2D Ising model: correlations via boundary value problems

Dmitry Chelkak (Steklov Institute, St. Petersburg & Université de Genève)

[ Sample of a critical 2D Ising configuration (with two disorders), © Clément Hongler (EPFL) ]

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2D Ising model: correlations via boundary value problems

- Nearest-neighbor Ising model in 2D
  - dimers and fermionic observables
  - discrete holomorphicity at criticality
  - spinor observables and spin correlations
- A classical computation revisited: explicit formulae for “diagonal” two-point correlations in $\mathbb{Z}^2$ via full-plane spinors
- Conformal covariance at criticality
  - Riemann boundary value problems for holomorphic spinors in continuum
  - Explicit formulae (CFT prediction)
  - Convergence (Ch.–Hongler–Izyurov)
  - Other fields (convergence, fusion rules)

Extended version: arXiv:1605.09035
**Nearest-neighbor Ising (or Lenz-Ising) model in 2D**

**Definition**: *Lenz-Ising model* on a planar graph $G^*$ (dual to $G$) is a random assignment of $+/−$ spins to vertices of $G^*$ (faces of $G$).

Q: I heard this is called a (site) percolation?

[sample of a honeycomb percolation]
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Q: I heard this is called a (site) percolation?
A: .. according to the following probabilities:

$$
P[\text{conf. } \sigma \in \{\pm 1\}^{V(G^*)}] \propto \exp \left[ \beta \sum_{e=\langle uv \rangle} J_{uv} \sigma_u \sigma_v \right] 
\propto \prod_{e=\langle uv \rangle : \sigma_u \neq \sigma_v} x_{uv},$$

where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv \rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$. 

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**Disclaimer:** no external magnetic field.

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\]

where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv \rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.

- It is also convenient to use the parametrization $x_{uv} = \tan(\frac{1}{2} \theta_{uv})$.
- Working with subgraphs of *regular lattices*, one can consider the *homogeneous model* in which all $x_{uv}$ are equal to each other.
Phase transition (e.g., on $\mathbb{Z}^2$)

E.g., Dobrushin boundary conditions: $+1$ on $(ab)$ and $-1$ on $(ba)$:

\begin{itemize}
\item Ising (1925): no phase transition in 1D $\leadsto$ doubts about 2+D;
\item Peierls (1936): existence of the phase transition in 2D;
\item Kramers-Wannier (1941): $x_{\text{self-dual}} = \sqrt{2} - 1 = \tan\left(\frac{1}{2} \cdot \frac{\pi}{4}\right)$;
\item Onsager (1944): sharp phase transition at $x_{\text{crit}} = \sqrt{2} - 1$.
\end{itemize}
At criticality (e.g., on $\mathbb{Z}^2$):

- Kaufman-Onsager (1948-49), Yang (1952): **scaling exponent $\frac{1}{8}$** for the magnetization. [via spin-spin correlations in $\mathbb{Z}^2$ at $x \uparrow x_{\text{crit}}$]

- At criticality, for $\Omega_\delta \rightarrow \Omega$ and $u_\delta \rightarrow u \in \Omega$, it should be $\mathbb{E}_{\Omega_\delta}[\sigma_{u_\delta}] \asymp \delta^{\frac{1}{8}}$ as $\delta \rightarrow 0$.

$x = x_{\text{crit}}$
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**Question:** Convergence of (rescaled) spin correlations and conformal covariance of their scaling limits in arbitrary planar domains:

$$
\delta^{-\frac{n}{8}} \cdot E_{\Omega_\delta} [\sigma_{u_{1,\delta}} \cdots \sigma_{u_{n,\delta}}] \rightarrow \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle_{\Omega} \\
= \langle \sigma_{\varphi(u_1)} \cdots \sigma_{\varphi(u_n)} \rangle_{\varphi(\Omega)} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}}
$$

- In the infinite-volume setup other techniques are available, notably “exact bosonization” approach due to J. Dubédat.
2D Ising model as a dimer model (on a non-bipartite graph) [Fisher, Kasteleyn ('60s+),..., Kenyon, Dubédat ('00s+),...]

