2D Ising model at and near criticality

what we can prove and what we still would like to understand

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Sample of a critical 2D Ising configuration

[with two disorders inserted] © Clément Hongler (EPFL, Lausanne)

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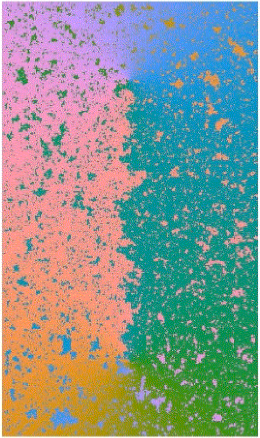
Ising model = random assignment of $+1/-1$ spins to lattice vertices (or faces)

Q: I heard this is called a percolation?



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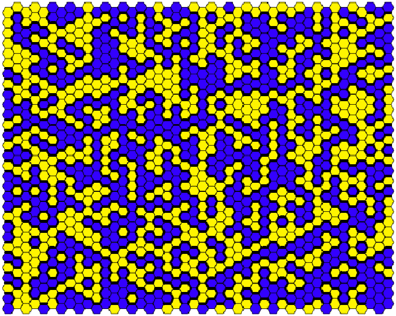
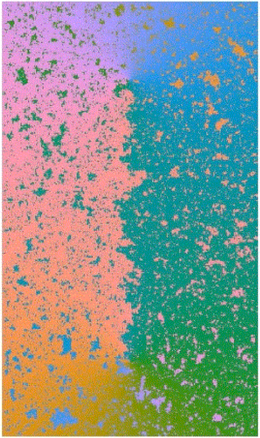


[sample of a honeycomb percolation]



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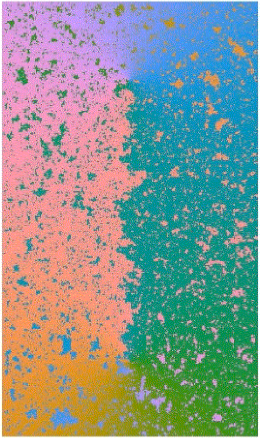
according to some *probabilities*:

$$\mathbb{P}[conf] \propto x^{\#(''+-'')},$$

where $x = e^{-2\beta J} = e^{-2J/kT} \in [0,1]$ has the same monotonicity as $T \in [0,+\infty]$.



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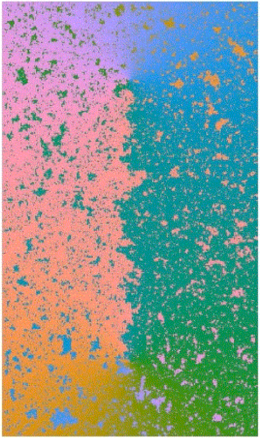
In other words, the *partition function* is

$$\mathcal{Z} = \sum_{\sigma \in \{\pm 1\}^{|V|}} \exp\left[-\beta \sum_{u \sim v} J_{uv} \sigma_u \sigma_v\right].$$



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Six "short" stories:

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6. Would like to understand: renormalization, near-critical regimes

Lenz-Ising model (1920–1941): phase transition in 1D and 2D

Lenz, 1920: $\mathbb{P}[conf] \propto x^{\#(''+-'')}$

$$\propto \exp(-\beta[J\sum_{n=0}^{N-1}\sigma_n\sigma_{n+1}+h\sum_{n=0}^{N}\sigma_n]);$$

- No external magnetic field: $h=0$;
- *Boundary conditions*: $\sigma_0=\sigma_N=+1$.

$\#0\ldots \qquad \ldots\#\lfloor rN\rfloor\ldots \qquad \ldots\#N$

$+1|+1||-1|-1|-1|\ldots|+1|+1|+1|-1|-1|\ldots|-1|+1|-1||+1|+1$

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NO PHASE TRANSITION IN 1D

Intuition: It costs only x^2 to have a pair $\ldots+1||-1\ldots-1||+1\ldots$ of "domain walls" surrounding $\sigma_{\lfloor rN\rfloor}$, so we see many of those.

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`Ising: ... I discussed the result of my paper widely with Professor Lenz and with Dr. Wolfgang Pauli, who at that time was teaching in Hamburg. There was some disappointment the linear model did not show the expected ferromagnetic properties.`

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1925–...: By analogy with 2×2 *transfer matrices* computations performed by Ising in 1D, it was believed that the model does not exhibit a phase transition in 2D and 3D as well (the size of transfer matrices in 2D is $2^N \times 2^N$, so nobody knew how to analyze them). More involved models to explain ferromagnetism were proposed.

