf)(x_0,y_0)]h_x+[(D_yf)(x_0,y_0)]h_y)$$

is an invertible operator in $\mathcal{L}(E\times F)$: its inverse can be explicitly written as

$$(k_x,k_y)\;\mapsto\;(k_x,[(D_yf)(x_0,y_0)]^{-1}(k_y-[(D_xf)(x_0,y_0)]k_x)).$$

Therefore, we can apply Theorem 3.1 to the mapping ψ and find a neighborhood $U=V_0\times W\ni(x_0,y_0)$ such that ψ is a C^m-diffeomorphism of U onto an open set $\psi(U)\subset E\times F$. By definition, for $(x,y)\in V_0\times W$, the equation $f(x,y)=0$ is equivalent to $\psi(x,y)=(x,0)$. Now let

$$V\;:=\;\{x\in V_0:(x,0)\in\psi(U)\}$$

($V\ni x_0$ is an open set in E since $\psi(U)$ is open in $E\times F$) and

$$g(x):=(\pi_F\circ\psi^{-1})(x,0)\;\text{ for }\;x\in V.$$

The proof is complete (the C^m-smoothness of g follows from that of ψ^{-1}). □

4. (Compact) smooth manifolds

We conclude this lecture by briefly discussing a notion of a *compact smooth manifold (embedded) in* $\mathbb{R}^N$ and will continue next time by discussing its link with an 'abstract' definition of compact smooth manifolds that was mentioned in the first part of the course.

Definition 4.1. *A compact set $M^n\subset\mathbb{R}^N$ (where $N>n$) is called a C^k-smooth manifold of dimension n if*

(1) *for each point $a\in M^n$ there exists an open neighborhood $a\in U\subset\mathbb{R}^N$ and a smooth mapping $f\in C^k(U;\mathbb{R}^{N-n})$ such that* $\operatorname{rank}(Df)(a)=N-n$ *and $M^n\cap U=\{x\in U:f(x)=0\}$.*

or, equivalently,

(2) *for each point $a\in M^n$ there exist a subset $J=\{j_1,\dots,j_n\}\subset[1,N]\subset\mathbb{N}$ of coordinates, an open neighborhood $a\in U=V\times W\subset\mathbb{R}^J\times\mathbb{R}^{[1,N]\smallsetminus J}$ and a smooth mapping $g\in C^k(V;W)$ such that $M^n\cap U=\{(x_J,g(x_J)),x_J\in V\}$.*

The equivalence (1)⇔(2) is a corollary of the implicit function theorem:

- '(1)⇒(2)': since $\operatorname{rank}(Df)(a)=N-n$, we can find a $(N-n)\times(N-n)$ minor in the matrix $(Df)(a)$ that admits a bounded inverse and denote by J the set of remaining coordinates;
- '(2)⇒(1)': one can simply take $f(x):=x_{[1,N]\smallsetminus J}-g(x_J)$.

We will continue discussing smooth manifolds in the next lecture.

December 07, 2020

We concluded the last lecture by discussing two equivalent descriptions of (compact) *smooth n-dimensional manifold* <u>*embedded*</u> <u>*into*</u> $\mathbb{R}^N$ (the word 'smooth' – here and below – means either C^k or C^∞): $M = M^n$ is a compact subset of $\mathbb{R}^N$ such that for each point $a \in M$ the following holds

(1) there exists an open neighborhood $a \in U_a \subset \mathbb{R}^N$ and a smooth function $f_a : U_a \to \mathbb{R}^{N-n}$ such that $\operatorname{rank}(Df_a)(a) = N - n$ and

$$M \cap U_a = \{x \in U_a : f_a(x) = 0\}$$

(in other words, $M \cap U$ is – locally – the zero set of a smooth function f_a);

(2) there exists $J \subset [1, N]$, $\#J = n$, an open neighborhood $a \in U_a = V_a \times W_a \subset \mathbb{R}^J \times \mathbb{R}^{[1,N]\smallsetminus J}$ and a smooth function $g_a : V_a \to W_a$ such that

$$M \cap U_a = \{(x_J, g_a(x_J));\ x_J \in V_a\}$$

(i.e., M is – locally – a graph of a smooth function $\mathbb{R}^J \to \mathbb{R}^{[1,N]\smallsetminus J}$).

The equivalence (1)⇔(2) easily follows from the implicit function theorem.

Recall also that in the first part of the course we also briefly discussed an 'abstract' definition of smooth n-dimensional topological manifolds, which does <u>not</u> require considering an ambient space $\mathbb{R}^N$:

(0) $M = M^n$ is called a (compact) *smooth* <u>*topological*</u> *manifold of dimension n* if M is a compact Hausdorff topological space and there exists a (finite, by compactness) open covering $M = \bigcup_{\alpha \in A} U_\alpha$ such that
- each $U_\alpha \subset M$ is homeomoprhic (by a mapping $\varphi_\alpha : U_\alpha \to B^n$) to the unit open ball $B^n \subset \mathbb{R}^n$ and
- all compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are (C^k or C^∞) smooth on their natural domains of definition $\varphi_\alpha(U_\alpha \cap U_\beta) \subset B^n$.

Recall that U_α are called *charts* and the collection $(U_\alpha)_{\alpha \in A}$ – an *atlas*. Also, note that one can speak about smooth (up to C^k) functions between topological manifolds:
- for $s \le k$, a mapping $f : M \supset U \to M'$ is called C^s-smooth if all the compositions $\varphi'_{\alpha'} \circ f \circ \varphi_\alpha^{-1}$ are C^s-smooth on their domains of definition (where $\varphi'_{\alpha'}$ denote the chart mappings on the manifold M').

Clearly, this definition does not depend on the choice of a chart of M at a point $a \in U$ neither on the chart of M' at $f(a)$ as we require that all compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ and $\varphi'_{\beta'} \circ {\varphi'}_{\alpha'}^{-1}$ are C^k-smooth and $k \ge s$.

It is easy to see that smooth manifolds embedded into $\mathbb{R}^N$ can be viewed as a *particular case* of the definition (0) of smooth topological manifolds:
- Indeed, in (2) one can choose $V_a \in \mathbb{R}^J$ to be an open ball and define $\varphi_a : U_a \cap M \to V_a$ to be the projection onto the coordinates $\mathbb{R}^J$. This is a homeomorphism of since $\varphi_a^{-1} = (\mathrm{id}, g_a)$.
- The compositions $\varphi_b \circ \varphi_a^{-1}$ are smooth on their domains of definitions is a triviality since φ_b is a projection of g_a on a (different) subset of coordinates.

In particular, one can speak about smooth functions defined on smooth manifolds embedded into $\mathbb{R}^N$. It is also not hard (though less trivial) to prove that (0)⇒(2) in the following sense:

Proposition 4.2. *Let $M = M^n$ be a compact C^k- or C^∞-smooth n-dimensional topological manifold. Then there exists $N \ge n$ (a priori, depending on M) and a compact smooth manifold $M_N \subset \mathbb{R}^N$ embedded into $\mathbb{R}^N$ such that M is homeomorphic and, moreover, C^k- or C^∞-, respectively, diffeomorphic to M_N.*

The latter means that both the mapping $M \to M_N$ and its inverse $M_N \to M$ are (C^k- or C^∞-, respectively) smooth as mappings between *topological* manifolds; recall that (2) can be viewed as a particular case of (0).

Proof. For a point $a \in M$, let $\varphi_a : M \supset U_a \to B^n$ be a homeomorphism such that $\varphi_a(a) = 0$. (To find φ_a, consider a chart (U, φ) on M such that $a \in U$, an open ball $B^n(\varphi(a), r) \subset B^n = B^n(0,1)$ and denote $\phi_a(\cdot) := \rho^{-1} \cdot (\varphi(\cdot) - \varphi(a))$; $U_a := \varphi_\alpha^{-1}(B^n(\varphi(a), r))$). By compactness, one can find a finite subcover of M by open sets $\varphi_a^{-1}(B(0, \frac{1}{2}))$, let $a_1, \ldots, a_m$ be the corresponding points in M. We now construct the mapping

$$\Phi = (\Phi_k)_{k=1,\ldots,m} : M^n \to \mathbb{R}^{(n+1)m}$$

as follows[1]:

$$\Phi_k(x) := (\eta(\|\varphi_{a_k}\|) \cdot \varphi_{a_k}(x)\,;\, \theta(\|\varphi_{a_k}(x)\|)) \in \mathbb{R}^n \times \mathbb{R},$$

where we declare $\Phi_k(x) := 0$ for $x \notin U_{a_k}$ and $\eta, \theta \in C_0^\infty(\mathbb{R}_+; [0,1])$ are such that

- $\eta(t) = 1$ if $t \le \frac{1}{2}$; η is strictly decreasing on $[\frac{1}{2}; \frac{3}{4}]$; $\eta(t) = 0$ if $t \ge \frac{3}{4}$;
- $\theta(t) = 1$ if $t \le \frac{1}{4}$; θ is strictly decreasing on $[\frac{1}{4}; \frac{3}{4}]$; $\theta(t) = 0$ if $t \ge \frac{3}{4}$.

Let us first check that Φ is a *bijection* from M onto $\Phi(M)$. Denote $V_k := \varphi_{a_k}^{-1}(B(0, \frac{1}{2}))$, recall that the open sets V_k, $k = 1, \ldots, m$, cover M.

- If $x, y \in V_k$, then $\Phi_k(x) = \Phi_k(y)$ implies $x = y$ since the first (n-dimensional) component of Φ_k equals φ_{a_k} on V_k.
- If $x \in V_k$ but $y \notin V_k$, then the second component of $\Phi_k(x)$ is strictly greater than $\theta(\frac{1}{2})$ whilst the first component of $\Phi_k(y)$ is smaller or equal than $\theta(\frac{1}{2})$.

For simplicity (and without loss of generality) assume that $k = 1$ and note that

$$\begin{aligned} \Phi(V_1) &= \Phi(M) \cap \{y \in \mathbb{R}^{(n+1)m} : y_{n+1} > \theta(\tfrac{1}{2})\} \\ &= \Phi(M) \cap \{y \in \mathbb{R}^{(n+1)m} :\ y_1^2 + \ldots + y_n^2 < \tfrac{1}{4},\ y_{n+1} > \theta(\tfrac{1}{2})\,\} \end{aligned}$$

and, moreover, on the set $\Phi(V_1)$ all the remaining coordinates are smooth functions of $\phi_{a_k}(x)$, $k = 2, \ldots, m$, and hence smooth functions of $\phi_{a_1}(x) = (y_1, \ldots, y_n)$ since all the compositions $\phi_{a_k} \circ \phi_{a_1}^{-1}$ are smooth (and $y_{n+1} = \theta(\|\varphi_{a_1}(x)\|)$ is also a smooth function of $y_1^2 + \ldots + y_n^2 = \|\varphi_{a_1}(x)\|^2$ on $\Phi(V_1)$).

Thus, there exists a smooth function $g : B(0, \frac{1}{2}) \to \mathbb{R}^{(n+1)m-n}$ such that

$$\begin{aligned} \Phi(V_1) &= \Phi(M) \cap \big[B(0, \tfrac{1}{2}) \times \big((\theta(\tfrac{1}{2}), +\infty) \times \mathbb{R}^{(n+1)(m-1)}\big)\big] \\ &= \{(y_1, \ldots, y_n\,,\, g(y_1, \ldots, y_n)),\ (y_1, \ldots, y_n) \in B(0, \tfrac{1}{2})\,\}. \end{aligned}$$

In particular, $\Phi(M)$ is a smooth (and compact as a continuous image of a compact topological space M) manifold embedded into $\mathbb{R}^N$.

The fact that smooth manifolds M and $\Phi(M)$ are diffeomorphic is a triviality since $(y_1, \ldots, y_n) = \varphi_{a_1}(x)$ on V_1, thus there is nothing to prove if we consider the chart $(V_1, \varphi(a_1))$ on M and the corresponding chart $(\Phi(V_1); \pi_{\mathbb{R}^n})$ on $\Phi(M)$. □

[1]Compared to the mess which appeared during the lecture with the bijection property, let us simply keep the information about all $\|\varphi_{a_k}(x)\|$ as additional coordinates and embed the topological manifold M into $\mathbb{R}^{(n+1)m}$ instead of $\mathbb{R}^{nm}$.

December 09, 2020

We start by a general remark on the definitions of a (compact) smooth manifold. In the last lecture we discussed the equivalence of the three viewpoints:

(0) 'abstract' definition (charts $\varphi_\alpha : M \supset U \to \mathbb{R}^n$ in a topological space M);
(1) $M \subset \mathbb{R}^N$ is (locally) the zero set of a smooth function $f : \mathbb{R}^N \to \mathbb{R}^{N-n}$;
(2) $M \subset \mathbb{R}^N$ is locally the graph of a smooth function $g : \mathbb{R}^n \to \mathbb{R}^{N-n}$.

The viewpoint (2) is probably the most intuitive one but it is worth noting that (0)–(2) are not equivalent in the other contexts due to the additional 'rigidity' present in other classes of functions (by which one can replace $\mathbb{R}$-smooth ones) and that (2) is actually the least appropriate for such generalizations. E.g., one uses

(0) with f being a polynomial mapping in order to define *algebraic manifolds* (and, further, algebraic varieties);
(1) to define $\mathbb{C}$-*manifolds* (aka *Riemann surfaces* if $n = 1$).

In both cases, one cannot reformulate the definition via (2): in the former case the (local) solution of polynomial equations is not polynomial; in the latter there is no way to embed an abstract $\mathbb{C}$-manifold into $\mathbb{C}^N$ (as in Proposition 4.2) because of the rigidity of complex-differentiable (=holomorphic=analytic) mappings.