- Partition function \( \mathcal{Z} = \sum_{\sigma \in \{\pm 1\}^{V(G^*)}} \prod_{e = \langle uv \rangle : \sigma_u \neq \sigma_v} \chi_{uv} \)

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- **Kac–Ward formula (1952–..., 1999–...)**: \( Z^2 = \det[\text{Id} - T] \),
  \[
  T_{e,e'} = \begin{cases} 
  \exp\left[\frac{i}{2}\text{wind}(e, e')\right] \cdot (x_e x_{e'})^{1/2} & \text{if } e' \neq \overline{e} \text{ prolongs } e; \\
  0 & \text{otherwise}. 
  \end{cases}
  \]
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 0 \end{cases} \]

  [ is equivalent to the **Kasteleyn theorem for dimers on** $G_F$ ]

  [ more details in arXiv:1507.08242 (w/ Cimasoni & Kassel) ]
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- Note that \( V(G_F) \cong \{\text{oriented edges and corners of } G\} \)

- **Local relations** for the entries \( K_{a,e}^{-1} \) and \( K_{a,c}^{-1} \) of the inverse Kasteleyn (or the inverse Kac–Ward) matrix: (an equivalent form of) the identity \( K \cdot K^{-1} = \text{Id} \)
Fermionic observables: combinatorial definition [Smirnov’00s]

For an oriented edge $a$ and a midedge $z_e$ (similarly, for a corner $c$),

$$F_G(a, z_e) := \bar{\eta}_a \sum_{\omega \in \text{Conf}_G(a, z_e)} \left[ e^{\frac{i}{2} \text{wind}(a \rightsquigarrow z_e)} \prod_{\langle uv \rangle \in \omega} x_{uv} \right]$$

where $\eta_a$ denotes the (once and forever fixed) square root of the direction of $a$.

- The factor $e^{\frac{i}{2} \text{wind}(a \rightsquigarrow z_e)}$ does not depend on the way how $\omega$ is split into non-intersecting loops and a path $a \rightsquigarrow z_e$.

- **Via dimers on $G_F$:** $F_G(a, c) = \bar{\eta}_c K_{c,a}^{-1}$

$$F_G(a, z_e) = \bar{\eta}_e K_{e,a}^{-1} + \bar{\eta}_e K_{\bar{e},a}^{-1}$$
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- When both $a$ and $e$ are “boundary” edges, the factor $\eta_a e^{-\frac{i}{2}\text{wind}(a \rightsquigarrow z_e)} = \pm \eta_{\bar{e}}$ is fixed and $F_G(a, z_e)$ becomes the partition function of the Ising model (on $G^*$) with Dobrushin boundary conditions.
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- **Local relations:** at criticality, can be thought of as some (strong) form of discrete Cauchy–Riemann equations.

- **Boundary conditions** $F(a, z_e) \in \overline{\eta}_e \mathbb{R}$ ($\overline{e}$ is oriented outwards) uniquely determine $F$ as a solution to an appropriate discrete Riemann-type boundary value problem.
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Fermionic observables per se can be used

- to construct (discrete) martingales for growing interfaces and then to study their convergence to SLE curves [Smirnov(2006), ..., Ch.–Duminil-Copin–Hongler–Kemppainen–Smirnov(2013)]
- to analyze the energy density field [Hongler–Smirnov, Hongler (2010)]

$$\varepsilon_e := \delta^{-1} \cdot [\sigma_e - \sigma_{e^+} - \varepsilon_e^\infty]$$

where $e^\pm$ are the two neighboring faces separated by an edge $e$
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- to analyze the energy density field [Hongler–Smirnov, Hongler (2010)]

$$\varepsilon_e := \delta^{-1} \left[ \sigma_{e^-} - \sigma_{e^+} - \varepsilon_{e^\infty} \right]$$

- but more involved ones are needed to study spin correlations and their limits [Ch.–Izyurov(2011), Ch.–Hongler–Izyurov(2012)]
Spinor observables and spin correlations

- spin configurations on $G^*$
  $\leftrightarrow$ domain walls on $G$
  $\leftrightarrow$ dimers on $G_F$
- Kasteleyn’s theory: $Z = \text{Pf}[K]$

$[K = -K^T$ is a weighted adjacency matrix of $G_F]$
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- Kasteleyn’s theory: $\mathcal{Z} = \text{Pf}[K]$ 
  [$K = -K^\top$ is a weighted adjacency matrix of $G_F$]

- Claim:
  \[
  \mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] = \frac{\text{Pf}[K_{[u_1, \ldots, u_n]}]}{\text{Pf}[K]},
  \]
  where $K_{[u_1, \ldots, u_n]}$ is obtained from $K$ by changing the sign of its entries on slits linking $u_1, \ldots, u_n$ (and, possibly, $u_{\text{out}}$) pairwise.

