

Crossing probabilities in the critical 2D Ising model

Dmitry Chelkak (PDMI Steklov, St.Petersburg)

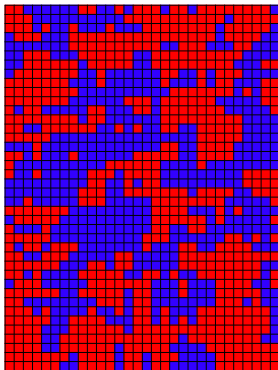
joint work with Stanislav Smirnov (Geneva)

arXiv:0910.2045: “*Universality in the 2D Ising model and conformal invariance of fermionic observables*”, 50pp.

CONFORMAL STRUCTURES AND DYNAMICS (CODY)

SEILLAC, FRANCE, MAY 2–8, 2010

2D Ising model: (square grid)



Spins $\sigma_i = +1$ or -1 .

Hamiltonian:

$$H = - \sum_{\langle ij \rangle} \sigma_i \sigma_j .$$

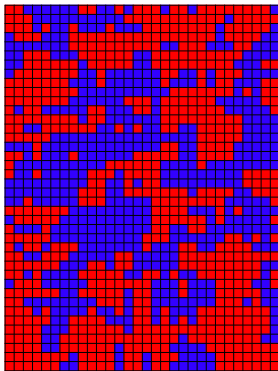
Partition function:

$$\mathbb{P}(\text{conf.}) \sim e^{-\beta H} \sim x^{\# \langle +- \rangle} ,$$

where

$$x = e^{-2\beta} \in [0, 1] .$$

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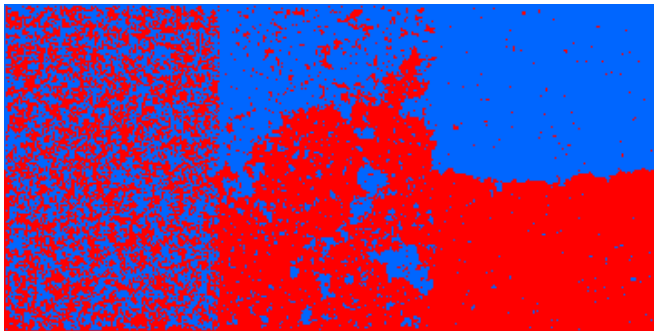
where

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Other “lattices” (planar graphs): $H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j .$

$$\mathbb{P}(\text{conf.}) \sim \prod_{\langle ij \rangle: \sigma_i \neq \sigma_j} x_{ij}, \quad x_{ij} \in [0, 1] .$$

Phase transition, criticality:



$$x > x_{\text{crit}}$$

$$x = x_{\text{crit}}$$

$$x < x_{\text{crit}}$$

(Dobrushin boundary values: two marked points a, b on the boundary; $+1$ on the arc (ab) , -1 on the opposite arc (ba))

[Peierls '36; Kramers-Wannier '41]: $x_{\text{crit}} = \frac{1}{\sqrt{2+1}}$

Conformal invariance:

Quantities (spin correlations, crossing probabilities, etc.)

[Cardy's formula for percolation, etc.]



Geometry (interfaces, loop ensembles, etc.)

[Schramm's SLEs, CLEs, etc.]

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“ \uparrow ”: SLE computations

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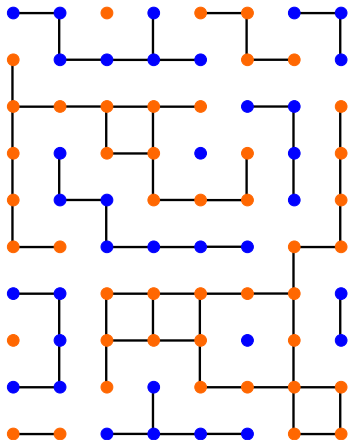
Geometry (interfaces, loop ensembles, etc.)
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“ \Uparrow ”: SLE computations

“ \Downarrow ”: Conformal martingale principle

Ref: S. Smirnov. *Towards conformal invariance of 2D lattice models*. [Proceedings of the international congress of mathematicians (ICM), Madrid, Spain, August 22–30, 2006]