4.1. **Tangent space and tangent bundle of a smooth manifold.** Let M^n be a smooth $\mathbb{R}$-manifold and first assume that we view it in the sense of (1) or (2) (i.e., as a smooth manifold 'embedded into $\mathbb{R}^N$; we emphasize this viewpoint by using the notation $M_N = M^n_N$ instead of $M = M^n$). In this case we can speak about a *tangent space* to M^n_N at a point $a \in M^n_N$ by defining

$$T_a M^n_N \ := \ \mathrm{Ker}[(Df)(a)] \ = \ \{(v, [(Dg)(a_J)]v),\, v \in \mathbb{R}^J\},$$

where the first definition relies upon (1) and the second upon (2); in this approach $T_a M^n_N$ is understood as an n-dimensional subspace of $\mathbb{R}^N$.

It is easy to see that $T_a M_N$ depends on M_N only and not on the choice of f or the set of coordinates $J \subset [1, N]$, $\#J = n$ (clearly, the choice of J (locally) defines g uniquely). Indeed, for all pairs f and g one has $f(x_J, g(x_J)) = 0$ and hence the chain rule gives

$$[(Df)(a)](v; [(Dg)(a_J)]v) = 0 \ \text{ for all } v \in \mathbb{R}^J,$$

i.e., $\mathrm{Ker}[(Df)(a)] \supset \{(v, [(Dg)(a_J)]v),\, v \in \mathbb{R}^J\}$. However, the non-degeneracy condition $\mathrm{rank}[(Df)(a)] = N - n$ can be written as $\dim \mathrm{Ker}[(Df)(a)] = n$. Therefore, these two spaces are equal since $\dim\{(v, [(Dg)(a_J)]v),\, v \in \mathbb{R}^J\} = \dim \mathbb{R}^J = n$ too.

Let us now give the definition of the tangent space $T_a M$ for smooth topological manifolds, using the preceding discussion as the motivation.

- Let $M = M^n$ be a smooth *topological* manifold of dimension n and let $a \in M$. Consider the set Γ_a of all smooth curves $\gamma : [-1, 1] \to M$ such that $\gamma(0) = a$ and introduce the equivalence relation
$$\gamma \sim \gamma_1 \ \text{ if } \ (\varphi_a \circ \gamma)'(0) = (\varphi_a \circ \gamma_1)'(0)$$
in a certain (and then in all, by the chain rule) chart $U_a \ni a$.

Definition 4.3. *The* tangent space $T_a M$ at $a \in M$ *is the set of equivalence classes* $\Gamma_a / \sim$ *equipped with the vector and topological structures of* $\mathbb{R}^n$ *by* $[\gamma] \leftrightarrow (\varphi_a \circ \gamma)'(0)$. *(These structures do not depend on the choice of the chart* φ_a *due to the chain rule.)*

Assume now that $f : M^n \supset U \to M_1^{n_1}$ is a C^1-mapping between smooth manifolds (in general, of different dimensions $n \neq n_1$). The simplest way to define the derivative of f at a point $a \in U$ is to consider a local chart (U_a, φ_a) of M at a, a local chart (V_b, ψ_b) of M_1 at $b := f(a)$ and to think about the mapping

$$\psi_b \circ f \circ \varphi_a^{-1} :\ \mathbb{R}^n \supset \varphi_a(U_a) \to \varphi_b(V_b) \subset \mathbb{R}^{n_1}$$

and about its derivative at the point $\varphi_a(a)$.

- However, one can do better and define $(Df)(a)$ in a *chart-invariant* way as a linear operator

$$(Df)(a) :\ T_aM \ \to\ T_{f(a)}M_1, \qquad \Gamma_a \ni \gamma \mapsto f \circ \gamma \in \Gamma_{f(a)}. \tag{4.1}$$

Indeed, the mapping $\gamma \mapsto f \circ \gamma$ can be re-written in local charts as

$$\varphi_a \circ \gamma \ \mapsto\ (\psi_b \circ f \circ \varphi_a^{-1}) \circ (\varphi_a \circ \gamma) \ =\ \psi_b \circ f \circ\ \gamma$$

and hence the derivative $D(\psi_b \circ f \circ \varphi_a^{-1})(\varphi(a)) : \mathbb{R}^n \to \mathbb{R}^{n_1}$ can be written as the linear operator

$$(\varphi_a \circ \gamma)'(0) \mapsto [D(\psi_b \circ f \circ \varphi_a^{-1})(\varphi(a))](\varphi_a \circ \gamma)'(0) = (\psi_b \circ f \circ \gamma)'(0),$$

which also proves that (4.1) is correctly defined as a linear mapping acting from $T_aM = \Gamma_a/\sim$ to $T_bM_1 = \Gamma_b/\sim$ (and not only as a mapping from Γ_a to Γ_b).

Remark 4.1. Let us emphasize that the tangent spaces T_aM and $T_{f(a)}M_1$ depend on the point a. This does not allow one to define *higher* derivatives of smooth mappings $f : M \to M_1$ in a *chart-invariant* way: replacing f by $\psi_b \circ f \circ \varphi_a^{-1}$ we identify all tangent spaces T_xM, $x \in U_a$, with each other (and similarly for tangent spaces T_yM_1, $y \in V_b$) and this *identification* is chart-dependent. This discussion naturally leads to the course *'Géométrie Différentielle'* so we stop it here.

Instead of identifying the tangent spaces T_aM, $a \in M$, with each other, one can view the *disjoint union* of them as a smooth manifold of the twice larger dimension.

Definition 4.4. *Let M be a C^k-smooth topological manifold of dimension n. The* tangent bundle *TM of M is a C^{k-1}-smooth topological manifold of dimension $2n$ defined as follows:*

- *as a set, $TM := \bigsqcup_{a\in M} T_aM = \{(a,v) :\ a \in M,\ v \in T_aM\}$;*
- *each chart $(U_\alpha; \varphi_\alpha)$ of M defines a chart $(\mathcal{U}_\alpha; \Phi_\alpha)$ of TM, where $\mathcal{U}_\alpha := \bigsqcup_{a\in U_\alpha} T_aM$ and the mapping $\Phi_\alpha : \mathcal{U}_\alpha \to B^n \times \mathbb{R}^n$ is defined as*

$$\Phi_\alpha : (a, [\gamma_a]) \mapsto (\varphi_\alpha(a), (\varphi_\alpha \circ \gamma)'(0)), \quad \gamma_a \in \Gamma_a$$

(and the topology in TM is induced by the mappings Φ_α).

It is easy to see that thus defined TM is a Hausdorff topological space (though never compact – because of the second 'vector' component – even if M was compact) and that the charts Φ_α are C^{k-1}-compatible:

$$\Phi_\beta \circ \Phi_\alpha^{-1} : (x\,, v) \mapsto \big(\, (\varphi_\beta \circ \varphi_\alpha^{-1})(x)\,, [D(\varphi_\beta \circ \varphi_\alpha^{-1})(x)](v)\,\big).$$

Remark 4.2. It directly follows from the definition that each C^k-smooth mapping $f : M^n \supset U \to M_1^{n_1}$ gives rise to a C^{k-1}-smooth mapping $Df : TM^n \supset TU \to TM_1^{n_1}$ defined as $(Df)(a,v) := (f(a), [(Df)(a)](v))$. However, let us emphasize once again that the tangent bundles TM^n and $TM_1^{n_1}$ are smooth manifolds of dimensions $2n$ and $2n_1$, respectively, thus the 'second derivative' $DDf : TTM^n \supset TTU \to TTM_1^{n_1}$ is a much more complicated object than D^2f for $f : \mathbb{R}^n \supset U \to \mathbb{R}^{n_1}$; cf. Remark 4.1

Quasi-détour. We conclude this section by sketching a proof of (a weak form of) the *Whitney embedding theorem* that says that one can always replace an unknown (depending on the manifold under consideration) dimension N in Proposition 4.2 by $N = 2n+1$. (In fact, one can always take $N = 2n$ and actually this can be further slightly improved – using very deep techniques – unless n is a power of 2 in which case the projective space $M^n = \mathbb{R}\mathrm{P}^n$ cannot be embedded into $\mathbb{R}^{2n-1}$.)

Theorem 4.5. *Let $M = M^n$ be a compact C^k-smooth topological manifold of dimension n with $k \geq 2$. Then, there exists a smooth topological manifold M_{2n+1} embedded into $\mathbb{R}^{2n+1}$ such that M is homeomorphic and C^k-diffeomorphic to M_{2n+1}.*

Sketch of the proof. We already know from Proposition 4.2 that it is enough to consider smooth manifolds embedded into a certain space $\mathbb{R}^N$ (where $N \gg n$ depends on a manifold). Thus, it remains to explain how one can decrease this dimension to $2n+1$. (Decreasing it to $2n$ is less trivial, letting alone the further improvements.) The key idea of the proof can be formulated as follows:

- If $M_N^n \subset \mathbb{R}^N$ is a smooth manifold of dimension n embedded into $\mathbb{R}^N$ with $N \geq 2n+2$, then there exists a *direction* $h \in S^{N-1} \subset \mathbb{R}^N$ such that the orthogonal projection $\pi_{h^\perp} : \mathbb{R}^N \to \mathbb{R}^{N-1}$ along the direction h is a diffeomorphism of $M_N^n \subset \mathbb{R}^N$ and $M_{N-1}^n := \pi_{h^\perp}(M_N^n) \subset \mathbb{R}^{N-1}$.

Given a direction $h \in S^{N-1}$ (i.e., $h \in \mathbb{R}^N$ such that $\|h\| = 1$), let us discuss what can go wrong when we replace M_N by its orthogonal projection $\pi_{h^\perp} M_N$. The first (less conceptual) problem is that a pair of distinct points $x, y \in M_N$ can have the same projections, which means that

$$\pm h = H(x,y) := \frac{x-y}{\|x-y\|}, \quad (x,y) \in (M_N \times M_N)' := (M_N \times M_N) \smallsetminus \{(x,x) : x \in M_N\}.$$

To rule out this scenario, note that $(M_N \times M_N)'$ is a smooth (non-compact) manifold of dimension $2n$ and that H is a C^1 (even C^k with $k \geq 2$) function on this manifold. Then a simple lemma shows that the Hausdorff dimension of the image of H cannot be greater than $2n$. Provided that $2n < N-1$, this means that there remains plenty of directions $h \in S^{N-1}$ such that the projection along h leads to a *bijective* correspondence of M_N and $M_{N-1} := \pi_{h^\perp}(M_N)$. Since M_N is compact, the continuous bijection $\pi_{h^\perp} : M_N \to M_{N-1}$ is (automatically) a homeomorphism.

A more conceptual obstacle is that, even if M_N and M_{N-1} are homeomorphic as subsets of $\mathbb{R}^N$ and $\mathbb{R}^{N-1}$, respectively, the projection M_{N-1} is not necessary a smooth manifold embedded in $\mathbb{R}^{N-1}$ if $h \in T_a M_N \subset \mathbb{R}^N$ for a certain $a \in M_N$.

Exercise: Prove that if $h \notin T_a M_N$ then there exists an open neighborhood of the point $\pi_{h^\perp} a \in U \subset \mathbb{R}^{N-1}$ such that $M_{N-1} \cap U$ is the graph of a smooth function and the projection $\pi_{h^\perp} : M_N \to M_{N-1}$ is a local C^k-diffeomorphism near a.

It remains to find $h \in S^{N-1} \smallsetminus \bigcup_{a \in M_N} T_a M_N \subset \mathbb{R}^N$. To this end, assume that M_N is locally the graph of a smooth function $V_a \ni x_J \mapsto g(x_J) \in \mathbb{R}^{[1,N] \smallsetminus J}$. Then,

$$\textstyle\bigcup_{x_J \in V_a} T_{(x_J, g(x_J))} M_N \;=\; \big\{ \big(v, [(Dg)(x_J)]v\big) : x_J \in V_a,\ v \in \mathbb{R}^J \big\} \subset \mathbb{R}^N$$

is a C^1-smooth (actually, C^{k-1}-smooth, this is where we use the fact that $k \geq 2$ and not just $k \geq 1$) image of a $2n$-dimensional open set $V_a \times \mathbb{R}^J$ and thus its Hausdorff dimension does not exceed $2n$. Taking a union over (finitely many) charts we see that it remains plenty of directions h which can be used to pass from M_N to M_{N-1}.

Note that the second part of the proof works for all $N \geq 2n+1$. Thus, to improve $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n}$ one needs to remove possible 'non-local' intersections in M_{2n}. $\square$

December 14, 2020

5. Ordinary differential equations: basics

Let E be a Banach space (later on, we will concentrate on the case $E=\mathbb{R}^N$), $\mathcal{O}\subset\mathbb{R}\times E$ be an open set, and $f:\mathcal{O}\to E$ be a *continuous* function[2]. Given a point $(t_0,x_0)\in\mathcal{O}$, consider the *Cauchy problem*

$$u'(t)=f(t,u(t)),\quad u(t_0)=x_0. \tag{5.1}$$

A particular case when f does not depend on the time variable t and $\mathcal{O}=\mathbb{R}\times U$, where U is an open set in E, is called *autonomous* differential equations/systems. In this case, the mapping $f:E\supset U\to E$ is called a *vector-field* on U.

Definition 5.1. *A function $u\in C^1(I;E)$ is called a* local solution *of* (5.1) *if $I\ni t_0$ is an open interval (or an open ray or $\mathbb{R}$), $u(I)\subset\mathcal{O}$ and* (5.1) *holds for all $t\in I$.*

It is worth emphasizing that

- we do not specify in advance the interval I on which u is defined.

Also, note that

- higher-order differential equations $u^{(k)}(t)=f(t,u(t),\ldots,u^{(k-1)}(t))$ can be re-written as $U'(t)=F(t,U(t))$, where $U(t):=(u(t),\ldots,u^{(k-1)}(t))\in E^k$ and $F(t,v_0,\ldots,v_{k-1}):=(v_1,v_2,\ldots,v_{k-1},f(t,v_0,\ldots,v_{k-1}))$;
- the setup is invariant under the time-reversal: if $f_-(t,x)=-f(2t_0-t,x)$, then $u_-(t):=u(2t_0-t)$ is a local solution of a similar Cauchy problem with f replaced by f_- and vice versa.