- More invariant way to think about entries of $K_{[u_1, \ldots, u_n]}$:
  double-covers of $G$ branching over $u_1, \ldots, u_n$
Spinor observables and spin correlations

Main tool: spinors on the double cover $[\Omega_\delta; u_1, \ldots, u_n]$.

$$F_{\Omega_\delta}(z) := \left[ Z_{\Omega_\delta}^+ [\sigma_{u_1} \ldots \sigma_{u_n}] \right]^{-1} \cdot \sum_{\omega \in \text{Conf}_{\Omega_\delta}(u_1, z)} \phi_{u_1,\ldots,u_n}(\omega, z) \cdot x_{\text{crit}}^{\#\text{edges}(\omega)},$$

$$\phi_{u_1,\ldots,u_n}(\omega, z) := e^{-\frac{1}{2} \text{wind}(p(\omega))} \cdot (-1)^{\#\text{loops}(\omega \setminus p(\omega))} \cdot \text{sheet}(p(\omega), z).$$

- wind $(p(\gamma))$ is the winding of the path $p(\gamma) : u_1 \rightarrow u_1 + \frac{\delta}{2} \sim z$;
- $\#\text{loops}$ – those containing an odd number of $u_1, \ldots, u_n$ inside;
- sheet $(p(\gamma), z) = +1$, if $p(\gamma)$ defines $z$, and $-1$ otherwise.

- Note that $F(z^\#) = -F(z^b)$ if $z^\#$, $z^b$ lie over the same edge of $\Omega_\delta$. 
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$$F_{\Omega_\delta}(z) := \left[ Z^+_{\Omega_\delta} [\sigma_{u_1} \cdots \sigma_{u_n}] \right]^{-1} \cdot \sum_{\omega \in \text{Conf}_{\Omega_\delta}(u_1^\rightarrow, z)} \phi_{u_1,\ldots,u_n}(\omega, z) \cdot x_{\text{crit}}^\#\text{edges}(\omega),$$

$$\phi_{u_1,\ldots,u_n}(\omega, z) := e^{-\frac{i}{2}\text{wind}(p(\omega))} \cdot (-1)^\#\text{loops}(\omega \setminus p(\omega)) \cdot \text{sheet}(p(\omega), z).$$

Claim:

$$F_{\Omega_\delta}(u_1 + \frac{3\delta}{2}) = \frac{\mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_1+2\delta} \cdots \sigma_{u_n}]}{\mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_1} \cdots \sigma_{u_n}]}$$

In other words, spatial derivatives of spin correlations are particular values of spinor observables.

- Remark: Both fermionic and spinor observables can be introduced using spin-disorder formalism of Kadanoff and Ceva.
“Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan \frac{1}{2} \theta \leq x_{\text{crit}} = \tan \frac{\pi}{8}$ and $D_n(x) := \mathbb{E}_{C^\wedge} [\sigma(0,0)\sigma(2n,0)]$

where $C^\wedge = \{(k,s): k, s \in \mathbb{Z}, k+s \in 2\mathbb{Z}\}$ is the $\frac{\pi}{4}$-rotated $\mathbb{Z}^2$.

**Theorem:** [B. Kaufman – L. Onsager ’48-49, C.N. Yang ’52]

$$\lim_{n \to \infty} D_n(x) = (1 - \tan^4 \theta)^{\frac{1}{4}} \sim \text{const} \cdot (x_{\text{crit}} - x)^{\frac{1}{4}} \text{ for } x < x_{\text{crit}}$$

[T.T.Wu'66] $D_n(x_{\text{crit}}) = \left(\frac{2}{\pi}\right)^n \prod_{s=1}^{n-1} \left(1 - \frac{1}{4s^2}\right)^{s-n} \sim \text{const} \cdot (2n)^{-\frac{1}{4}}$

Classical reference for many explicit computations (1973):

B.M. McCoy and T.T. Wu “*The two-dimensional Ising model*”
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Let $x = \tan \frac{1}{2}\theta \leq x_{\text{crit}} = \tan \frac{\pi}{8}$ and $D_n(x) := E_{\mathbb{C}^\diamond} [\sigma_{(0,0)} \sigma_{(2n,0)}]$ where $\mathbb{C}^\diamond = \{(k, s) : k, s \in \mathbb{Z}, k+s \in 2\mathbb{Z}\}$ is the $\frac{\pi}{4}$-rotated $\mathbb{Z}^2$.