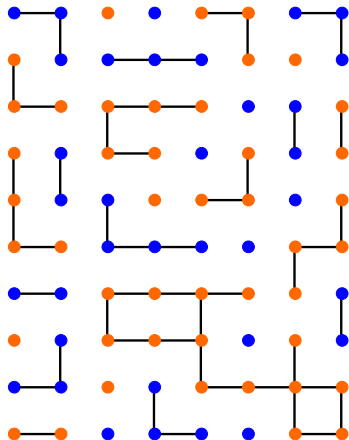
Spin- and FK-Ising models (random cluster representation):



$$\mathbb{P}(\text{spins conf.}) \sim x^{\#\langle+-\rangle}$$

$$= \prod_{\langle ij \rangle} [x + (1-x) \cdot \chi_{s(i)=s(j)}]$$

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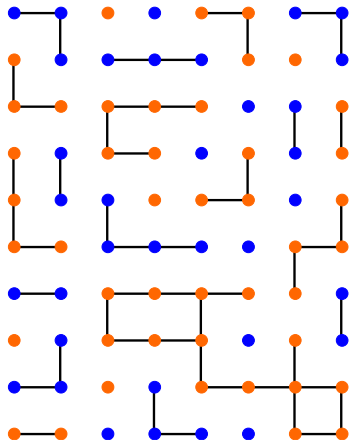


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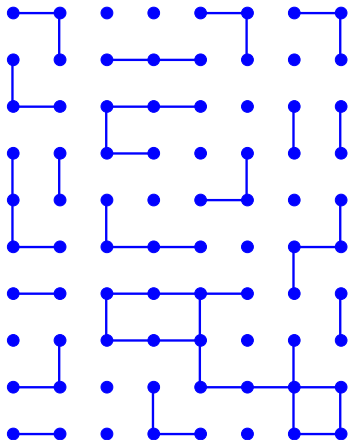
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equal spins (but not all)

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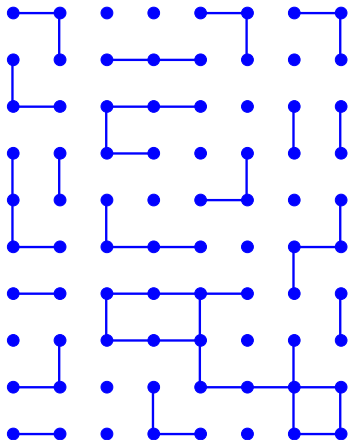
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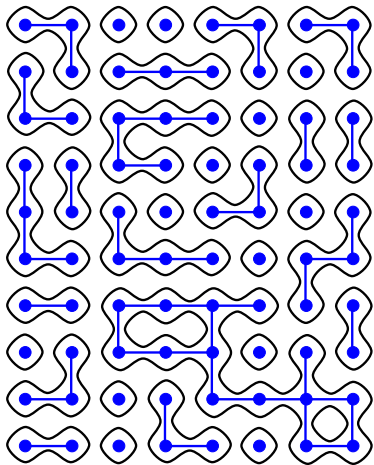


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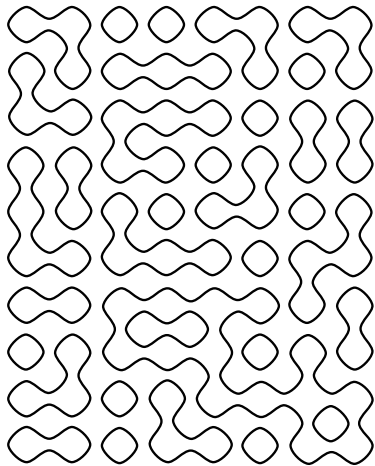
$$\sqrt{2}^{\#\text{loops}} [(1-x)/(x\sqrt{2})]^{\#\text{open}}$$

since

$$\#\text{loops} - \#\text{open edges}$$

$$= 2\#\text{clusters} + \text{const}$$

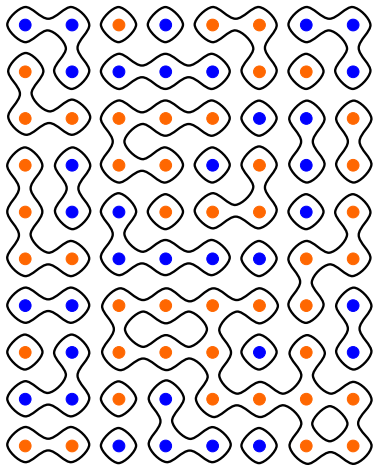
Spin- and FK-Ising models (random cluster representation):