Lemma 5.2. *A function u is a local solution of the Cauchy problem* (5.1) *if and only if $u\in C^0(I;E)$, $u(I)\subset\mathcal{O}$ and the following* integral equation *is fulfilled:*

$$u(t)=x_0+\int_{t_0}^t f(s,u(s))ds\quad \textit{for all } t\in I. \tag{5.2}$$

In particular, if (5.2) *holds, then $u\in C^1(I;E)$.*

Proof. This is a direct corollary of the fundamental theorem of calculus. □

Remark 5.1. One can also consider differential equations on smooth manifolds. In this case, f should be thought of as a function $f:\mathbb{R}\times M\supset\mathcal{O}\to TM$ (or as $f:M\supset U\to TM$ for autonomous equations) such that $f(t,x)\in T_xM$ and the equation (5.1) should be, as usual, understood via local charts φ_α of M as

$$(\varphi_\alpha\circ u)'(t)=(\varphi_\alpha\circ\gamma^{(u(t))})'(0),\quad\text{where}\quad[\gamma^{(u(t))}]=f(t,u(t))$$

is an equivalence class of smooth curves $\gamma^{(u(t))}:(-1,1)\to M$ passing through the point $u(t)=\gamma^{(u(t))}(0)$. Clearly, this differential equation is chart-independent: as usual, if one replaces a local chart φ_α by another one φ_β, this simply results in applying the invertible linear operator $D(\varphi_\beta\circ\varphi_\alpha^{-1})(\varphi_\alpha(u(t)))$ to both sides.

There are several basic questions on the Cauchy problem (5.1):

[2] It is worth noting that a continuous function is always locally bounded: for each $(t_0,x_0)\in\mathcal{O}$ there exist small enough $\tau,\rho>0$ such that $\sup_{(t,x)\in\overline{B}(t_0,\tau)\times\overline{B}(x_0,\rho)}\|f(t,x)\|<+\infty$. However, if E is infinite-dimensional, then f can be unbounded on larger sets $\overline{B}(t_0,T)\times\overline{B}(x_0,R)\subset\mathcal{O}$ as the closed unit ball in E is not compact.

(1) **Local existence:** under which conditions on f can one guarantee that a local solution exists? We do not discuss this in detail, still let us mention the following results:
   - **Peano's theorem for $E = \mathbb{R}^N$.** *For finite-dimensional spaces E, the continuity assumption $f \in C(\mathcal{O}; E)$ is already sufficient for the existence of local solutions.*
   - However, this is not true for infinite-dimensional Banach spaces E: it can[3] happen that no local solution of (5.1) exists. The reason is that all the proof of the classical Peano theorem relies on a certain compactness argument, which fails in infinite-dimensional spaces unless additional assumptions on f are imposed (e.g., Peano theorem holds provided that f is a *compact mapping*, i.e., that it sends closed balls in $\mathcal{O}$ into pre-compact subsets of E); see a *détour* after Theorem 5.3.

(2) **Local uniqueness:** under which assumptions on f can one guarantee that a solution of the Cauchy problem (5.1) is locally unique? (More precisely, the local uniqueness means that if $u_{1,2}$ are two solutions of (5.1) defined on intervals $I_{1,2}$, respectively, then there exists an open interval $t_0 \in I \subset I_1 \cap I_2$ such that $u_1(t) = u_2(t)$ for all $t \in I$).

   A classical theorem (which is usually attributed to Picard (and Lindelöf) in the English-German-Polish-Russian tradition, and to Cauchy and Lipschitz in the French one) is that the **local Lipschitness of f in x:**

$$\|f(t,x) - f(t,y)\| \le C_{\tau,\rho} \cdot \|x - y\| \quad \text{for all} \quad t \in \overline{B}(t_0, \tau),\ x, y \in \overline{B}(x_0, \rho) \qquad (5.3)$$

   (together with the continuity of f) is sufficient for *both* the local existence and uniqueness; see Theorem 5.3 below.

(3) **Maximal solutions,** more precisely their behavior near the endpoints of the maximal existence interval. To give a definition, *assume that the local uniqueness property holds at all points of $\mathcal{O}$.* Then, it is easy to see that
   - if $u_{1,2}$ are two local solutions of the same Cauchy problem (5.1), then $u_1(t) = u_2(t)$ for all $t \in I_1 \cap I_2$ (and not only for $t \in I \subset I_1 \cap I_2$).
     [*Proof.* the set $\{t \in I_1 \cap I_2 : u_1(t) = u_2(t)\}$ is obviously closed in $I_1 \cap I_2$ but is also open as we can apply the local uniqueness property for the Cauchy problem with the initial data (t, x), $x := u_1(t) = u_2(t)$.]

   This observation (provided that the local uniqueness holds everywhere in $\mathcal{O}$) allows one to define
   - the *maximal existence interval* $I_{\max} = I_{\max}(t_0, x_0)$ of a local solution of (5.1) simply as the union of all intervals I_β on which all possible local solutions u_β of (5.1) are defined;
   - and the *maximal solution* $u_{\max} \in C^1(I_{\max}; E)$ of (5.1) by setting $u_(t) := u_\beta(t)$ for $t \in I_\beta$; recall that all these local solutions agree with each other provided we have the local uniqueness property.

   If $\mathcal{O} = I \times U$ and especially for autonomous equations (in which case $\mathcal{O} = \mathbb{R} \times U$), it is natural to ask what can prevent a maximal solution to be defined on the whole I; e.g., how $u(t)$ behaves if $t \to \sup I_{\max} < \sup I$.

[3]The first example of such a differential equation was given in 1949 by Jean Dieudonné in his short note *Deux exemples singuliers d'equations différetielles* (available online).

Before discussing general theorems, it is instructive to consider the following toy example: an autonomous differential equation in the one-dimensional space $E = \mathbb{R}$

$$u' = |u|^\alpha, \tag{5.4}$$

where $\alpha \neq 0$ is a fixed parameter. Note that $u(t) \equiv 0$ is always a solution of this equation provided that $\alpha > 0$.

(a) Let $\alpha = 1$. Then the solutions of (5.4) are Ce^t and $-Ce^{-t}$, where $C > 0$ (the sign appears due to the absolute value in the right-hand side of (5.4)). This is the best possible situation: we have the local existence and uniqueness at all points, and $I_{\max} = \mathbb{R}$ for all solutions.

(b) Let $\alpha > 1$. Then the non-zero solution of (5.4) read as

$$u(t) = |(\alpha-1)(T-t)|^{-1/(\alpha-1)} \cdot \operatorname{sign}(T-t), \quad \text{where} \ \ T = T(t_0, x_0) \in \mathbb{R}.$$

In this case we still have the local existence and uniqueness at all points but $I_{\max} = (-\infty, T)$ for solutions started at $x_0 > 0$ and $I_{\max} = (T, +\infty)$ for those with $x_0 < 0$.

(c) Let $0 < \alpha < 1$. Similarly to the previous case, local solutions of (5.4) with $x_0 \neq 0$ are

$$u(t) = |(1-\alpha)(t-T)|^{1/(1-\alpha)} \cdot \operatorname{sign}(t-T), \quad \text{where} \ \ T = T(t_0, x_0) \in \mathbb{R}.$$

However, now *there is no local uniqueness property if* $x_0 = 0$. In this case each local solution can be extended to a solution defined on $I = \mathbb{R}$ but we prefer not to speak about $I_{\max}$ as this extension is not unique[4].

Remark 5.2. Note that the right-hand side $f(x) = |x|^\alpha$ of (5.4) is not Lipschitz at the point $x_0 = 0$ but is still α-Hölder, where α can be arbitrary close to 1. This example illustrates the fact that the Lipschitzness of f (in the space variable x) is really crucial for the local uniqueness.

(d) Finally, let $\alpha < 0$, in this case the right-hand side $f : U \to \mathbb{R}$ of (5.4) is defined only on $U = \mathbb{R}_+ \cup \mathbb{R}_-$, the solutions are as in (c) and $I_{\max} = (T, +\infty)$ or $I_{\max} = (-\infty, T)$ depending on the sign of x_0. This situation is very similar to (b) except that instead of the 'blow-up' $u(t) \to \pm\infty$ as $t \to T$ we now have $u(t) \to 0 \notin U$ as $t \to T$.

We now move back to a general setup. Assume that $(t_0, x_0) \in \mathcal{O}$ and that f is continuous (and hence locally bounded) and locally Lipschitz in x near the point (t_0, x_0), namely that for certain $\tau, \rho > 0$ such that $\overline{B}(t_0, \tau) \times \overline{B}(x_0, \rho) \subset \mathcal{O}$ we have

$$\|f(t,x)\| \le M_{\tau,\rho} \qquad \text{for all } (t,x) \in \overline{B}(t_0,\tau) \times \overline{B}(x_0,\rho);$$
$$\|f(t,x) - f(t,y)\| \le C_{\tau,\rho}\|x-y\| \quad \text{for all } t \in \overline{B}(t_0,\tau) \text{ and } x, y \in \overline{B}(x_0,\rho).$$

Theorem 5.3 (Picard(–Lindelöf)/Cauchy–Lipschitz). *Under the above assumptions, there exists* $\varepsilon = \varepsilon(\tau, \rho, M_{\tau,\rho}, C_{\tau,\rho}) > 0$ *such that the Cauchy problem* (5.1) *has(!) a unique(!) solution on an interval* $I = I_\varepsilon(t_0) := (t_0 - \varepsilon, t_0 + \varepsilon)$.

[4]Even without the local uniqueness one can define *maximal* solutions $u_{\max}$ of (5.1) by requiring that there is no other solution u of (5.1) defined on a strictly larger interval $I \supsetneq I_{\mathrm{Imax}}$ such that $u(t) = u_{\max}(t)$ for all $t \in I_{\max}$. However, if a local solution can be extended to a maximal one in a non-unique way, then the corresponding intervals $I_{\max}$ can depend on the choice of $u_{\max}$.

Proof. Assume that $\varepsilon \le \tau$ is chosen so that $\varepsilon \cdot M_{\tau,\rho} \le \frac{1}{2}\rho$ and $\varepsilon \cdot C_{\tau,\rho} \le \frac{1}{2}$, and consider a (non-linear) mapping

$$A : C(\overline{I}; \overline{B}(x_0, \tfrac{1}{2}\rho)) \ni u \mapsto Au \in C(\overline{I}; E), \quad (Au)(t) := x_0 + \int_{t_0}^{t} f(s, u(s))ds.$$

Note that A maps $C(\overline{I}; \overline{B}(x_0, \frac{1}{2}\rho))$ into itself since

$$\|(Au)(t)\| \ \le\ |t - t_0| \cdot M_{\tau,\rho} \le\ \tfrac{1}{2}\rho \ \text{ for all } t \in \overline{I}.$$

Moreover, A is a $\frac{1}{2}$-Lipschitz contraction since

$$\begin{aligned}\|(Au)(t) - (Av)(t)\| \ &\le\ (t - t_0) \cdot \int_{t_0}^{t} \|f(t, u(s)) - f(t, v(s))\| dt \\ &\le\ |t - t_0| C_{\tau,\rho} \cdot \sup\nolimits_{s\in\overline{I}} \|u(s) - v(s)\| \ \le\ \tfrac{1}{2}\|u - v\| \text{ for all } t \in \overline{I}.\end{aligned}$$

Therefore, the fixed point principle applies and there exists a unique function $u_0 \in C(\overline{I}; \overline{B}(x_0, \frac{1}{2}\rho))$ such that $Au_0 = u_0$, which is nothing but the integral reformulation (5.2) of the Cauchy problem (5.1).

Concerning the uniqueness, the fixed point argument given above, a priori, does not forbid the existence of another solution with $\|u(t)\| > \frac{1}{2}\rho$ for a certain $t \in I_\varepsilon(t_0)$. However, it implies the *local* uniqueness: for each such a solution there exists an interval $I \ni t_0$ such that $u(t) = u_0(t)$ for all $t \in I$ (since $\|u(t) - x_0\| \le \frac{1}{2}\rho$ for t close enough to t_0). Then, the uniqueness of the solution on the whole interval $I_\varepsilon(t_0)$ follows by the same argument as in the discussion of maximal solutions: the set $\{t \in I_\varepsilon(t_0) : u(t) = u_0(t)\}$ is both closed an open. □

Détour[5]. If the Lipschitness assumption on f is dropped, then one can still use the same idea in order to prove the *existence* of a local solution of the Cauchy problem (5.1) relying upon another fixed point theorem, e.g., upon

Schauder's fixed point theorem. *If $\overline{B} \subset E$ is a convex closed subset of a Banach space and $A : \overline{B} \to \overline{B}$ is a continuous mapping such that $A(\overline{B})$ is pre-compact in E, then A has a fixed point, i.e., there exists $u \in \overline{B}$ such that $A(u) = u$.*

Note that the Schauder fixed point theorem, in particular, generalizes the Brouwer fixed point theorem in which $\overline{B} = \overline{B}^n \subset \mathbb{R}^n$ is a closed *finite*-dimensional ball and thus no additional compactness assumption is required.

To apply this theorem to the existence of solutions of the Cauchy problem (5.1) with continuous f acting in a *finite*-dimensional space (this is the classical Peano theorem mentioned at the beginning of this section), one should prove that the image of the mapping A is compact in the space $C(\overline{I}; \overline{B}(x_0, \frac{1}{2}\rho))$. This is a more-or-less straightforward corollary of the Arzelá-Ascoli theorem since the functions Au are actually, by the definition of A, uniformly Lipschitz in the time variable t.

However, in infinite-dimensional Banach spaces E such a proof (and the local existence of solutions of the Cauchy problem (5.1) itself) *fails.* The reason is that, though the functions Au are still uniformly Lipschitz, the set of values $\{(Au)(t) \,|\, u \in C(\overline{I}; \overline{B}(x_0, \frac{1}{2}\rho))\}$ at a fixed point $t \neq t_0$ is not necessarily compact in E (in the finite-dimensional setup, this is a triviality since these values are uniformly bounded). Therefore, the Arzelá–Ascoli theorem cannot be applied without additional assumptions on f besides its continuity.