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Answer *[Ising, 1925]*: NO PHASE TRANSITION IN 1D.

[Peierls, 1936]: THERE IS A PHASE TRANSITION IN 2(+)D.

Intuition (*combinatorics*): Consecutive "$+-$" contours surrounding a given site should be longer and longer. Each costs us $x^{\#\mathrm{edges}}$, so it is not affordable to have many, provided x is small enough.

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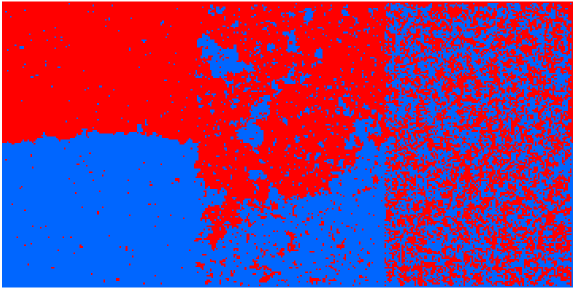
[Kramers-Wannier, 1941]: Combinatorial duality argument based on the algebraic trick due to van der Waerden $\Rightarrow$ a prediction for the critical value $x_{\mathrm{crit}}=\tan\frac{\pi}{8}=\sqrt{2}-1$ on the square lattice.

Onsager's solution (1944–1952) and orthogonal polynomials



$x < x_{\mathrm{crit}}$ $x \approx x_{\mathrm{crit}}$ $x > x_{\mathrm{crit}}$

[Dobrushin boundary values: two marked points a, b on the boundary; -1 on the arc (ab), $+1$ on the opposite arc (ba)]

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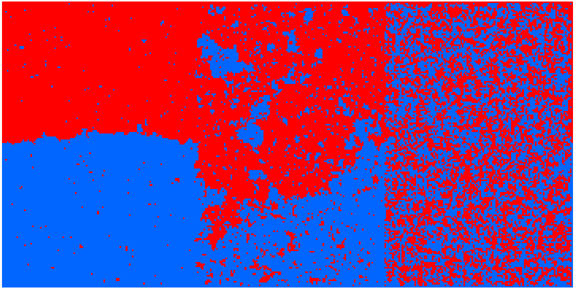

[Onsager, 1944]: diagonalization of $2^N \times 2^N$ transfer matrices in 2D
(involves highly nontrivial algebraic structure of those)

$\Rightarrow$ an explicit formula for the free energy of 2D Ising model

$\Rightarrow$ first breakthrough results about the (near-)critical behavior

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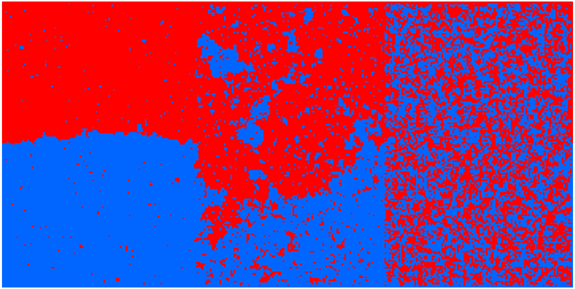

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[Kaufman-Onsager, 1948-49, unpublished]: some spin-spin expectations $\Rightarrow$ *scaling exponent* $\frac{1}{8}$ for the magnetization

$\mathbb{E}[\sigma_*] \asymp (x_{\mathrm{crit}} - x)^{\frac{1}{8}}$ as $\begin{matrix} x \to x_{\mathrm{crit}}, \\ N = \infty, \end{matrix}$ or $\mathbb{E}[\sigma_*] \asymp N^{-\frac{1}{8}}$ as $\begin{matrix} N \to \infty, \\ x = x_{\mathrm{crit}}. \end{matrix}$

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[Yang, 1952, Phys. Rev.]: "The spontaneous magnetization of a two-dimensional Ising model", first published rigorous derivation

[Szegö '1952, Comm. Sém. Math. Univ. Lund] "On certain Hermitian forms associated with the Fourier series of a positive function"

Historical comments: [R. J. Baxter, arXiv:1103.3347 & 1211.2665]