Self-dual case ($x = x_{\text{crit}}$):

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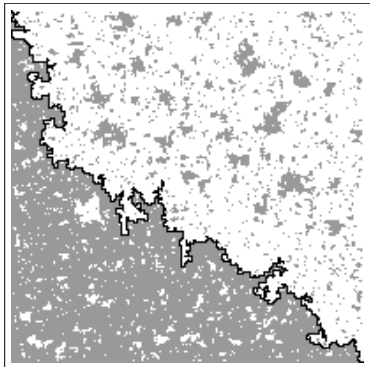
Then

$$\begin{aligned} \mathbb{P}_{\text{spin}}(s(i) = s(j)) \\ = \frac{1}{2}(1 + \mathbb{P}_{\text{FK}}(i \leftrightarrow j)) \end{aligned}$$

Convergence to SLE. Square lattice (Smirnov):

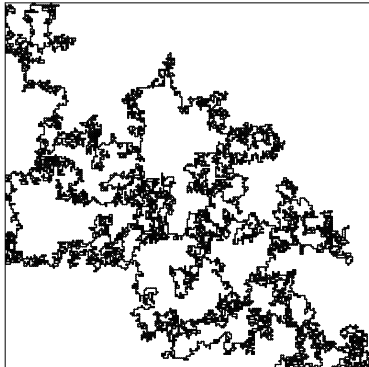
SPIN-ISING THEOREM:

Interface \rightarrow $SLE(3)$



FK-ISING THEOREM:

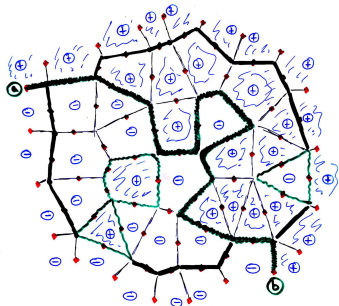
Interface \rightarrow $SLE(16/3)$



Universality. Isoradial graphs/rhombic lattices:

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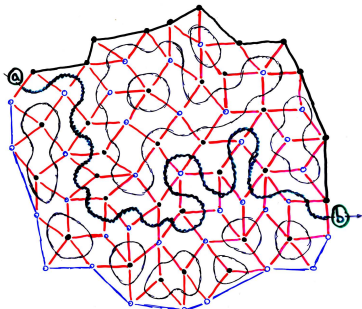
Interface \rightarrow SLE(3)



$$Z = \sum_{\text{config.}} \prod_{z: \oplus \leftrightarrow \ominus} \tan \frac{\theta(z)}{2}$$

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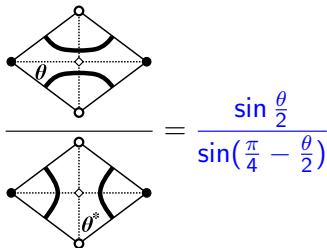
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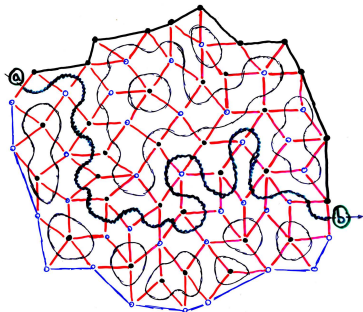
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FK-ISING LOCAL WEIGHTS:



$$\frac{\text{Rhombus with angle } \theta \text{ and two arcs}}{\text{Rhombus with angle } \theta^* \text{ and two arcs}} = \frac{\sin \frac{\theta}{2}}{\sin(\frac{\pi}{4} - \frac{\theta}{2})}$$



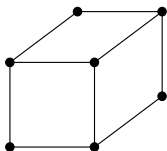
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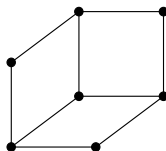
FK-ISING LOCAL WEIGHTS:
$$\frac{\text{Diagram 1}}{\text{Diagram 2}} = \frac{\sin \frac{\theta}{2}}{\sin(\frac{\pi}{4} - \frac{\theta}{2})} =: r(\theta)$$

satisfies $r(0) = 0$ and $Y - \Delta$ invariance: if $\alpha + \beta + \gamma = \frac{\pi}{2}$, then

$$1 = r(\alpha)r(\beta) + r(\alpha)r(\gamma) + r(\beta)r(\gamma) + \sqrt{2} \cdot r(\alpha)r(\beta)r(\gamma).$$