[5] This was only very briefly mentioned during the lecture. Note that this type of ideas is extremely important when proving the existence of solutions of *équations aux dérivées partielles*.

December 16, 2020

6. Global solutions and Gronwall's lemma

We now move to the discussion of the behavior of maximal solutions near the end-points of their maximal existence intervals. For simplicity, let $\mathcal{O} = I \times U$, where $U \subset E$ is an open set and $I \subset \mathbb{R}$ is an open interval (recall that we have, trivially, $I = \mathbb{R}$ for autonomous equations as f does not depend on t). In this case, a maximal solution $u \in C^1(I_{\max}; U)$ of the differential equation $u'(t) = f(t, u(t))$ is called *global* if $I_{\max} = I$.

Proposition 6.1. *Let $f : I \times U \to E$ be continuous and locally Lipschitz in x. Assume that u is a maximal solution and $T_{\max} := \sup I_{\max} < \sup I$. Then, for each compact $K \subset U$ there exists $T_K < T_{\max}$ such that $u(t) \notin K$ for all $t > T_K$. In other words, a maximal solution with $T_{\max} < \sup I$ has to leave all compact subsets of U as $t \to T_{\max}$. (By time-reversal, the same holds if $T_{\min} := \inf I_{\max} > \inf I$.)*

Proof. On the contrary, assume that there exists a sequence of times $t_n \uparrow T_{\max}$ such that $x_n := u(t_n) \in K$. Using the compactness of K and taking a subsequence, we can assume that $x_n \to x_* \in K \subset U$ as $n \to \infty$. The function f is continuous and locally Lipschitz in x in a vicinity of the point $(T_{\max}, x_*) \in I \times U$. It follows from Theorem 5.3 that there exists $\tau, \rho, \varepsilon > 0$ such that the Cauchy problem (5.1) admits a local solution for each initial data $(t_n, x_n) \in \overline{B}(T_{\max}, \frac{1}{2}\tau) \times \overline{B}(x_*, \frac{1}{2}\rho)$ and that this solution exists for at least time $\varepsilon > 0$ which does not depend on (t_n, x_n). This leads to a contradiction provided that n is chosen large enough so that $T_{\max} - t_n < \varepsilon$. □

Let us now assume that $U = E$. If E is finite-dimensional and $T_{\max} < \sup I$, then the fact that a maximal solution u exists from all compacts $K \subset E$ simply means that $\|u(t)\| \to \infty$ as $t \to T_{\max}$. However, if E is infinite-dimensional, then the behavior can be more complicated unless we impose more assumptions on f. In particular, it can[6] happen that $\|u(t)\|$ remains bounded as $t \to T_{\max}$. Loosely speaking, this is related to the fact that the continuity and local Lipschitzness (in x) of the function f does not imply that this function is uniformly Lipschitz (or even bounded) on *bounded* closed subsets of $I \subset E$, which are not compact anymore.

The following theorem gives a simple sufficient condition under which all solutions of (5.1) are global. Actually, the idea of its proof and 'technical' Lemma 6.3 are more important than the result itself.

Theorem 6.2. *Let $f : I \times E \to E$ be a* globally *Lipschitz function, i.e. assume that $\|f(t,x) - f(t,y)\| \le C\|x - y\|$ for all $t \in I$ and all $x, y \in E$. Then, all maximal solutions of the differential equation $u'(t) = f(t, u(t))$ are global.*

Trivially, one can generalize this result to the case when the Lipschitz constant $C = C(t)$ depends on t in, e.g., a continuous way so that $\max_{t \in \overline{J}} C(t) < +\infty$ for all closed segments $\overline{J} \subset I$. Indeed, in this case Theorem 6.2 implies that $I_{\max} \supset J$ for all closed segments $\overline{J}$ and hence $I_{\max} = I$.

The proof of Theorem 6.2 is based upon Lemma 6.3, known as Gronwall's lemma. Before giving, a (stronger) 'integral' version that we use below, let us first formulate its (weaker) 'differential' variant.

[6]E.g., see the note *Deux exemples singuliers d'equations différetielles,* Jean Dieudonné (1949).

- Let a function $w \in C^1([0,T])$ and constants $a > 0$ and $b \in \mathbb{R}$ be such that the differential *inequality* $w'(s) \le aw(s) + b$ holds for all $s \in [0,T]$. Then, $w(t) \le w_0(t)$ for all $t \in [0,T]$, where

$$w_0(t) \ := \ e^{at} \cdot (w(0) + \tfrac{b}{a}) - \tfrac{b}{a}$$

solves the *equation* $w_0'(s) = aw_0(s) + b$ with the initial data $w_0(0) = w(0)$.

By integrating the assumption $w'(s) \le aw(s) + b$ we see that

$$w(t) \ \le \ w(0) + \int_0^t (aw(s) + b)ds \ \text{ for all } t \in [0,T]. \tag{6.1}$$

It turns out that this (weaker) inequality is sufficient for the same conclusion.

Lemma 6.3 (Gronwall). *Let $a > 0$, $b \in \mathbb{R}$, and $w \in C([0,T];\mathbb{R})$ be such that the inequality* (6.1) *holds on $[0,T]$. Then, $w(t) + \frac{b}{a} \le e^{at} \cdot (w(0) + \frac{b}{a})$ for all $t \in [0,T]$.*

Proof. For $t \in [0,T]$, denote

$$v(t) \ := \ e^{-at} \cdot \left(w(0) + \frac{b}{a} + \int_0^t (aw(s)+b)ds \right).$$

It is easy to see that the condition (6.1) can be written as

$$\begin{aligned} e^{at} \cdot v'(t) \ &= \ -a \cdot \left(w(0) + \frac{b}{a} + \int_0^t (aw(s)+b)ds \right) + (aw(t) + b) \\ &= \ a \cdot \left(w(t) - w(0) - \int_0^t (aw(s)+b)ds \right) \ \le \ 0. \end{aligned}$$

Therefore, we have

$$w(t) + \tfrac{b}{a} \ \overset{(6.1)}{\le} \ e^{at} \cdot v(t) \ \le \ e^{at} \cdot v(0) \ = \ e^{at} \cdot (w(0) + \tfrac{b}{a}),$$

as claimed. □

Proof of Theorem 6.2. Let $u \in C(I_{\max}; E)$ be the maximal solution of the Cauchy problem (5.1) with the initial data $u(t_0) = x_0$. Assume, by contradiction, that $T_{\max} := \sup I_{\max} < \sup I$ and denote (see also Remark 6.1 below)

$$w(t) := \|u(t) - x_0\| \quad \text{for} \quad t \in [t_0, T_{\max}).$$

Due to the global Lipschitzness of the function f we have

$$\begin{aligned} \|u'(t)\| \ = \ \|f(t,u(t))\| \ &\le \ \|f(t,u(t)) - f(t,x_0)\| + \|f(t,x_0)\| \\ &\le \ C \cdot \|u(t) - x_0\| + \|f(t,x_0)\| \\ &\le \ Cw(t) + M, \quad \text{where} \ \ M := \max\nolimits_{t \in [t_0, T_{\max}]} \|f(t,x_0)\|, \end{aligned}$$

note that $M < +\infty$ due to the continuity of f since the second argument of $f(t,x_0)$ does not change. It is not hard to deduce from this inequality that

$$w(t) - w(t_0) \le \int_{t_0}^t (Cw(s) + M)ds \quad \text{for all } t \in [t_0, T_{\max}). \tag{6.2}$$

Loosely speaking, this corresponds to saying that $w'(t) \le \|u'(t)\|$; however a technical problem is that the function $w(t)$ is not necessarily differentiable. To be on a safe side, one can do the following: for each partition $t_0 = s_0 < s_1 < \ldots < s_N = t$ of the segment $[t_0, t]$ note that

$$\begin{aligned} w(s_{k+1}) - w(s_k) &= \|u(s_{k+1}) - x_0\| - \|u(s_k) - x_0\| \\ &\le \|u(s_{k+1}) - u(s_k)\| \le \max_{s\in[s_k,s_{k+1}]} \|u'(s)\| \cdot (s_{k+1} - s_k) \\ &\qquad\qquad \le \max_{s\in[s_k,s_{k+1}]}(Cw(s)+M) \cdot (s_{k+1} - s_k), \end{aligned}$$

where we use the 'bounded increments' Lemma 1.8 in the second line. Therefore,

$$w(t) - w(t_0) \le \sum_{k=0}^{N-1} \max_{s\in[s_k,s_{k+1}]}(Cw(s)+M) \cdot (s_{k+1} - s_k)$$

and refining the partition s_k one gets the Riemann integral in the right-hand side.

Gronwall's lemma applied to the inequality (6.2) gives the estimate

$$w(t) \le (M/C) \cdot \big(e^{C(t-t_0)} - 1\big) \quad \text{for all} \quad t \in [t_0, T_{\max}).$$

In particular, the norm $\|u(t)\|$ remains bounded as $t \to T_{\max}$. This already concludes the proof in the finite-dimensional case $E = \mathbb{R}^N$ as in this case we should have $\|u(t)\| \to \infty$ as $t \to T_{\max} < \sup I_{\max}$ due to Proposition 6.1.

Even if E is infinite-dimensional, the global Lipschitzness of f implies that the function

$$\|f(t, x)\| \le C(\|x - u(t)\| + w(t)) + M$$

also remains uniformly bounded in a (fixed size) vicinity of the trajectory $(t, u(t))$ as $t \to T_{\max}$ and thus we have a contradiction with Theorem 5.3: for each $t < T_{\max}$ the maximal solution $u(t)$ admits a continuation on the interval $I_\varepsilon(t) = (t-\varepsilon, t+\varepsilon)$, where ε does not depend on $t \to T_{\max}$. □

Remark 6.1. In the case $E = \mathbb{R}^n$ (or in a Hilbert space), there is a standard trick to avoid the technical discussion related to a possible non-smoothness of the function $\|u(t) - x_0\|$ and to simplify the proof. To this end, consider the function

$$w(t) := \|u(t) - x_0\|^2, \quad t \in [t_0, T_{\max})$$

instead of $\|u(t) - x_0\|$. Note that this function is differentiable and that

$$w'(t) = 2\langle u'(t), u(t) - x_0\rangle \le 2\|u'(t)\| \cdot \|u(t) - x_0\| = 2\|u'(t)\| \cdot w(t).$$

and hence

$$w'(t) \le 2(C\|u(t) - x_0\| + M) \cdot \|u(t) - x_0\| \le (2C + 1) \cdot w(t) + M^2$$

(because of the Cauchy–Schwarz inequality applied to the term $2M\|u(t) - x_0\|$). The rest of the proof goes as above by applying the Gronwall lemma to the last inequality instead of (6.2).

We now come back to a general situation $f : \mathbb{R} \times E \supset \mathcal{O} \to E$. Another useful corollary of Lemma 6.3 is the following lemma, which claims the *stability* of solutions with respect to the initial data.

Lemma 6.4. *Let* $(t_0, x_1), (t_0, x_2) \in \mathcal{O}$ *and* $u_{1,2} : I_{1,2} \to E$ *be solutions of the Cauchy problem* (5.1) *with initial data* $u_{1,2}(t_0) = x_{1,2}$*. If*

$$\|f(s, u_2(s)) - f(s, u_1(s))\| \le C\|u_2(s) - u_1(s)\| \tag{6.3}$$

for all $s \in I_1 \cap I_2$*, then* $\|u_2(t) - u_1(t)\| \le e^{C|t-t_0|} \cdot \|x_2 - x_1\|$ *for all* $t \in I_1 \cap I_2$.

Proof. Denote $w(t) := \|u_2(t) - u_1(t)\|$. As in the proof of Theorem 6.2 in what concerns technical details, we have

$$w(t) - w(0) \;\le\; \int_{t_0}^{t} \|u_2'(s) - u_1'(s)\|ds \;\le\; \int_{t_0}^{t} Cw(s)ds \quad \text{for} \quad t > t_0\,.$$

Therefore, $w(t) \le e^{C(t-t_0)} \cdot w(0)$ due to the Gronwall lemma. A similar estimate for $t < t_0$ follows by the time-reversal. □

We will continue discussing differential equations on January 04, 06, 11 and 13. Merry Christmas, Happy New Year and stay safe!

January 04, 2021

7. Dependence on the initial data

From now onwards assume that

$$\begin{array}{l}\text{a continuous function } f : \mathbb{R} \times E \supset \mathcal{O} \to E \text{ is bounded and} \\ \text{uniformly Lipschitz in } x \text{ on all } \textit{bounded closed} \text{ sets } F \subset \mathcal{O}.\end{array} \tag{7.1}$$

In particular, this holds[7] true provided that E is finite-dimensional and f is continuous and locally Lipschitz in x (i.e., under usual assumptions required for the local existence and uniqueness of solutions). However, if E is infinite-dimensional, then bounded closed sets are not compact and one can view (7.1) as an additional 'regularity-type' assumption on the right-hand side of the differential equation (5.1).

Given $t_0 \in I$, let

- $I_{\max}(t_0, x_0)$ denote the maximal existence interval of the solution of (5.1);
- $\mathcal{D}_{t_0} := \bigcup_{x \in E : (t_0, x) \in \mathcal{O}} (I_{\max}(t_0, x) \times \{x\}) \subset \mathbb{R} \times E$;
- a mapping $\varphi_{t_0} : \mathcal{D}_{t_0} \to E$, sometimes called the *flow* of the differential equation $u'(t) = f(t, u(t))$, be defined as $\varphi_{t_0}(t, x) := u_{(t_0,x)}(t)$, where $u_{(t_0,x)}$ is the solution of the Cauchy problem with the initial data $u_{(t_0,x)}(t_0) = x$. For shortness, we will also use the notation $\varphi_{t_0}^t(x) := \varphi_{t_0}(t, x)$.