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**Historical comments:** [see R.J. Baxter, arXiv:1103.3347 & 1211.2665 for more details]

Onsager: ... I have found a general formula for the evaluation of Toeplitz matrices. The only thing I did not know was how to fill out the holes in the mathematics and show the epsilons and the deltas and all of that...

... we talked to Kakutani and Kakutani talked to Szego, and the mathematicians got there first.
“Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan \frac{1}{2} \theta \leq x_{\text{crit}} = \tan \frac{\pi}{8}$ and $D_{n+1}(x) := \mathbb{E}_{\mathcal{C}^\diamond} [\sigma(-\frac{3}{2},0)\sigma(2n+\frac{1}{2},0)]$

Local relations: $F_{\mathcal{C}^\diamond}(d) = \frac{m}{4} \sum_{d' \sim d} F_{\mathcal{C}^\diamond}(d'), \quad m := \sin(2\theta) \leq 1.$

[Above, we focus on purely real values of the spinor observable on one particular type of corners.] **Note that** $m = 1$ **iff** $x = x_{\text{crit}}$. 
“Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan \frac{1}{2} \theta \leq x_{\text{crit}} = \tan \frac{\pi}{8}$ and $D_{n+1}(x) := \mathbb{E}_{C_\circ} \left[ \sigma(-\frac{3}{2},0) \sigma(2n+\frac{1}{2},0) \right]$

Local relations: $F_{C_\circ}(d) = \frac{m}{4} \sum_{d' \sim d} F_{C_\circ}(d')$, $m := \sin(2\theta) \leq 1$.

For $s \geq 0$, denote $Q_{n,s}(e^{it}) := \sum_{k \in \mathbb{Z}: k+s \in 2\mathbb{Z}} e^{\frac{1}{2}ikt} F_{C_\circ}(k, s)$.

Local relations $\Rightarrow Q_{n,s}(e^{it}) = \left(\frac{m}{2} \cos \frac{t}{2}\right) \cdot (Q_{n,s-1}(e^{it})+Q_{n,s+1}(e^{it}))$.

Boundedness as $s \to \infty \Rightarrow Q_{n,1}(e^{it}) = \left[ \frac{1-(1-(m \cos \frac{t}{2})^2)^{\frac{1}{2}}}{m \cos \frac{t}{2}} \right] Q_{n,0}(e^{it})$. 
“Diagonal” correlations in $\mathbb{Z}^2$: classical computation revisited

Let $x = \tan \frac{1}{2} \theta \leq x_{\text{crit}} = \tan \frac{\pi}{8}$ and $D_{n+1}(x) := \mathbb{E}_C \diamond [\sigma(-\frac{3}{2},0) \sigma(2n+\frac{1}{2},0)]$

Combinatorics of spinor observables $\Rightarrow$ the following values on $\mathbb{R}$:

$$D_{n+1} Q_{n,0}(e^{it}) = 0 + D_n + \ldots + D^*_n e^{int} + 0$$

$$w(e^{it}) \cdot D_{n+1} Q_{n,0}(e^{it}) = \ldots + D_{n+1} + 0 + q^2 D^*_{n+1} e^{int} + \ldots$$

where $w(e^{it}) = |1 - q^2 e^{it}|$, $q := \tan \theta \leq 1$ and $D^*_n := D_n(\tan(\frac{\pi}{4} - \theta))$. 
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Let $x = \tan \frac{1}{2} \theta \leq x_{\text{crit}} = \tan \frac{\pi}{8}$ and $D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\infty} [\sigma(-\frac{3}{2},0) \sigma(2n+\frac{1}{2},0)]$

Therefore, the values of these full-plane spinor observables on the real line are coefficients of certain orthogonal polynomials $Q_n$ wrt $w(e^{i\theta})$ [which are simply Legendre polynomials if $x = x_{\text{crit}}$].