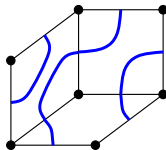
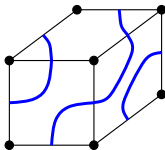


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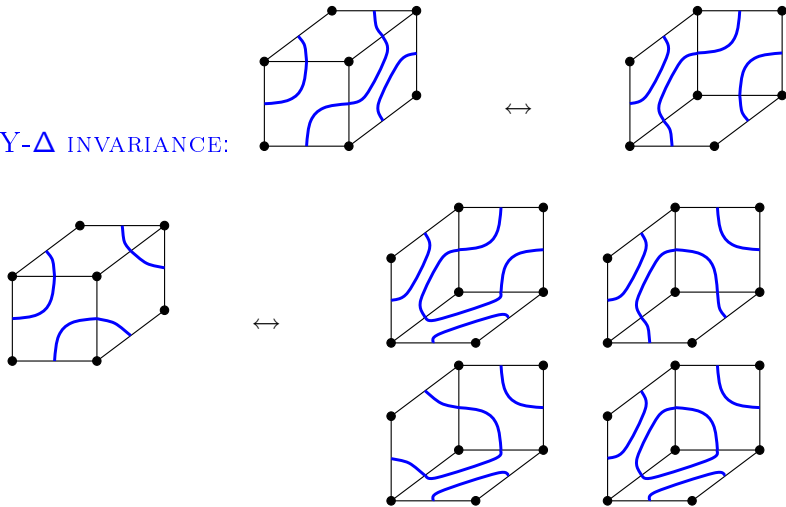
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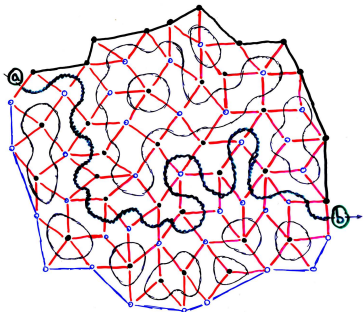


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Conformal martingale (discrete fermionic observable):

FK-ISING THEOREM:

Interface \rightarrow $SLE(16/3)$



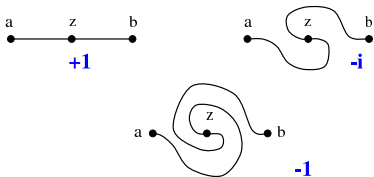
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Conformal martingale (discrete fermionic observable):

Discrete holomorphic observable having the martingale property:

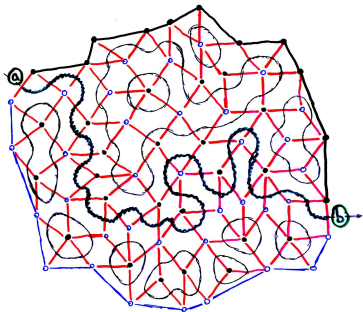
$$F^\delta = \mathbb{E} \chi[z \in \gamma] \cdot e^{-\frac{i}{2} \cdot \text{wind}(\gamma, b \rightarrow z)},$$

where $z \in \diamond$.



FK-ISING THEOREM:

Interface \rightarrow *SLE(16/3)*



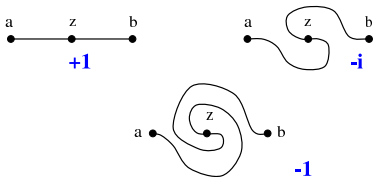
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Boundary Value Problem:

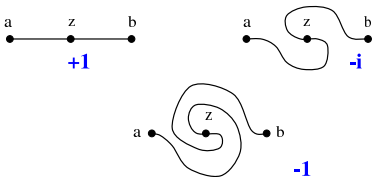
- ▶ $F(z)$ is holomorphic in Ω ;
- ▶ $\text{Im}[F(\zeta)(\tau(\zeta))^{\frac{1}{2}}] = 0$
for $\zeta \in \partial\Omega \setminus \{a, b\}$,
where $\tau(\zeta)$ goes from a
to b ;
- ▶ (mult.) normalization.