For autonomous differential equations $u'(t) = f(u(t))$ the dependence of the flow φ_{t_0} on t_0 is marginal: $\varphi_{t_0}(t, x) = \varphi^{t-t_0}(x)$, where $\varphi^s(x) := \varphi_0^s(x) = \varphi_0(s, x)$.

Example. Consider an autonomous equation $u'(t) = (u(t))^2 - 1$ in $E = \mathbb{R}$. Then,

- ∘ $u(t) \equiv \pm 1$ are constant solutions;
- ∘ if $x_0 \in [-1, 1]$, then, due to the local uniqueness, the solution cannot cross the lines ± 1, thus it is global, i.e., exists for all $t \in \mathbb{R}$;
- ∘ if $x_0 > 1$ (similarly, if $x_0 < -1$), then the solution blows up in a finite time;
- ∘ in fact, all solutions of this equation can be written explicitly (exercise) as $u(t) = (1 + ce^{2t})/(1 - ce^{2t})$, where $c = (u(0) - 1)/(u(0) + 1) \in \mathbb{R} \cup \{\infty\}$. This means that

$$\varphi^t(x) \;=\; \frac{x - \tanh t}{1 - x \tanh t}\,, \qquad \mathcal{D}_0 = \{(t, x) \in \mathbb{R}^2 : x \tanh t < 1\}.$$

---

[7]Indeed, if f is not Lipschitz in x on a *compact* set $F \subset \mathcal{O}$, then one can find two sequences of points $(t_n, x_n), (t_n, y_n) \in F$ such that $\|f(t_n, x_n) - f(t_n, y_n)\|/\|x_n - y_n\| \to \infty$ as $n \to \infty$. Passing to a subsequence we can assume that $t_n \to t_*$, $x_n \to x_*$ and $y_n \to y_*$ as $n \to \infty$, which directly leads to a contradiction in both cases $x_* \ne y_*$ (trivially) and $x_* = y_*$ (because of the local Lipschitzness of f in x near the point $(t_*, x_*) \in F \subset \mathcal{O}$).

Proposition 7.1. *Under the 'usual' assumptions* (7.1), *the following is fulfilled: (i) the set $\mathcal{D}_{t_0} \subset \mathbb{R}\times E$ is open and (ii) the mapping φ_{t_0} is locally Lipschitz on $\mathcal{D}_{t_0}$.*

Proof. (i) Let $(t_0,x_0)\in\mathcal{O}$ and $[t_0,t_1]\subset I\max(t_0,x_0)$; the case $t_1<t_0$ is similar. We need to prove that $[t_0,t_1]\in I_{\max}(t_0,x)$ for all x sufficiently close to x_0.

Let $u_0(s):=\varphi_{t_0}(s,x_0)$ be the solution of the Cauchy problem with $u_0(t_0)=x_0$. For each $t\in[t_0,t_1]$ there exists $\rho(t)>0$ such that $\overline{B}((t,u_0(t));4\rho(t))\subset\mathcal{O}$, where $\overline{B}$ stands for the closed ball in the *Cartesian product* $\mathbb{R}\times E$ equipped with the norm $\|(s,w)-(t,u)\|:=|s-t|+\|w-u\|$. The trajectory $\{(s,u_0(s))\}_{s\in[t_0,t_1]}$ is a continuous image of a compact and so is compact. Hence, we can find a finite cover

$$\{(s,u_0(s))\}_{s\in[t_0,t_1]} \ \subset\ \textstyle\bigcup_{k=1,\dots,N} B((s_k,u_0(s_k));\rho(s_k)),\quad \text{where}\ \ s_k\in[t_0,t_1].$$

Define a closed set $T\subset\mathcal{O}$ (a 'tube' around the trajectory $(s,u_0(s))$) by

$$\mathrm{T}\ :=\ \{(s,w):\ s\in[t_0,t_1],\ \|w-u_0(s)\|\le r\},\qquad r:=\min\nolimits_{k=1,\dots,N}\rho(s_k).$$

Since $\|(s,w)-(s_k,u_0(s_k))\|\le\|w-u(s)\|+\|(s,u(s))-(s_k,u(s_k))\|$ we have

$$\mathrm{T}\ \subset\ \textstyle\bigcup_{k=1\dots,N} B((s_k,u_0(s_k));2\rho(s_k)).$$

Assume now that $\|x-x_0\|\le r$ is such that $t_1<T_{\max}(t_0,x):=\sup I_{\max}(t_0,x)$. Since the function f is bounded and uniformly Lipschitz in x on a bigger set

$$F\ :=\ \bigcup_{k=1\dots,N}\overline{B}((s_k,u_0(s_k));4\rho(s_k))\ \supset\ \bigcup_{(s,w)\in\mathrm{T}}\overline{B}((s,w);2r),$$

the solution should exit the tube T <u>strictly</u> <u>before</u> then it stops existing:

$$(s,u(s))\notin\mathrm{T}\ \ \text{for a certain}\ s\in(t_0,T_{\max}(t_0,x)). \tag{7.2}$$

(Otherwise, there is a contradiction with the local existence: if $(s,u(s))\in\mathrm{T}$, then $I_{\max}(t_0,x)\supset[s,s+\delta)$, where $\delta>0$ does no depend on $s\to T_{\max}(x)$.)

Finally, let $\|f(s,w_2)-f(s,w_1)\|\le C\|w_2-w_1\|$ for $(s,w_1),(s,w_2)\in\mathrm{T}$ and

$$\|x-x_0\|\ \le\ \varepsilon:=re^{-C(t_1-t_0)}.$$

We now claim that $t_1<T_{\max}(t_0,x)$, i.e., that the solution $u(t):=\varphi_{t_0}(t,x)$ of the Cauchy problem with the initial data $u(t_0)=x$ exists for all $s\in[t_0,t_1]$. Indeed, if

$$\inf\{s\in[t_0,T_{\max}(t_0,x)):\ (7.2)\ \text{holds}\}\ =:\ t_{\mathrm{exit}}\ <\ t_1,$$

then Lemma 6.4 implies that

$$\|u(t_{\mathrm{exit}})-u_0(t_{\mathrm{exit}})\|\ <\ e^{C(t_1-t_0)}\cdot\|u(t_0)-u_0(t_0)\|\ \le\ e^{C(t_1-t_0)}\cdot\varepsilon\ =\ r,$$

which contradict to the definition of t_{exit}. Therefore, we have $t_1\le t_{\mathrm{exit}}<T_{\max}(t_0,x)$.
(ii) Consider a point $(t,x)\in\mathcal{D}_{t_0}$ and let $t<t_1<\sup I_{\max}(t_0,x)$. Repeating the arguments given above, we see that $\|\varphi_{t_0}(t,y)-\varphi_{t_0}(t,x)\|\le e^{C(t_1-t_0)}\|y-x\|$ provided that $\|y-x\|\le\varepsilon(x)$; in other words the flow φ_{t_0} is Lipschitz in x near the point (t,x). The uniform Lipschitzness of $\varphi_{t_0}(t,y)$ in t trivially follows from the local boundedness of f, which gives $\|\varphi_{t_0}(t',y)-\varphi_{t_0}(t,y)\|\le M\|t'-t\|$ for all t' close enough to t and all y such that $\|y-x\|\le\varepsilon(x)$, where M denotes the maximum of f on an appropriate closed bounded subset of $\mathcal{O}$. □

Assume now that $f(t,x)$ is *differentiable* in x and, similarly to (7.1), that

both mappings $f:\mathbb{R}\times E\supset\mathcal{O}\to E$ and $D_xf:\mathcal{O}\to\mathcal{L}(E)$ are continuous and bounded on all *bounded closed* sets $F\subset\mathcal{O}$. (7.3)

(trivially, if E is finite-dimensional, then the continuity of f and D_xf is enough).

Theorem 7.2. *Under assumption* (7.3) *we have* $\varphi_{t_0} \in C^1(\mathcal{D}_{t_0}; E)$*. The derivative* $\Phi_{(t_0,x_0)}(t) := D_x\varphi_{t_0}(t, x_0) \in \mathcal{L}(E)$ *solves the linear differential equation*

$$\Phi'(t) \ = \ [D_x f](t, \varphi_{t_0}(t, x_0)) \circ \Phi(t) \tag{7.4}$$

with initial data $\Phi(t_0) = \mathrm{Id}$ *(recall that* $\frac{\partial}{\partial t}\varphi_{t_0}(t, x) = f(t, \varphi_{t_0}(t, x))$ *by definition).*

To prove this theorem we need first to discuss basics of linear differential equations, this is done in Section 8. However, let us first make two comments:

Remark 7.1. (i) The equation (7.4) can be formally derived as follows:

$$\begin{aligned}\tfrac{\partial}{\partial t}[D_x\varphi_{t_0}(t, x_0)] \ &\overset{[???]}{=} \ D_x[\tfrac{\partial}{\partial t}\varphi_{t_0}(t, x_0)] \\ &= \ D_x[f(t, \varphi_{t_0}(t, x_0))] \ = \ [D_x f](t, \varphi_{t_0}(t, x_0)) \circ D_x\varphi_{t_0}(t, x_0).\end{aligned}$$

(Note that, by definition, $\varphi_{t_0}(t_0, x) = x$ and hence $D_x\varphi_{t_0}(t_0, x) = \mathrm{Id}$.) Justifying this formal computation is not straightforward. In fact, the proof of Theorem 7.2 given below goes in a different way and gives (7.4) directly.

(ii) Similarly, if the function f is n times continuously differentiable in x, then so is the flow φ_{t_0}. One can prove this statement by iteratively applying Theorem 7.2 to the derivatives $D_x^k\varphi_{t_0}(t, x)$; we will not discuss technical details in these lectures.

8. Linear differential equations and Duhamel's principle

Let us consider a linear differential equation

$$u'(t) = A(t)u(t) + b(t), \quad t \in I \subset \mathbb{R}, \tag{8.1}$$

where $A \in C(I; \mathcal{L}(E))$ and $b \in C(I; E)$. The right-hand side is a globally Lipschitz function of $u(t)$. Therefore, all maximal solutions of the equation (8.1) are global (i.e., exist on the whole interval I) due to Theorem 6.2.

Homogeneous case ($b(t) \equiv 0$): resolvent. Consider the following $\mathcal{L}(E)$-valued (we now look for a function $R_{t_0} : I \to \mathcal{L}(E)$ instead of $u : I \to E$) Cauchy problem

$$R'_{t_0}(t) = A(t)R_{t_0}(t), \quad R_{t_0}(t_0) = \mathrm{Id}; \tag{8.2}$$

note that the right-hand side is still globally Lipschitz in R and hence this Cauchy problem has a global solution $R_{t_0} \in C^1(I; \mathcal{L}(E))$. The operator-valued solution $R_{t_0}(t)$ (or $R^t_{t_0}$ or $R(t, t_0)$) of the Cauchy problem (5.1) is called the *resolvent* of the homogeneous linear differential equation $u'(t) = A(t)u(t)$. It is easy to see that

- if $u'(t) = A(t)u(t)$, then $u(t) = R(t, t_0)u(t_0)$ (indeed, the right-hand side satisfies the same differential equation and has the same value at $t = t_0$);
- the identity $R(t_3, t_1) = R(t_3, t_2)R(t_2, t_1)$ holds for all $t_1, t_2, t_3 \in I$ (indeed, as operator-valued functions of t_3 both sides solve the same equation $R'(t) = A(t)R(t)$ with the same initial data at $t = t_2$);
- in particular, $R(s, t)R(t, s) = \mathrm{Id}$ for all $s, t \in I$.

Inhomogeneous case: Duhamel's principle.

Proposition 8.1. *Let* $u(t)$ *solves the differential equation* (8.1) *and* $t_0 \in I$*. Then,*

$$u(t) = R(t, t_0)u(t_0) + \textstyle\int_{t_0}^t R(t, s)b(s)ds, \tag{8.3}$$

where the resolvent $R(t, t_0) := R_{t_0}(t)$ *is defined by* (8.2)*.*

We will start the next lecture with a proof of Proposition 8.1 and then will derive Theorem 7.2 from this formula.

January 06, 2021

We begin this lecture with the proofs of Proposition 8.1 (Duhamel's principle) and Theorem 7.2 (differentiability of the flow $\varphi_{t_0}(t,x)$ defined by a differential equation with a differentiable right-hand side $f(t,x)$). We will continue discussing linear differential equations after the latter proof.

Proof of Proposition 8.1 (Duhamel's principle). Let $v(t) := R(t_0,t)u(t)$ or, equivalently, $u(t) = R(t,t_0)v(t)$ (this definition can be understood as follows: if we solve the *homogeneous* differential equation $w'(s) = A(s)w(s)$ with the initial data $w(t) = u(t)$, then $w(t_0) = v(t)$). Then,

$$u'(s) = A(s)R(s,t_0)v(s) + R(s,t_0)v'(s) = A(s)u(s) + R(s,t_0)v'(s),$$

which means that $v'(s) = R(t_0,s)b(s)$ and hence $v(t) = v(t_0) + \int_{t_0}^t R(t_0,s)b(s)ds$. This directly implies (8.3) since $v(t_0) = u(t_0)$ and $R(t,t_0)R(t_0,s) = R(t,s)$. □

Proof of Theorem 7.2 (differentiability of the flow $\varphi_{t_0}(t,x)$). For shortness, assume that $t_0 = 0$ and let $t > 0 = t_0$ (the case $t < t_0$ is similar). Denote $\varphi^t(x) := \varphi_{t_0}(t,x)$ and $A(t) := [D_x f](t,\varphi^t(x_0))$.