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Subtle point: asymptotics of Toeplitz determinants $\det[f_{j-k}]_{0,0}^{n,n}$ $\leftrightsquigarrow$ orthogonal polynomials w.r.t the weight $f(e^{i\theta}) = \sum_{s\in\mathbb{Z}} f_s e^{is\theta}$

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`... we talked to Kakutani and Kakutani talked to Szego, and the mathematicians got there first.`

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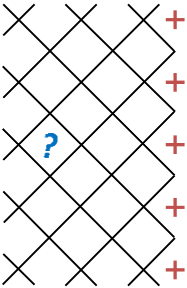
Magnetization in the zig-zag half-plane at criticality: *[Ch.–Hongler, unpublished]*

$$\mathbb{E}^{+}_{\mathbb{H}_\diamond}[\sigma_{2n}] = \left(\frac{2}{\pi}\right)^n \cdot \prod_{\ell=1}^{2n-1} \left(1 - \frac{1}{4\ell^2}\right)^{\lfloor \frac{1}{2}\ell \rfloor - n}$$

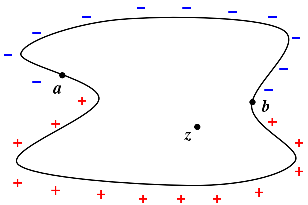


[links with the spectral theory of Jacobi matrices are available for the 'layered' Ising model in $\mathbb{H}_\diamond$]

Conformal Field Theory predictions (1984–1990s)

1952–1984: essential combinatorial simplifications (reduction to the dimer model) were done and many scaling exponents explicitly computed in the plane or the half-plane *[McCoy–Wu, 1973]*.

[Belavin–Polyakov–Zamolodchikov, 1984]: scaling limits of correlation functions should be *conformally covariant*.



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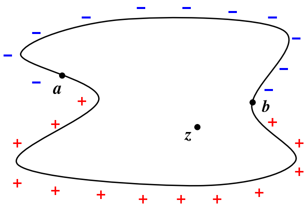
For instance, if $\Omega_\delta \to \Omega$ as $\delta \to 0$, it should be

$$\delta^{-\frac{1}{8}} \mathbb{E}^{ab}_{\Omega_\delta}[\sigma(z^\delta)] \quad \to \quad \mathcal{C} \cdot \langle \sigma_z \rangle^{ab}_{\Omega},$$

with $\langle \sigma_z \rangle^{ab}_{\Omega} = |\phi'(z)|^{\frac{1}{8}} \langle \sigma_{\phi(z)} \rangle^{\phi(a)\phi(b)}_{\Omega'}$
for all conformal mappings $\phi : \Omega \to \Omega'$.



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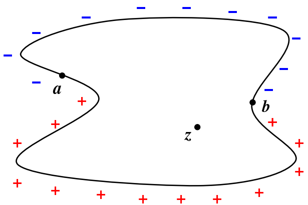

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Intuition: scaling covariance + rotational invariance [**?**]
+ locality of the model [**?** $\Rightarrow$ **?**] conformal covariance

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Together with some other "algebraic" assumptions (finite number of primary fields, concrete scaling exponents, ...), this allows one to identify all the scaling limits of correlation functions as (particular) solutions to some PDEs provided by

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For instance, it should be *[Cardy, 1984; Burkhardt–Guim, 1993]* $\langle \sigma_{z_1} \dots \sigma_{z_k} \rangle_\Omega^+ = \prod_{s=1}^k |\phi'(z_s)|^{\frac{1}{8}} \cdot \langle \sigma_{\phi(z_1)} \dots \sigma_{\phi(z_k)} \rangle_{\Omega'}^+$ and

$$\langle \sigma_{z_1} \dots \sigma_{z_k} \rangle_{\mathbb{H}}^+ = \prod_{s=1}^{k} (2\,\mathrm{Im}\, z_s)^{-\frac{1}{8}} \times \left[2^{-\frac{k}{2}} \sum_{\mu \in \{\pm 1\}^k} \prod_{s<m} \left| \frac{z_s - z_m}{z_s - \overline{z}_m} \right|^{\frac{\mu_s \mu_m}{2}} \right]^{\frac{1}{2}}$$

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In two words: CFT provides remarkable "algebraic" techniques (e.g., some special Virasoro algebra representations play an extremely important role) that eventually lead to very *concrete formulae* for correlation functions. **Case closed. Wonderful!**

But...

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But... should one prove that discrete correlation functions indeed have conformally covariant limits as $\delta \to 0$? ... [it depends] ...