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Solution: $F(z) = \sqrt{\Phi'(z)}$,

$\Phi : (\Omega; a, b) \rightarrow (S, -\infty, +\infty)$,

$$S = \mathbb{R} \times (0, 1).$$

Universality. Convergence to SLE (FK-Ising):

F^δ is a *discrete holomorphic* martingale. Then:

- ▶ Take a “discrete integral” $H^\delta := \text{Im} \int (F^\delta)^2(z) d^\delta z$ (miraculously, it is well defined);
- ▶ H^δ is NOT discrete harmonic, so prove that it is “approximately” harmonic;

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This needs some work (see arXiv:0910.2045, 0810.2188).

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- ▶ Then deduce convergence of an interface to *SLE(16/3)* from the convergence of the martingale observable.

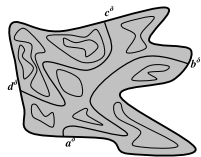
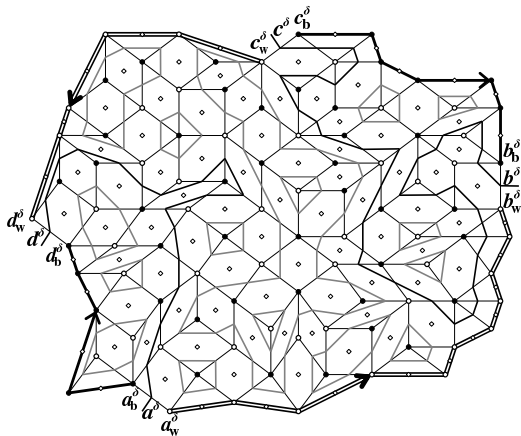
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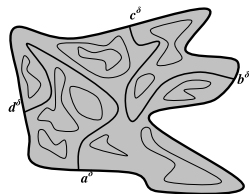
INTERFACES \rightarrow SLE(16/3). In which topology?

- ▶ Convergence of driving forces in the Loewner equation. Directly follows from the convergence of observable.
- ▶ Convergence of curves themselves. Needs some a priori information (estimates of some **crossing probabilities**). (Aizenman, Burchard, '99; Kemppainen, Smirnov '09)

FK-Ising crossing probability:

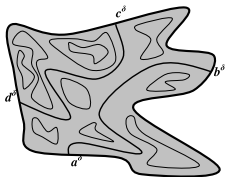


P^δ vs. Q^δ



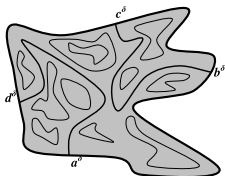
FK-Ising crossing probability:

P^δ



THEOREM: For all $r, R, t > 0$ there exists $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ such that if $B(0, r) \subset \Omega^\delta \subset B(0, R)$ and either both $\omega(0; \Omega^\delta; a^\delta b^\delta), \omega(0; \Omega^\delta; c^\delta d^\delta)$ or both $\omega(0; \Omega^\delta; b^\delta c^\delta), \omega(0; \Omega^\delta; d^\delta a^\delta)$ are $\geq t$ (i.e., quadrilateral Ω^δ has no neighboring small arcs), then

Q^δ

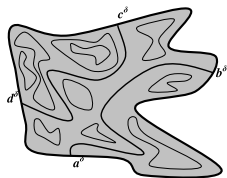


$$|P^\delta - P(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)| \leq \varepsilon(\delta)$$

(uniformly w.r.t. Ω^δ and \diamond^δ), where P depends only on the conformal modulus of $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)$.

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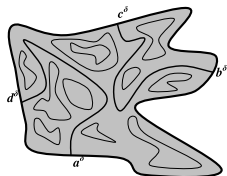
P^δ



In the half-plane \mathbb{H} : for $u \in [0, 1]$,

$$\begin{aligned} P(\mathbb{H}; [1-u, 1] \leftrightarrow [\infty, 0]) \\ = \frac{\sqrt{1 - \sqrt{1-u}}}{\sqrt{1-\sqrt{u}} + \sqrt{1-\sqrt{1-u}}}. \end{aligned}$$

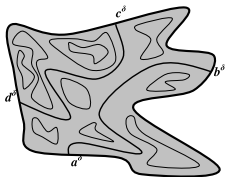
Q^δ



This is a special case of a hypergeometric formula for crossings in a general FK model. In the Ising case it becomes algebraic and furthermore can be rewritten in several ways.