Let $0 < t < T_{\max} := \sup I_{\max}(0,x_0)$, the case $t < 0$ is similar. It follows from Lemma 6.4 and Proposition 7.1 that there exist $\varepsilon, C > 0$ such that

$$\|\varphi^s(x) - \varphi^s(x_0)\| \ \le\ e^{Cs}\cdot\|x - x_0\| \quad \text{uniformly in } x \in \overline{B}(x_0,\varepsilon) \text{ and } s \in [0,t].$$

Note that we have (see Lemma 1.8)

$$\begin{aligned}&\|f(s,\varphi^s(x)) - f(s,\varphi^s(x_0)) - [A(s)](\varphi^s(x) - \varphi^s(x_0))\|\\ &\qquad\le\ \sup\nolimits_{y\in[\varphi^s(x_0),\varphi^s(x)]}\ \|[D_x f](s,y) - A(s)\|\cdot\|\varphi^s(x) - \varphi^s(x_0)\|.\end{aligned}$$

Moreover, it easily follows from the continuity of $D_x f$ and the compactness of the trajectory $\{\varphi^s(x_0), s \in [0,t]\} \subset E$ that, as $\|x - x_0\| \to 0$,

$$\sup\nolimits_{y\in[\varphi^s(x_0),\varphi^s(x)]}\ \|[D_x f](s,y) - A(s)\| \to 0 \ \textit{uniformly} \text{ in } s \in [0,t].$$

Denote $u(s,x) := \varphi^s(x) - \varphi^s(x_0)$. It follows from the preceding discussion that

$$u'(s,x)\ =\ f(s,\varphi^s(x)) - f(s,\varphi^s(x_0))\ =\ A(s)u(s,x) + b(s,x)$$

where $b(s,x) = o(\|x - x_0\|)$ uniformly in $s \in [0,t]$. We now apply Duhamel's formula (see Proposition 8.1) and conclude that

$$\begin{aligned} u(t,x)\ &=\ R(t,0)u(0,x) + \textstyle\int_0^t R(t,s)b(s,x)ds\\ &=\ [\Phi_{(t_0,x_0)}(t)](x - x_0) + o(\|x - x_0\|),\end{aligned}$$

where $R(t,s)$ denotes the resolvent of the linear equation $\Phi'(t) = A(t)\Phi(t)$. Note that this equation is nothing but (7.4), which we use to *define* $\Phi_{(t_0,x_0)}(t) := R(t,0)$. The proof is complete. □

Example. Before going further, let us consider a toy example of a linear equation coming from everybody's childhood (as at first (quadratic) approximation this example describes the response of a swing to a periodic force $\sin t$):

$$u''(t) = -u(t) + \varepsilon\sin t, \quad u(0) = u'(0) = 0$$

(where $\varepsilon \in \mathbb{R}$ can be thought of as a (small) parameter), which can be rewritten as

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon \sin t \end{pmatrix}, \quad u_1(0) = u_2(0) = 0.$$

It is easy to see that

$$R(t,s) \;=\; \exp\left[(t-s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \;=\; \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}$$

and hence the solution is given by

$$u(t) \;=\; \varepsilon \textstyle\int_0^t \sin(t-s)\sin s\, ds \;=\; \frac{1}{2}\varepsilon(-t\cos t + \sin t).$$

Note a *resonance effect:* the solution grows linearly in t but if we replace the external force by 1 or by $\sin \omega t$ with $\omega \neq \pm 1$, then $u(t)$ remains bounded for all t.

Proposition 8.2. *The following identity holds:* $\det R(t,t_0) = \exp\left[\int_{t_0}^t \mathrm{Tr}(A(s))ds\right]$.

Remark 8.1. If $A(s) = A$ does not depend on s, then $R(t,t_0) = \exp[(t-t_0)A]$ and the identity is trivial since $\det(\exp M) = \exp(\mathrm{Tr}\, M)$ for all matrices $M \in \mathbb{C}^{n\times n}$ (this is straightforward by considering the Jordan normal form of M). However, let us emphasize that, in general,

$$R(t,t_0) \;\neq\; \exp\left[\textstyle\int_{t_0}^t A(s)ds\right]$$

since $(\exp[M(t)])' \neq M'(t)\exp[M(t)]$.

Proof. Let $R(t,t_0) = [r_1(t),\dots,r_n(t)]$, where $r_k : I \to \mathbb{R}^n$ solves the equation $u'(t) = A(t)u(t)$ with the initial data $r_k(t_0) = e_k$, the k-th basis vector of $\mathbb{R}^n$. Since $\det \mathbb{R}(t,t_0)$ is a multi-linear function of $r_(t),\dots,r_n(t)$, we have

$$\begin{aligned}
&(\det R(t,t_0))'/\det R(t,t_0)\\
&= \textstyle\sum_{k=1}^n \det[r_1(t),\dots,r_{k-1}(t),A(t)r_k(t),r_{k+1}(t),\dots,r_n(t)]\,/\det R(t,t_0)\\
&= \textstyle\sum_{k=1}^n \det\big(R(t_0,t)\cdot[r_1(t),\dots,r_{k-1}(t),A(t)r_k(t),r_{k+1}(t),\dots,r_n(t)]\big)\\
&= \textstyle\sum_{k=1}^n \det[e_1,\dots,e_{k-1},R(t_0,t)A(t)R(t,t_0)e_k,e_{k+1},\dots,e_n]\\
&= \mathrm{Tr}[R(t_0,t)A(t)R(t,t_0)] \;=\; \mathrm{Tr}[R(t,t_0)R(t_0,t)A(t)] \;=\; \mathrm{Tr}[A(t)].
\end{aligned}$$

The claim is now trivial since $\det R(t_0,t_0) = \det \mathrm{Id} = 1$. □

Quasi-détour. Hamiltonian systems. This is an important class of autonomous differential equations (or systems of equations in a *phase space* $u = u(t) \in E = \mathbb{R}^{2n}$) which originated in the work of Hamilton (1805–1865) on the classical mechanics.

- Let $u = (q,p)$, where $q = q(t) \in \mathbb{R}^n$ is called (generalized) *positions* and $p = p(t) \in \mathbb{R}^n$ are called (generalized) *momenta* of a system.
- Let $\mathcal{H} : \mathbb{R}^{2n+1} \to \mathbb{R}$ be a smooth function called *Hamiltonian*, in classical mechanics $\mathcal{H}(t,q,p) = \mathcal{H}(q,p)$ is the energy of a system in a state (q,p).
- A Hamiltonian system of differential equations in $\mathbb{R}^{2n}$ is

$$\begin{aligned} q_k'(t) &= [\partial\mathcal{H}/\partial p_k](t,q(t),p(t)), \\ p_k'(t) &= -[\partial\mathcal{H}/\partial q_k](t,q(t),p(t)), \end{aligned} \tag{8.4}$$

  or, equivalently,

$$u'(t) \;=\; \Omega \cdot {}^t\nabla_u \mathcal{H}(t,u(t)), \qquad \Omega := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \tag{8.5}$$

  (recall that in these notes we view the gradient $\nabla_u \mathcal{H}$ as a 'row' vector).

Example. Consider a function $\mathcal{H}(q,p) := \sum_{k=1}^{n} \frac{1}{2m_k} p_k^2 + V(q_1, \ldots, q_k)$; the two terms are the kinetic and the potential energy of a system. Equations (8.4) are nothing but the Newton's laws of motion. It is worth noting that the homogeneous equation $u''(t) = -u(t)$ mentioned above can be obtained in this way (with $n=1$) if we set $q(t) := u(t)$, $p(t) := u'(t)$ and $\mathcal{H}(q,p) := \frac{1}{2}(p^2+q^2)$. (For a 'real' circular pendulum, the potential is $V(q) = 1 - \cos q = \frac{1}{2}q^2 + O(q^4)$, this is why above we said that $u''(t) = -u(t)$ should be viewed as a first (quadratic) approximation.)

Simple fact. If the Hamiltonian $\mathcal{H}(t,q,p) = \mathcal{H}(q,p)$ does not depend on time, then the value $\mathcal{H}(q(t),p(t))$ does not change along the trajectories: indeed,

$$\frac{d}{dt}\mathcal{H}(q(t),p(t)) \;=\; \nabla\mathcal{H}(q(t),p(t)) \cdot \Omega\, {}^t\nabla\mathcal{H}(q(t),p(t)) \;=\; 0$$

(recall that, from a physics perspective, $\mathcal{H}(q,p)$ is nothing but the energy of a system in the state (q,p), so this fact corresponds to the conservation of energy).

A much deeper fact (which is also true for time-dependent Hamiltonians) is

Theorem 8.3 (Liouville). *The flow $\varphi_{t_0}^t$ of a Hamiltonian system conserves the volume in the phase space: the determinant of the Jacobian $\mathrm{J}[\varphi_{t_0}^t] = 1$.*

We cannot discuss the proof of Liouville's theorem in these notes except in the trivial case of quadratic Hamiltonians:

Proposition 8.4. *Liouville's theorem holds provided that $\mathcal{H}(t,u) = \langle u, H(t)u\rangle$, where $H = {}^tH \in C(I, \mathbb{R}^{2n\times 2n})$.*

Proof. Note that we have ${}^t\nabla_u \mathcal{H}(t,u) = 2H(t)u$, thus equation (8.5) reads as $u'(t) = 2\Omega H(t)u(t)$. It remains to apply Proposition 8.2 since

$$\mathrm{Tr}[\Omega M] = \mathrm{Tr}[{}^t(\Omega M)] = \mathrm{Tr}[-M\Omega] = 0 \;\text{ if }\; M = {}^tM. \qquad \square$$

Détour[8]. Heat equation in $\mathbb{R}^n$. Formally(!!), one can view the classical (inhomogeneous) *heat equation*

$$\frac{\partial u}{\partial t}(t,x) \;=\; \Delta u(t,x) + b(t,x), \quad u(0,x) = u_0(x).$$

(where $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is an unknown function) as a linear differential equation $u'(t) = \Delta u(t) + b(t)$ for a function $u \in C^1(\mathbb{R}_+; E)$, e.g., with $E = L^2(\mathbb{R}^n)$. An obvious problem of this approach is that the Laplacian $u \mapsto \Delta u$ is not a bounded linear operator: it is not even defined on the whole space $E = L^2(\mathbb{R}^n)$. However, this can be eventually overcome due to the following observation:

> the *resolvent* $R(t,0) = \exp(t\Delta)$ (defined, e.g., via the spectral theory of self-adjoint operators) belongs to $\mathcal{L}(E)$ for $t \geq 0$ and satisfies $\|R(t,0)\|_{\mathcal{L}(E)} \leq 1$ (this follows from the fact that $\mathrm{spec}(-\Delta) = \mathbb{R}_+$).
>
> (In fact, the resolvent $\exp(t\Delta)$ has even much nicer properties: for each $t > 0$ it maps $L^2(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.)

In particular, Duhamel's principle applies to the heat equation as well as to other equations (e.g., to the classical *wave equation*). This discussion naturally leads towards basics of the course *'Equations aux Dérivées Partielles'*.

[8] This discussion was totally omitted during the lecture.

January 11, 2021

9. Lyapunov stability of stationary points

Let $E = \mathbb{R}^n$ be a finite-dimensional space and consider an autonomous differential equation (system)

$$u'(t) = f(u(t)), \quad \text{where } \ f \in C^1(U; E) \tag{9.1}$$

is a C^1-smooth *vector-field* on $U \subset E$. Two important[9] cases of such systems are

- ◦ Hamiltonian equations (see last lecture), for which $f = \Omega\, {}^t\nabla\mathcal{H}$;
- ◦ *gradient-descent* equations for which $f = -\,{}^t\nabla\mathcal{E}$, where $\mathcal{E} \in C^2(U;\mathbb{R})$ (which are often used to find a (local) minimum of a given function Ψ).

Let $\varphi^t(x) = \varphi_{t_0}(t_0+t, x)$, $x \in U$, $t \in I_{\max}(x) := I_{\max}(0,x)$ be the flow defined by the equation (9.1); recall that $\varphi^t \in C^1(U;U)$.

- The curves $(\varphi^t(x))_{t\in I_{\max}(x)}$ are called *integral curves* of the vector-field f.

Sometimes, one also calls the decomposition of U into integral curves of (9.1) the *phase plot* of the equation/system. Let $x_0 \in U$. If $f(x_0) \neq 0$, then the integral curves passing near x_0 are close to straight lines going in the direction $f(x_0)$.

- If $f(x_0) = 0$, then $\varphi^t(x_0) = x_0$ and x_0 is called a *stationary point* of f.

Further, a stationary point x_0 is called

- *stable* if for each $C > 0$ there exist $\varepsilon = \varepsilon(C) > 0$ such that $T_{\max}(x) = +\infty$ and $\|\varphi^t(x) - x_0\| \le C$ for all $x \in \overline{B}(x_0,\varepsilon) \subset U$ and $t \ge 0$;
- *asymptotically stable* if one also has $\varphi^t(x) \to x_0$ as $t \to +\infty$ for all $x \in \overline{B}(x_0,\varepsilon_0)$ provided that $\varepsilon_0 > 0$ is small enough;
- *exponentially stable* if, in addition to the above, there exist $\alpha, C > 0$ such that $\|\varphi^t(x) - x_0\| \le Ce^{-\alpha t}\|x - x_0\|$ for all $x \in \overline{B}(x_0,\varepsilon_0)$ and $t \ge 0$.

Near a stationary point x_0 the equation (9.1) can be written as

$$\tfrac{d}{dt}(u(t) - x_0) = A(u(t) - x_0) + o(\|u(t) - x_0\|), \quad \text{where } A := [Df](x_0) \in \mathbb{R}^{n\times n}.$$

One can consider a *linear approximation* $v'(t) = Av(t)$ of this equation. Clearly, if A has zero eigenvalues, then the behaviour of trajectories near the corresponding eigenspace cannot be modeled by this linear approximation so we assume that $\lambda_k \neq 0$. To develop an intuition, let us consider small-dimensional examples. A general perspective, which we will not(!) justify, is the following: the trajectories of the original equation (9.1) and those of its linearization $v'(t) = Av(t)$ near x_0 have 'the same structure' provided that $\operatorname{Im}\lambda_k \neq 0$ for all $k = 1, \ldots, n$.