Conformal Field Theory predictions (1984–1990s)

[Belavin–Polyakov–Zamolodchikov, 1984]: scaling limits of correlation functions should be *conformally covariant*.

For instance, it should be *[Cardy, 1984; Burkhardt–Guim, 1993]* $\langle \sigma_{z_1} \ldots \sigma_{z_k} \rangle_\Omega^+ = \prod_{s=1}^k |\phi'(z_s)|^{\frac{1}{8}} \cdot \langle \sigma_{\phi(z_1)} \ldots \sigma_{\phi(z_k)} \rangle_{\Omega'}^+$ and

$$\langle \sigma_{z_1} \ldots \sigma_{z_k} \rangle_{\mathbb{H}}^+ = \prod_{s=1}^k (2\,\mathrm{Im}\, z_s)^{-\frac{1}{8}} \times \Big[2^{-\frac{k}{2}} \sum_{\mu \in \{\pm 1\}^k} \prod_{s<m} \Big| \frac{z_s - z_m}{z_s - \overline{z}_m} \Big|^{\frac{\mu_s \mu_m}{2}} \Big]^{\frac{1}{2}}$$

Theorem (Ch., Hongler, Izyurov, *Ann. Math. 2015*):

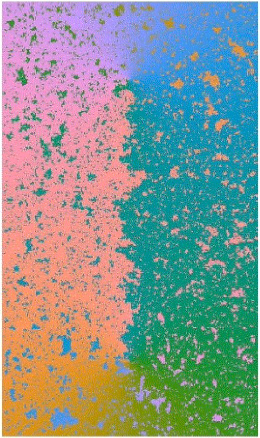
If $\Omega_\delta \to \Omega$ as $\delta \to 0$, $\delta^{-\frac{n}{8}} \mathbb{E}_{\Omega_\delta}^+ [\sigma(z_1) \ldots \sigma(z_k)] \to \langle \sigma_{z_1} \ldots \sigma_{z_k} \rangle_\Omega^+$.

Geometry: conformal loop ensembles [Sheffield–Werner, 2012]

Question: What could be a good candidate for the *scaling limit of loops* and interfaces surrounding Ising clusters?

• [single interfaces (e.g., with Dobrushin $+1/-1$ boundary conditions): *Schramm's* SLE_κ *curves*]

In one line: non-self-intersecting 2D curves, were *introduced by Oded Schramm in 2000*, are defined dynamically via the classical Loewner evolution *[1923]* with a 1D white noise input, can be analyzed combining *geometrical complex analysis* and *stochastic calculus*.



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• collection of the outermost loops (say, for all "+" boundary conditions)

Intuition: Distribution of loops should
(a) be *conformally invariant*
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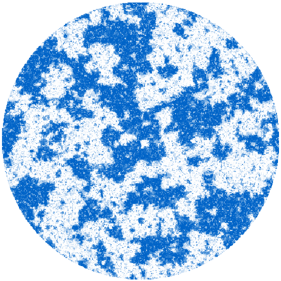

a sample with **free b.c.**, © C. Hongler

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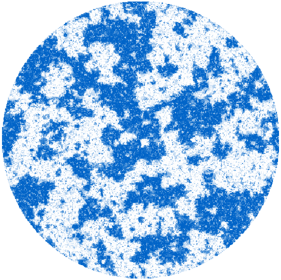

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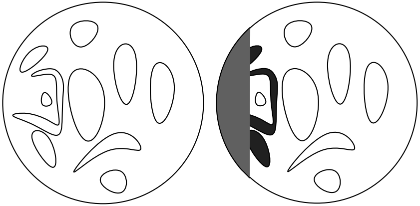

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Given the set of loops intersecting $D_2 \setminus D_1$, the conditional law of the remaining loops is an independent CLE in each component of the (interior of the) complement of this set.