FK-Ising crossing probability:

P^δ

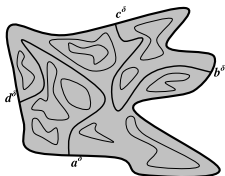


In the unit disc \mathbb{D} : for $\theta \in [0, \frac{\pi}{2}]$,

$$\frac{P(\mathbb{D}; [-e^{-i\theta}, -e^{i\theta}] \leftrightarrow [e^{-i\theta}, e^{i\theta}])}{P(\mathbb{D}; [e^{i\theta}, -e^{-i\theta}] \leftrightarrow [-e^{i\theta}, e^{-i\theta}])}$$

$$= \frac{\sin \frac{\theta}{2}}{\sin(\frac{\pi}{4} - \frac{\theta}{2})} =: r(\theta).$$

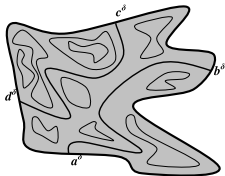
Q^δ



Remark: *This macroscopic formula formally coincides with the relative weights corresponding to two different possibilities of crossings inside microscopic rhombi in the FK-Ising model on isoradial graphs.*

FK-Ising crossing probability:

P^δ

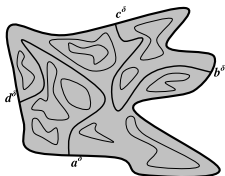


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Q^δ

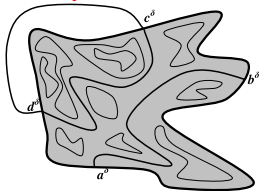


Remark: In particular, the $Y - \Delta$ relation holds, i.e.,

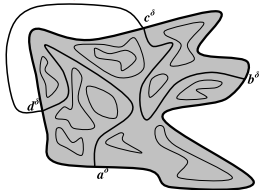
$$r(\alpha + \beta) = \frac{r(\alpha) + r(\beta) + \sqrt{2} \cdot r(\alpha)r(\beta)}{1 - r(\alpha)r(\beta)}$$

FK-Ising crossing probability. External coupling.

$$\frac{\sqrt{2}P^\delta}{\sqrt{2}P^\delta + Q^\delta}$$

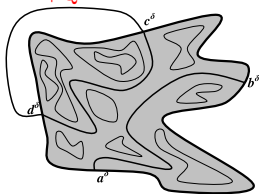


$$\frac{Q^\delta}{\sqrt{2}P^\delta + Q^\delta}$$

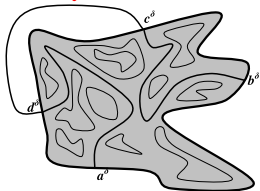


FK-Ising crossing probability. External coupling.

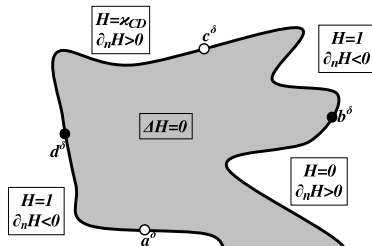
$$\frac{\sqrt{2}P^\delta}{\sqrt{2}P^\delta + Q^\delta}$$



$$\frac{Q^\delta}{\sqrt{2}P^\delta + Q^\delta}$$

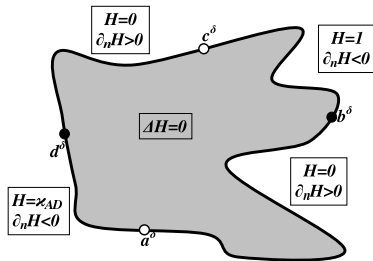


Construct a discrete holomorphic observable F_{CD}^δ . Then for an (almost) discrete harmonic function $H_{CD} = \text{Im} \int (F_{CD}^\delta(z))^2 d^\delta z$:

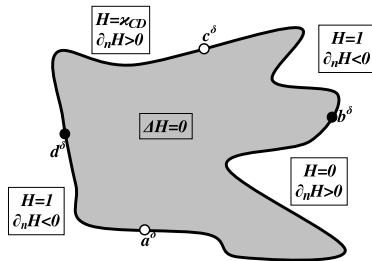


FK-Ising crossing probability. External coupling.