- Let $n = 1$. If $A > 0$, then the solution grows as $t \to +\infty$ whilst, if $A < 0$, then the solution decays exponentially fast as $t \to +\infty$. Note that this is exactly what happens with solutions of the equation $u'(t) = (u(t))^2 - 1$ near the stationary points $u = \pm 1$: the stationary point $u = +1$ is unstable, the stationary point $u = -1$ is exponentially stable.

---

[9]**Détour** (this discussion was totally omitted during the lecture). It is worth noting that, at lest formally, the classical *heat equation* $u_t = \Delta u$, which was already mentioned during the last lecture, can be thought of as a gradient-descent equation with $\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^n}\|\nabla u(x)\|^2 dx$. Indeed, a formal integration by parts implies that $[\nabla\mathcal{E}(u)]h = \int\langle\nabla u(x), \nabla h(x)\rangle dx = -\int \Delta u(x)h(x)dx$.

In a similar manner, the classical *wave equation* $u_{tt} = \Delta u$ can be, at least formally viewed as a Hamiltonian system with the Hamiltonian $\mathcal{H}(u,v) := \frac{1}{2}\int_{\mathbb{R}^n}(\|\nabla u(x)\|^2 + (v(x))^2)dx$.

However, let us emphasize that it is not at all easy to adapt the finite-dimensional discussion to these equations; we mention them here only in order to make links with other courses.

- Now let $n=2$. The real 2×2 matrix A has either two real eigenvalues or two complex-conjugated ones. Assume for simplicity that $\lambda_1\neq\lambda_2$. Changing the basis in $\mathbb{R}^2$ appropriately, we can assume that
$$A=\begin{pmatrix} a_1 & 0 \\ 0 & a_2\end{pmatrix} \quad \text{or} \quad A=\begin{pmatrix} a & b \\ -b & a\end{pmatrix},$$
where $\lambda_1=a+ib$ and $\lambda_2=a-ib$ in the second case. In the first case (two real eigenvalues) we have the following situations:
  - If both $0>a_{1,2}=\lambda_{1,2}$, then all solutions decay as $t\to+\infty$ whilst, if both $0<\lambda_{1,2}=a_{1,2}$, then all solutions grow as $t\to+\infty$. Note that $\exp(ta_2)=[\exp(ta_1)]^{a_2/a_1}$, so the solutions of the linearized equation looks like power-low curves. Stationary points with such local behavior are called *stable/unstable nodes*.
  - If $a_1<0<a_2$, then the picture is different: the solution started at a vector $v(0)={}^t(v_1(0),0)$ exponentially decays whist all other solution grow as $t\to+\infty$. Such stationary points are called *saddle points*.
  - The name *saddle point* comes from considering the gradient-descent flow $u'(t)=-[{}^t\nabla\mathcal{E}](u(t))$: local minima of the function $\mathcal{E}:\mathbb{R}^2\to\mathbb{R}$ give rise to stable nodes, local maxima – to unstable nodes, and the saddle points are those points where $[{}^t\nabla\mathcal{E}](x_0)=0$ but the Hessian $[\nabla^t\nabla\mathcal{E}](x_0)$ is not sign-definite.

  In the second case (two complex-conjugated eigenvalues) the solutions can be written explicitly as
$$\begin{aligned} v_1(t) &= e^{at}(v_1(0)\cos bt+v_2(0)\sin bt),\\ v_2(t) &= e^{at}(-v_1(0)\sin bt+v_2(0)\cos bt).\end{aligned}$$
  - The solutions of the linearized equation are logarithmic spirals, either going towards the origin if $a<0$ or diverging from it if $a>0$. Such stationary points are called *stable/unstable foci*.
  - If $a=0$, then the trajectories of $v'(t)=Av(t)$ are circles (or ellipses in the original coordinate system). Such stationary points are called *centers*. However, let us emphasize that this picture is <u>not</u> stable when we add smaller terms to the linearization $v'(t)=A(t)$: the trajectories $u(t)-x_0$ <u>can</u> diverge from $v(t)$ (and of the stationary point x_0) as $t\to+\infty$ due to a kind of a resonance effect mentioned during the last lecture and produced by lower terms in the expansion of f near x_0.
- Clearly, when the dimension n increases, more and more different scenarios appear depending on the properties of eigenvalues of A. However, the case $n=2$ is already instructive enough: provided that $\operatorname{Re}\lambda_k\neq 0$ for all k, the picture in $\mathbb{R}^n$ can be loosely viewed as a direct sum of two- and one-dimensional pictures in the corresponding eigenspaces of A.

Definition 9.1. *(i) A function $\Phi:U\to\mathbb{R}$ is called a* Lyapunov function *for the autonomous differential equation* (9.1) *if $\nabla\Phi(x)\cdot f(x)\le 0$ for all $x\in U$.*
(ii) A function $\mathcal{H}:U\to\mathbb{R}$ is called a first integral *if $\nabla\mathcal{H}(x)\cdot f(x)=0$.*

Lemma 9.2. *A function $\Phi:U\to\mathbb{R}$ is a Lyapunov function for* (9.1) *if and only if $\frac{d}{dt}\Phi(\varphi^t(x))\le 0$ for all trajectories. Similarly, a function $\mathcal{H}:U\to\mathbb{R}$ is a first integral if $\mathcal{H}(\varphi^t(x))$ remains constant along the trajectories (this is why the name).*

Proof. By the chain rule, we have $\frac{d}{dt}\Phi(\varphi^t(x)) = \nabla\Phi(\varphi^t(x)) \cdot f(\varphi^t(x))$. □

In particular,

- for gradient-descent systems the function $\mathcal{E}$ is a Lyapunov function;
- for autonomous Hamiltonian systems the Hamiltonian $\mathcal{H}$ is a first integral and hence a Lyapunov function.

Proposition 9.3. *If x_0 is a strict (i.e., $[D^2\Phi](x_0) \ge \beta^2 \operatorname{Id}$, $\beta > 0$) local minimum of a Lyapunov function Φ, then x_0 is a stable stationary point.*

Proof. Let $\Phi(x_0) + \frac{1}{2}\beta^2\|x - x_0\|^2 \le \Phi(x) \le \Phi(x_0) + 2B^2\|x - x_0\|^2$ near x. Then,

$$\overline{B}(x_0, \varepsilon) \ \subset\ \{x : \Phi(x) \le \Phi(x_0) + 2B^2\varepsilon^2\} \ \subset\ \overline{B}(x_0, 2B\beta^{-1} \cdot \varepsilon)$$

if $\varepsilon > 0$ is small enough. Since $\Phi(\varphi^t(x))$ is a non-increasing function, we have $\|\varphi^t(x) - x_0\| \le C$ if $\|x - x_0\| \le \varepsilon := \frac{1}{2}\beta B^{-1} \cdot C$ provided that C is small enough. □

The following theorem is a standard criterion of stability of stationary points.

Theorem 9.4. *Let x_0 be a stationary point of the autonomous equation* (9.1) *and all eigenvalues λ_k, $k = 1, \dots, n$, of the matrix $[Df](x_0)$ satisfy* $\operatorname{Re}\lambda_k \le -\alpha < 0$. *Then, x_0 is a stable and, moreover, an exponentially stable stationary point.*

Proof. Let $A = [Df](x_0)$. Considering the Jordan normal form of A we can find an invertible *complex-valued* matrix Q such that $QAQ^{-1} = \Lambda + E$, where $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ and $\|E\| \le \frac{1}{4}\alpha$. Indeed, [10]

Let $\Phi(x) := \|Q(x - x_0)\|^2 = {}^t(x - x_0)^t\overline{Q}Q(x - x_0)$. Then,

$$\begin{aligned} \nabla\Phi \cdot h \ &=\ {}^th^t\overline{Q}Q(x - x_0) + {}^t(x - x_0)^t\overline{Q}Qh \\ &=\ 2\operatorname{Re}[{}^t(x - x_0)^t\overline{Q}Qh] \end{aligned}$$

and hence, since $QA = (\Lambda + E)Q$,

$$\begin{aligned} \nabla\Phi(x) \cdot f(x) \ &=\ 2\operatorname{Re}[{}^t(x - x_0)^t\overline{Q}QA(x - x_0)] + o(\|x - x_0\|^2) \\ &=\ 2\operatorname{Re}[{}^t(x - x_0)^t\overline{Q}(\Lambda + E)Q(x - x_0)] + o(\|x - x_0\|^2) \\ &\le -2\alpha \cdot \|Q(x - x_0)\|^2 + \tfrac{1}{2}\alpha\|Q(x - x_0)\| + o(\|x - x_0\|^2) \\ &\le\ -\alpha \cdot \|Q(x - x_0)\|^2 \ =\ -\alpha \cdot \Phi(x) \end{aligned}$$

provided that $\|x - x_0\| \le \varepsilon_0$ and $\varepsilon_0 > 0$ is chosen small enough. In particular, Φ is a Lyapunov function which has a strict minimum at x_0, therefore x_0 is a stable stationary point. Moreover, we have $\frac{d}{dt}\Phi(\varphi^t(x)) \le -\alpha\Phi(\varphi^t(x))$, which implies (via Gronwall's lemma) that $\Phi(\varphi^t(x)) \le e^{-\alpha t}\Phi(x)$. Since Q is an invertible matrix, we also have $C^{-1}\|x - x_0\|^2 \le \Phi(x) \le C\|x - x_0\|^2$ for a certain constant $C > 0$, which means that $\|\varphi^t(x) - x_0\| \le Ce^{-\alpha t}\|x - x_0\|$, i.e., the exponential stability. □

---

[10] Indeed, one can handle non-trivial Jordan cells by noting that

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda \end{pmatrix} = \operatorname{diag}\{1, \varepsilon, \varepsilon^2, \dots\} \begin{pmatrix} \lambda & \varepsilon & 0 & 0 \\ 0 & \lambda & \varepsilon & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda \end{pmatrix} \operatorname{diag}\{1, \varepsilon^{-1}, \varepsilon^{-2}, \dots\}.$$

January 13, 2021

10. Vector-fields and derivations on smooth manifolds

Let M be a C^∞-smooth manifold and consider the space $C^\infty(M)$ of smooth $\mathbb{R}$-valued functions on M.

Definition 10.1. *A linear mapping $\mathcal{D}: C^\infty(M) \to C^\infty(M)$ is called a* derivation *if it satisfies the Leibnitz rule: for all $f, g \in C^\infty(M)$ we have $\mathcal{D}(fg) = f\mathcal{D}(g) + g\mathcal{D}(f)$.*

It is easy to see that this definition automatically implies that

- $\mathcal{D}$ vanishes on constant functions: $\mathcal{D}(1) = 0$ since $\mathcal{D}(1 \cdot 1) = 2\mathcal{D}(1)$;
- $\mathcal{D}$ is a *local* operation: if $f(b) = 0$ for all $b \in U_a \supset M$, then $[\mathcal{D}f](a) = 0$ (and hence, by linearity, if $f_1(b) = f_2(b)$ for all $b \in U_a$, then $[\mathcal{D}f_1](a) = [\mathcal{D}f_2](a)$). Indeed, if $\phi \in C^\infty(M)$ is chosen so that $\phi(a) = 0$ and $\phi|_{M \smallsetminus U_a} = 1$, then $f = \phi f$ and the Leibnitz rule gives $[\mathcal{D}f](a) = [\mathcal{D}(\phi f)](a) = 0$.

Let $v : M \to TM$, $a \mapsto v(a) \in T_aM$, be a smooth vector-field on M. Denote by $\varphi_v^t = \varphi^t : M \to M$ the flow defined by the differential equation $u'(t) = v(u(t))$. It is easy to see that

$$[\mathcal{D}_v f](a) \ := \ \frac{d}{dt} f(\varphi_v^t(a))\big|_{t=0} \tag{10.1}$$

defines a derivation on M. An important fact that we prove below (see Theorem 10.3) is that *all* derivations on M can be obtained in this way, i.e.,

$$\text{there exists a } \textit{bijection} \ \ \{\text{derivations}\} \ \longleftrightarrow \ \{\text{smooth vector-fields}\}.$$

Recall that smooth manifolds are defined via homeomorphisms (called charts)

$$\varphi_\alpha : M \supset U_\alpha \to \mathrm{B}^n := B(0,1) \subset \mathbb{R}^n$$

such that the compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are C^∞-mappings between subsets of $\mathbb{R}^n$.

- Given $f \in C^\infty(M)$ and a chart φ_α, denote $f_\alpha := f \circ \varphi_\alpha^{-1} \in C^\infty(\mathrm{B}^n; \mathbb{R})$. This is the same function but considered in $\mathrm{B}^n \subset \mathbb{R}^n$ instead of $U_\alpha \subset M$.

Further, recall that the tangent space T_aM is formally defined as the space of equivalence classes of smooth curves $\gamma : (-1,1) \to M$ passing through a. Given a chart φ_α, this space is identified with $\mathbb{R}^n$ by considering curves $\varphi_\alpha \circ \gamma$ instead of γ.

- For a smooth vector-field v on M and a chart φ_α such that $a \in U_\alpha$, let a vector-field $v_\alpha : \mathrm{B}^n \to \mathbb{R}^n$ be defined as $v_\alpha(x) := (\varphi_\alpha \circ \gamma)'(0)$, where $\gamma \in v(\varphi_\alpha^{-1}(x))$ (recall that the latter is an equivalence class of smooth curves passing through the point $\gamma(0) = \varphi_\alpha^{-1}(x) \in M$).
- If we replace φ_α by another chart φ_β, then
$$v_\beta((\varphi_\beta \circ \varphi_\alpha^{-1})(x)) = [D(\varphi_\beta \circ \varphi_\alpha^{-1})(x)]\, v_\alpha(x) \tag{10.2}$$
- By definition, the differential equation $u'(t) = v(u(t))$ on M reads as $u_\alpha'(t) = v_\alpha(u_\alpha(t))$ in a chart φ_α, where $u_\alpha := u \circ \varphi_\alpha^{-1}$; it is easy to see from (10.2) that local solutions of this differential equation do not depend on the choice of a chart φ_α used to define them. In particular, we have (by the chain rule) the following formula:
$$[\mathcal{D}_v f](\varphi_\alpha^{-1}(x)) \ = \ \sum_{k=1}^n (v_\alpha(x))_k \cdot \frac{\partial f_\alpha}{\partial x_k}(x), \quad x \in \mathrm{B}^n \subset \mathbb{R}^n, \tag{10.3}$$
where $(v_\alpha(x))_k$ denotes the k-th component of the vector $v_\alpha(x) \in \mathbb{R}^n$.