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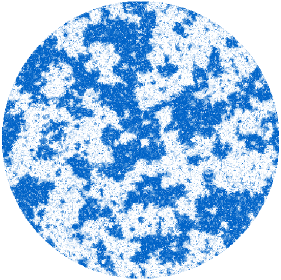

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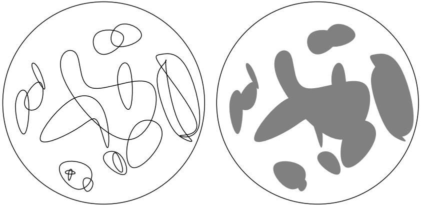

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Loop-soup construction:

• sample a (countable) set of Brownian loops using some natural *conformally-friendly* Poisson process of intensity c

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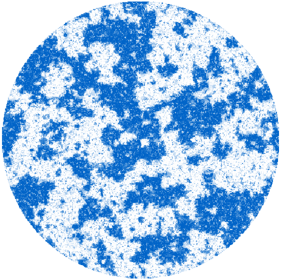

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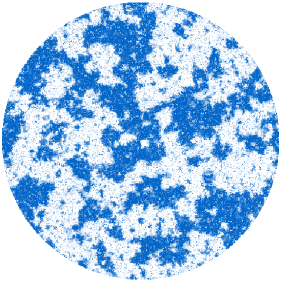

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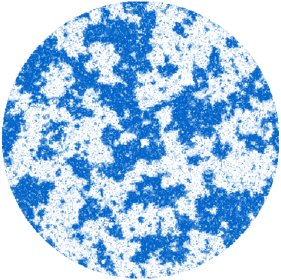

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but should one prove that discrete interfaces/loops indeed have conformally invariant limits as $\delta \to 0$?

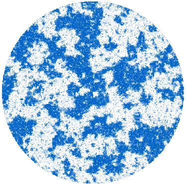
... [again, it depends] ...

From boundary value problems for discrete holomorphic functions to convergence of correlations and interfaces

Conformal Field Theory

Assuming conformal covariance of correlation functions appearing in the limit, they should form one of "algebraic structures", parameterized by a central charge.

Lattice models (e.g., Ising)



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Assuming conformal invariance of curves and loops appearing in the limit, there exists a unique family of "loop ensembles", parameterized by some intensity.

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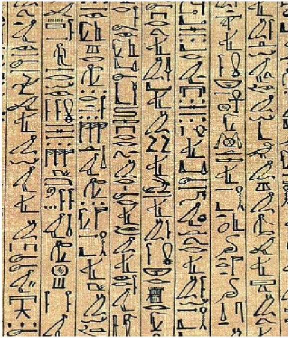

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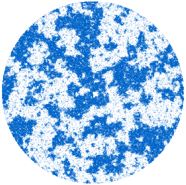

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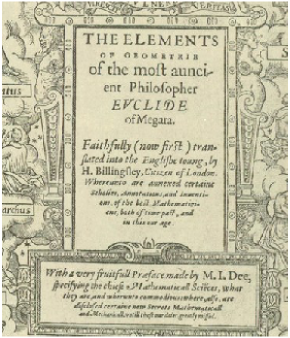

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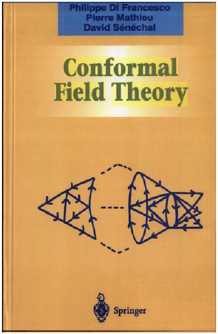
From boundary value problems for discrete holomorphic functions to convergence of correlations and interfaces



Deep interactions 'in continuum', cf.

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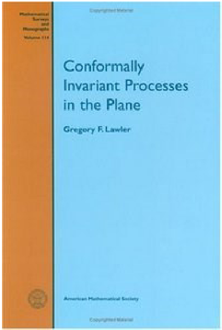
[..........]



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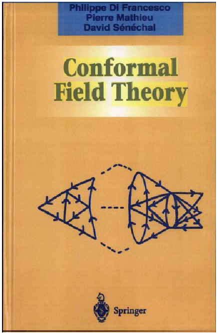

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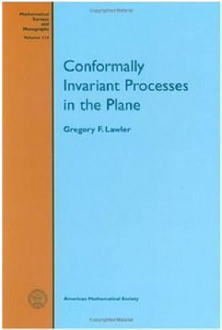

[..........]

But can one prove that these beautiful 'algebraic' and 'geometric' structures indeed arise in the limit of some lattice model as $\delta \to 0$ (e.g., the Ising model, which contains a lot of 'integrability' inside)?