Construct a discrete holomorphic observable F_{AD}^δ . Then for an (almost) discrete harmonic function $H_{AD} = \text{Im} \int (F_{AD}^\delta(z))^2 d^\delta z$:

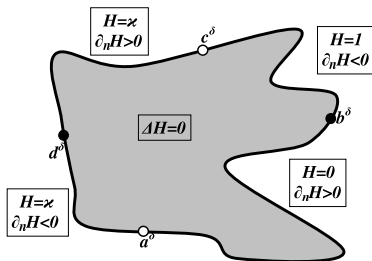


Construct a discrete holomorphic observable F_{CD}^δ . Then for an (almost) discrete harmonic function $H_{CD} = \text{Im} \int (F_{CD}^\delta(z))^2 d^\delta z$:



FK-Ising crossing probability. Conformal mapping.

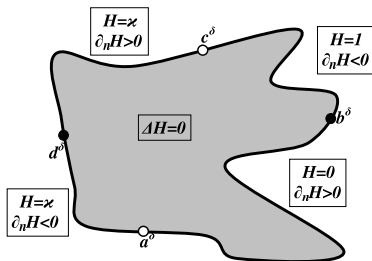
For some linear combination of observables $F^\delta := \alpha F_{AD}^\delta + \beta F_{CD}^\delta$ and $H = \text{Im} \int (F^\delta(z))^2 d^\delta z$ one has:



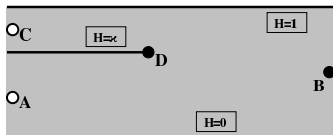
where the value κ^δ is determined by the ratio of crossing probabilities P^δ / Q^δ .

FK-Ising crossing probability. Conformal mapping.

For some linear combination of observables $F^\delta := \alpha F_{AD}^\delta + \beta F_{CD}^\delta$ and $H = \text{Im} \int (F^\delta(z))^2 d^\delta z$ one has:



Uniformization:

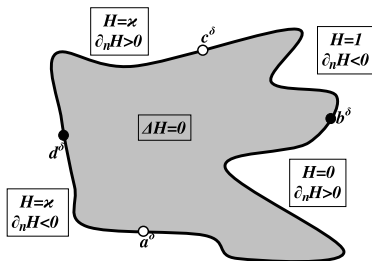


where the value κ^δ is determined by the ratio of crossing probabilities P^δ / Q^δ .

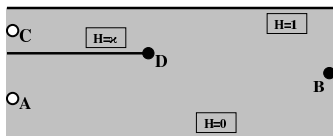
κ is uniquely determined by the conformal modulus of $(\Omega^\delta, a^\delta, b^\delta, c^\delta, d^\delta)$

FK-Ising crossing probability. Conformal mapping.

For some linear combination of observables $F^\delta := \alpha F_{AD}^\delta + \beta F_{CD}^\delta$ and $H = \text{Im} \int (F^\delta(z))^2 d^\delta z$ one has:



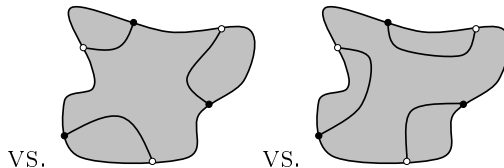
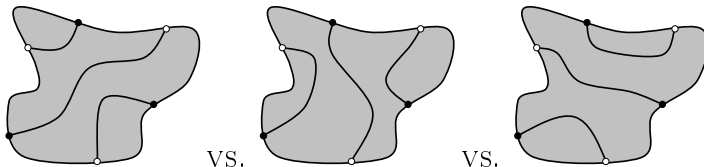
Uniformization:



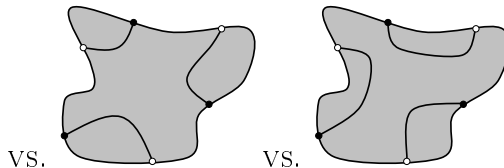
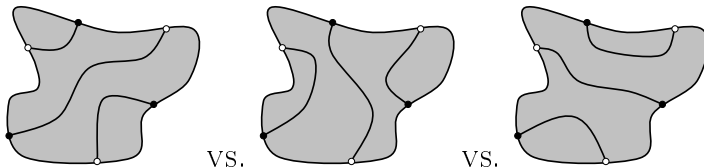
where the value κ^δ is determined by the ratio of crossing probabilities P^δ / Q^δ .

Convergence $H^\delta \rightarrow H$ for rough domains needs some work (see arXiv:0910.2045).

FK-Ising crossing probabilities: more points?



FK-Ising crossing probabilities: more points?



THANK YOU!