We need a simple fact, which is usually called Hadamard's lemma:

Lemma 10.2. *Let f be a (C^m-)smooth function on the unit ball $\mathrm{B}^n \subset \mathbb{R}^n$. Then, there exists ($C^{m-1}$-)smooth functions g_k such that $f(x) = f(0) + \sum_{k=1}^n x_k g_k(x)$. In particular, one can take $g_k(x) := \int_0^1 (\partial f/\partial x_k)(tx)dt$.*

Proof. This is nothing but the identity $f(x) - f(0) = \int_0^1 [(Df)(tx)](x)\, dt$. □

Theorem 10.3. *Let $\mathcal{D}$ be a derivation on M. Then, there exists unique smooth vector-field on M such that $\mathcal{D} = \mathcal{D}_v$, where the derivation $\mathcal{D}_v$ is defined by* (10.1)

Proof. Let $f \in C^\infty(M)$ and consider a chart $\varphi_\alpha : M \supset U_\alpha \to \mathrm{B}^n \subset \mathbb{R}^n$. We can apply Lemma 10.2 to the function $f_\alpha := f \circ \varphi_\alpha^{-1} : \mathrm{B}^n \to \mathbb{R}$ and write

$$f_\alpha(x) \;=\; f_\alpha(0) + \textstyle\sum_{k=1}^n x_k g_k(x), \quad x \in \mathrm{B}^n,$$

or, if we assume that $\varphi_\alpha(a_0) = 0$,

$$f(a) \;=\; f(a_0) + \textstyle\sum_{k=1}^n (\pi_k \circ \varphi_\alpha)(a)(g_k \circ \varphi_\alpha)(a), \quad a \in U_\alpha, \tag{10.4}$$

where $\pi_k : x \mapsto x_k$ is the k-th coordinate function on B^n. Assume for a second that we can view all functions in the identity (10.4) as being defined on the whole manifold M and not only in U_α. Then, the Leibnitz rule for $\mathcal{D}$ implies that

$$(\mathcal{D}f)(a_0) \;=\; \textstyle\sum_{k=1}^n [\mathcal{D}(\pi_k \circ \varphi_\alpha)](a_0) \cdot (g_k \circ \varphi_\alpha)(a_0).$$

Note that $(g_k \circ \varphi_\alpha)(a_0) = g_k(0) = (\partial f_\alpha/\partial x_k)(0)$. Therefore, if we *define*

$$v_\alpha(a) \;:=\; [\mathcal{D}(\pi_k \circ \varphi_\alpha)](a), \quad a \in U_\alpha, \tag{10.5}$$

then the formula (10.3) holds. Clearly, v_α is smooth on U_α since $\mathcal{D}$ maps smooth functions to smooth functions. It remains

(i) to fix a technical issue that functions $\pi_k \circ \varphi_\alpha$ and $g_k \circ \varphi_\alpha$ are defined only on U_α and not on the whole manifold M;
(ii) to prove that definitions (10.5) of v_α and v_β in two different charts φ_α and φ_β agree with each other in the sense of (10.3).

To fix (i), note that we can multiply functions $(\pi_k \cdot \varphi_\alpha)$ and $(g_k \cdot \varphi_\alpha)$ by a smooth function $\phi \in (C^\infty)$ chosen so that $\phi \equiv 1$ near a_0 and $\phi \equiv 0$ outside U_α, provided that we also replace f by $\phi^2 f$. Since derivation $\mathcal{D}$ is a local operation (see first comments after Definition 10.1), this multiplication does not change anything in the computation made above.

Finally, to check (ii), note that we already know from the formula (10.3) that $(\mathcal{D}f)(a) = 0$ if $Df(a) = 0$, i.e., if $\partial f_\alpha/\partial x_k(\varphi_\alpha(a)) = 0$ for all $k = 1, \ldots, n$. Let $y_1, \ldots, y_n$ be the coordinates in another chart φ_β. We need to check that

$$[\mathcal{D}(\pi_s \circ \varphi_\beta)](a) \;=\; \textstyle\sum_{k=1}^n (\partial y_s/\partial x_k)(\varphi_\alpha(a))[\mathcal{D}(\pi_k \circ \varphi_\alpha)](a).$$

By linearity of $\mathcal{D}$, this is equivalent to say that

$$\big[\mathcal{D}\big(\pi_s \circ \varphi_\beta - \textstyle\sum_{k=1}^n (\partial y_s/\partial x_k)(\varphi_\alpha(a)) \cdot (\pi_k \circ \varphi_\alpha)\big)\big](a) \;=\; 0.$$

The result follows since $[D(\pi_s \circ \varphi_\beta - \sum_{k=1}^n (\partial y_s/\partial x_k)(\varphi_\alpha(a)) \cdot (\pi_k \circ \varphi_\alpha))](a) = 0$. □

Lemma 10.4. *If $\mathcal{D}_v$ and $\mathcal{D}_w$ are derivations on M, then so is $\mathcal{D}_v \circ \mathcal{D}_w - \mathcal{D}_w \circ \mathcal{D}_v$.*

Proof. We only need to check that $\mathcal{D}_v \circ \mathcal{D}_w - \mathcal{D}_w \circ \mathcal{D}_v$ satisfies the Leibnitz rule: it holds due to

$$\begin{aligned}(\mathcal{D}_v \circ \mathcal{D}_w)(fg) &= \mathcal{D}_v(f \cdot \mathcal{D}_w g + g \cdot \mathcal{D}_w) \\ &= f \cdot (\mathcal{D}_v \circ \mathcal{D}_w)g + (\mathcal{D}_v f) \cdot (\mathcal{D}_w g) + (\mathcal{D}_v g) \cdot (\mathcal{D}_w f) + g \cdot (\mathcal{D}_v \circ \mathcal{D}_w) f\end{aligned}$$

and a similar formula for $(\mathcal{D}_w \circ \mathcal{D}_v)(fg)$. □

Lemma 10.4 together with Theorem 10.3 allow to give the following definition

Definition 10.5. *Let v, w be smooth vector-fields. A smooth vector-field $[v, w]$, called the* Lie bracket *of v and w is defined by the identity $\mathcal{D}_v \circ \mathcal{D}_w - \mathcal{D}_w \circ \mathcal{D}_v = \mathcal{D}_{[v,w]}$.*

(In algebra, a Lie bracket is an anti-symmetric bilinear operation satisfying the *Jacobi identity* $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$, which is starightforward if $[v, w]$ is defined as a commutator of two mappings defined by v and w, respectively.)

Recall that $(\mathcal{D}_w f)(a) = \lim_{t\to 0} \frac{1}{t}(f(\varphi_w^t(a)) - f(a))$ and hence

$$\mathcal{D}_{[v,w]} f(x) \;=\; \lim_{s,t\to 0} \frac{f((\varphi_w^t \circ \varphi_v^s)(a)) - f((\varphi_v^s \circ \varphi_w^t)(a))}{st}$$

In other words, the Lie bracket $[v, w]$ describes the non-commutativity of the two flows φ_w^t and φ_v^s. An alternative way of writing the same formula is

$$\mathcal{D}_{[v,w]} f(a) \;=\; \frac{\partial^2}{\partial s \partial t} f(\varphi_w^{-t} \circ \varphi_v^{-s} \circ \varphi_w^t \circ \varphi_v^s(a))\big|_{s=t=0}, \tag{10.6}$$

where we replaced a by $(\varphi_w^{-t} \circ \varphi_v^s)(a)$ in the previous formula and changed the signs of both s and t. (It is worth mentiong that no technical issues with exchanging the limits etc arise since we work with C^∞-smooth functions, so all convergences are actually uniform and all these ratios are smooth functions themselves.)

This discussion naturally leads to the course *Géométrie Différentielle* and we stop it here: recall that the subject of these notes is simply to develop a basement (language, basic notions etc) for more advanced courses.

Quasi-détour. The last topic to briefly mention is a very particular case when the manifold M is a *matrix Lie group*, i.e. a certain subgroup of $\mathbb{R}^{n\times n}$ which is also a topological manifold. We will focus on a concrete (simplest) case

$$M \;=\; \mathrm{SL}_n(\mathbb{R}) \;=\; \{G \in \mathbb{R}^{n\times n} : \det G = 1\}$$

but a similar discussion applies to all such groups.

- Let us consider the tangent space $T_{\mathrm{Id}}M$ to M at the identity element, which is called the *Lie algebra* $\mathfrak{sl}_n(\mathbb{R})$ corresponding to the Lie group $\mathrm{SL}_n(\mathbb{R})$. Since $\det(\mathrm{Id} + tA + o(t)) = 1 + t \operatorname{Tr} A + o(t)$, this tangent space admits an explicit description:

  $$T_{\mathrm{Id}}M \;=\; \mathfrak{sl}_n(\mathbb{R}) \;=\; \{A \in \mathbb{R}^{n\times n} : \operatorname{Tr} A = 0\}.$$

  (Indeed, note that we can view $M = \mathrm{SL}_n(\mathbb{R})$ as a smooth (n^2-1)-dimensional manifold embedded into the Euclidean space $\mathbb{R}^{n^2}$. All matrices $A \in T_{\mathrm{Id}}M$ should satisfy the equation $\operatorname{Tr} A = 0$ and this space already has dimension $n^2 - 1$, so there cannot be additional conditions.)

To justify the name 'Lie algebra' for the vector-space $\mathfrak{sl}_n(\mathbb{R})$, we need to introduce a Lie bracket $\mathfrak{sl}_n(\mathbb{R}) \times \mathfrak{sl}_n(\mathbb{R}) \to \mathfrak{sl}_n(\mathbb{R})$, which can be done in a 'brute force' way by declaring $[A,B] := AB - BA$, note that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. However, this construction can be understood in a much more conceptual way.

- Let $A \in \mathfrak{sl}_n(\mathbb{R}) = T_{\mathrm{Id}}M$. Note that the mapping

$$v_A : G \mapsto v_A(G) := GA \in T_G M$$

defines a smooth vector-field on $M = \mathrm{SL}_n(\mathbb{R})$ (indeed, it is easy to see that $T_G M = \{B \in \mathbb{R}^{n\times n} : \mathrm{Tr}(G^{-1}B) = 0\}$ and hence $GA \in T_G M$ iff $A \in T_{\mathrm{Id}}M$).

Thus, inside a huge set of all smooth vector-fields on M we now have a reasonably small subset of vector-fields v_A associated with the elements of the tangent space $T_{\mathrm{Id}}M$ (note that the group structure of M is absolutely crucial to define v_A). We can now try to compute the Lie bracket of two such vector-fields v_A, v_B and wonder whether the result is also associated to a certain element of $T_{\mathrm{Id}}M$ or not. As the following computation shows, the answer is affirmative. Moreover, the two Lie brackets $[v_A, v_B]$ and $[A,B] = AB - BA$ are the same.

Proposition 10.6. *The set of vector-fields $\{v_A,\ A \in \mathfrak{sl}_n(\mathbb{R})\}$ on $\mathrm{SL}_n(\mathbb{R})$ is closed under the operation of taking the Lie bracket and is isomorphic to the Lie algebra $\mathfrak{sl}_n(\mathbb{R})$, i.e., $[v_A, v_B] = v_{[A,B]}$ for all $A, B \in \mathfrak{sl}_n(\mathbb{R})$.*

Proof. Note that $\varphi^t_{v_A}(G) = G\exp(tA)$ since to construct the flow $\varphi^t_{v_A}$ we simply need to solve a linear differential equation $U'(t) = U(t)A$ with constant A. Therefore, if f is smooth function on $M = \mathrm{SL}_n(\mathbb{R})$ and $G \in M$, then for all $A, B \in \mathrm{sl}_n(\mathbb{R})$ we have (by expanding exponentials into series)

$$\begin{aligned}(\varphi^{-t}_{v_B} \circ \varphi^{-s}_{v_A} \circ \varphi^{t}_{v_B} \circ \varphi^{s}_{v_A})(G) &= G \cdot \exp(sA)\exp(tB)\exp(-sA)\exp(-tB) \\ &= G \cdot (\mathrm{Id} + st \cdot (AB - BA) + O(s^2 t) + O(st^2)) \\ &= \varphi^{st}_{v_C}(G) + O(s^2 t) + O(st^2), \quad \text{where } \ C := [A,B].\end{aligned}$$

Therefore, formula (10.6) implies that $(\mathcal{D}_{[v_A,v_B]}f)(G) = (\mathcal{D}_C f)(G)$. Since this identity holds for all functions $f \in C^\infty(M)$ and all $G \in M$, we are done. □

This discussion provides a glimpse of an analysis on Lie groups: if we want to think about higher derivatives of functions defined on, e.g., (subsets of) $\mathrm{SL}_n(\mathbb{R})$, then, instead of commuting partial derivatives $\partial/\partial x_k$ which we used for functions defined on $\mathbb{R}^n$, it makes sense to consider all derivations $\mathcal{D}_{v_A}$, $A \in \mathfrak{sl}_n(\mathbb{R})$ simultaneously and to benefit from the fact that the non-commutativity of these derivations can be expressed by similar derivations. Obviously, this discussion (as well as many much more inmportant things about Lie groups and algebras) also goes far beyond the scope of our class.

We stop here and hope that this introduction into the general topology and basics of the differential calculus will help you with other – more interesting – subjects.

The end