From boundary value problems for discrete holomorphic functions to convergence of correlations and interfaces

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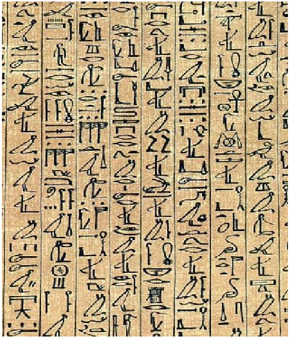

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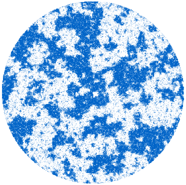
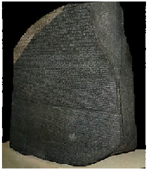
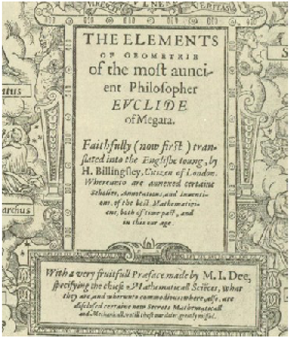




[Rosetta Stone]

Conformal Geometry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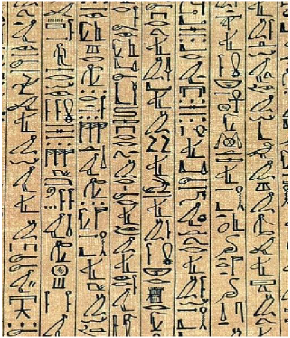

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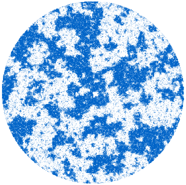

Main tool: discrete holomorphic functions

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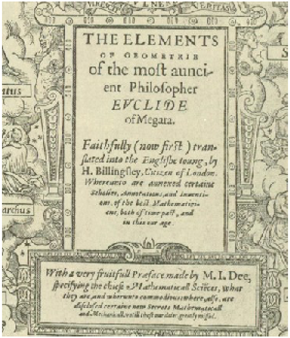

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From boundary value problems for discrete holomorphic functions to convergence of correlations and interfaces

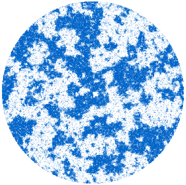
Conformal Field Theory

Assuming conformal covariance of correlation functions appearing in the limit, they should form one of "algebraic structures", parameterized by a central charge.

[Ising model, 2006–...]:

proofs of convergence for re-scaled correlation functions (fermions, energy densities, spins, ...)

Lattice models (e.g., Ising)



Main tool: discrete holomorphic functions

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Assuming conformal invariance of curves and loops appearing in the limit, there exists a unique family of "loop ensembles", parameterized by some intensity.

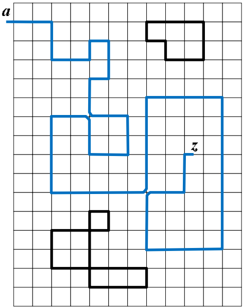
[Ising model, 2006–...]:

proofs of convergence for interfaces and their ensembles (various b.c. and topologies)

From boundary value problems for discrete holomorphic functions to convergence of correlations and interfaces

Main tool: discrete holomorphic functions

$$F_a^\delta(z) := \sum_{\mathrm{loops}+[a \rightsquigarrow z]} x^{\#\mathrm{edges}} e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z)}.$$



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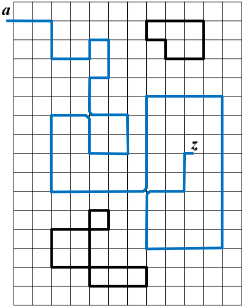

- "discrete fermions" played a crucial role in many aspects of the planar Ising model starting with the very first derivations;
- existence of "holomorphic fields" provided a strong evidence for the conformal invariance of the limit and its CFT description

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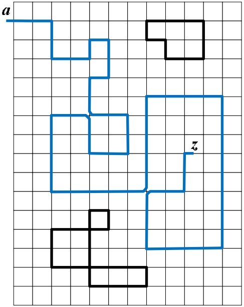

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- existence of "holomorphic fields" provided a strong evidence for the conformal invariance of the limit and its CFT description
- still, much (hard) work is needed to understand how to use these structures for the rigorous analysis when $\Omega_\delta \to \Omega$ as $\delta \to 0$, especially in rough domains formed by fractal interfaces.

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Some papers/preprints (convergence of correlations):

• basic observables: *[Smirnov '06]*, universality: *[Ch.,Smirnov '09]*
• energy density field: *[Hongler,Smirnov '10]*, *[Hongler '10]*

• spinor version, some ratios of spin correlations: *[Ch.,Izyurov '11]*
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• mixed correlations in multiply-connected Ω's *[on the way]*
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From boundary value problems for discrete holomorphic functions to convergence of correlations and interfaces

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Strategy of proving the convergence of correlation functions:

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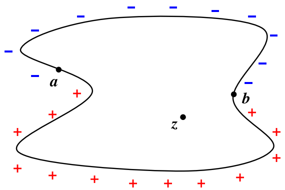

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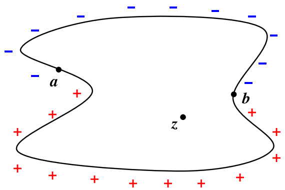

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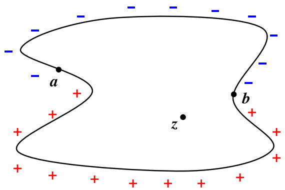

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• prove the convergence $\gamma_\delta \to \gamma$ and recover the law of γ using this family of martingales [some probabilistic techniques are needed].

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Example: to handle $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_z]$, one should consider the following b.v.p.:

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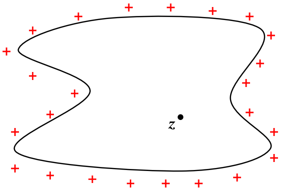

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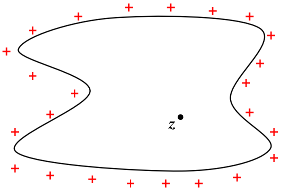

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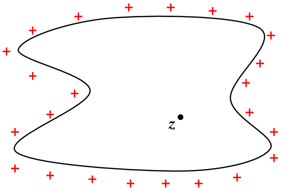

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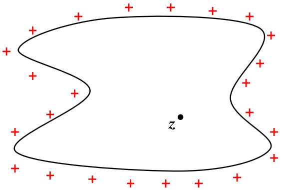

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Conformal exponent $\frac{1}{8}$: for any conformal map $\phi : \Omega \to \Omega'$,

- $f_{[\Omega,a]}(w) = f_{[\Omega',\phi(a)]}(\phi(w)) \cdot (\phi'(w))^{1/2}\,;$
- $\mathcal{A}_\Omega(z) = \mathcal{A}_{\Omega'}(\phi(z)) \cdot \phi'(z) + \frac{1}{8} \cdot \phi''(z)/\phi'(z)\,.$

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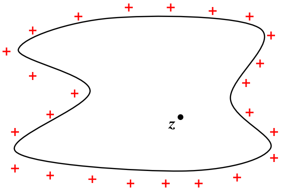

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Technical issues: • to find proper combinatorics in discrete;
- to handle tricky boundary conditions (Dirichlet for $\int \mathrm{Re}[f^2 dz]$);
- to prove convergence, incl. near singularities [complex analysis];
- to recover the normalization of $\mathbb{E}^+_{\Omega_\delta}[\sigma_z]$ [probabilistic techniques].

What we still would like to understand

∼90 years after the Lenz-Ising model was first suggested, even for regular 2D lattices, there are many hard questions remaining, especially for mathematicians who once got there...

• *renormalization: not only nearest-neighbor interactions* and/or the "massive" regime $T - T_{\mathrm{crit}} \sim m \cdot \delta$ as $\delta \to 0$.

[recent progress by Giuliani–Greenblatt–Mastropietro '12]
(energy density field in $\mathbb{C}$, *spin field remains a challenge*)

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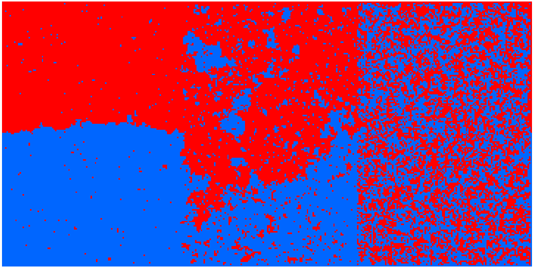

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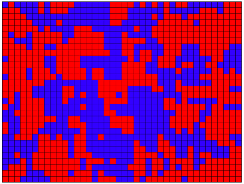

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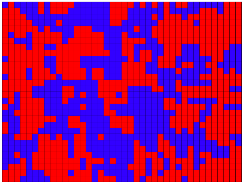

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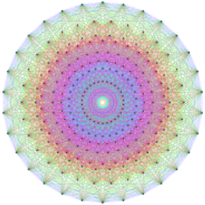

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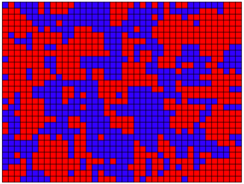

Merci!