$\implies$ one can express $D_{n+1}, D_{n+1}^*$ via $D_n, D_n^*$ and norms of $Q_n$, where $w(e^{i\theta}) = |1 - q^2 e^{i\theta}|$, $q := \tan \theta \leq 1$ and $D_n^* := D_n(\tan(\frac{\pi}{4} - \theta))$. 
Conformal covariance of spin correlations at criticality

- Three local primary fields: 
  \(1, \sigma\) (spin), \(\varepsilon\) (energy density);
  Scaling exponents: \(0, \frac{1}{8}, 1\).

- CFT prediction:
  
  If \(\Omega_\delta \to \Omega\) and \(u_{k,\delta} \to u_k\) as \(\delta \to 0\), then
  
  \[
  \delta^{-\frac{n}{8}} \cdot \mathbb{E}^+_{\Omega_\delta} [\sigma_{u_1,\delta} \cdots \sigma_{u_n,\delta}] \to C_\sigma \cdot \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle^+_{\Omega}
  \]

  where \(C_\sigma\) is a lattice-dependent constant,

  \[
  \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle^+_{\Omega} = \langle \sigma_{\varphi(u_1)} \cdots \sigma_{\varphi(u_n)} \rangle^+_{\Omega'} \cdot \prod_{s=1}^n |\varphi'(u_s)|^{\frac{1}{8}}
  \]

  for any conformal mapping \(\varphi : \Omega \to \Omega'\), and

  \[
  \left[ \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle^+_{\mathbb{H}} \right]^2 = \prod_{1 \leq s \leq n} (2 \text{Im } u_s)^{-\frac{1}{4}} \times \sum_{\mu \in \{\pm 1\}^n} \prod_{s < m} \left| \frac{u_s - u_m}{u_s - \overline{u}_m} \right|^{\frac{\mu_s \mu_m}{2}}
  \]
Conformal covariance of spin correlations at criticality

- Three local primary fields: $1$, $\sigma$ (spin), $\varepsilon$ (energy density);
  Scaling exponents: $0$, $\frac{1}{8}$, $1$.

- **Theorem:** [Ch.–Hongler–Izyurov]
  If $\Omega_\delta \to \Omega$ and $u_{k, \delta} \to u_k$ as $\delta \to 0$, then
  $$\delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega, \delta}^+ [\sigma_{u_1, \delta} \cdots \sigma_{u_n, \delta}] \to C_\sigma \cdot \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle_\Omega^+$$

  where $C_\sigma$ is a lattice-dependent constant,
  $$\langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle_\Omega^+ = \langle \sigma_{\varphi(u_1)} \cdots \sigma_{\varphi(u_n)} \rangle_{\Omega'}^+ \cdot \prod_{s=1}^n |\varphi'(u_s)|^{\frac{1}{8}}$$

  for any conformal mapping $\varphi : \Omega \to \Omega'$, and

  $$\left[ \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle_{\mathbb{H}}^+ \right]^2 = \prod_{1 \leq s \leq n} (2 \text{Im } u_s)^{-\frac{1}{4}} \times \sum_{\mu \in \{\pm 1\}^n} \prod_{s < m} \left| \frac{u_s - u_m}{u_s - \overline{u}_m} \right|^{\mu_s \mu_m}$$
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  \[
  \delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_1,\delta} \cdots \sigma_{u_n,\delta}] \rightarrow_{\delta \rightarrow 0} C_n^\sigma \cdot \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle_\Omega^+
  \]

General strategy: • **in discrete:** encode spatial derivatives as values of discrete holomorphic spinors $F^\delta$ that solve some **discrete Riemann-type boundary value problems**;

- **discrete$\rightarrow$continuum:** prove convergence of $F^\delta$ to the solutions $f$ of the similar continuous b.v.p. [non-trivial technicalities];
- **continuum$\rightarrow$discrete:** find the limit of (spatial derivatives of) using the convergence $F^\delta \rightarrow f$ [via coefficients at singularities].
Conformal covariance of spin correlations at criticality

Example: to handle $\mathbb{E}_{\Omega}^{+}[\sigma_u]$, one should consider the following b.v.p.:

- $f(z^\#) \equiv -f(z^\flat)$, branches over $u$;
- $\text{Im} \left[ f(\zeta) \sqrt{n(\zeta)} \right] = 0$ for $\zeta \in \partial \Omega$;
- $f(z) = \frac{1}{\sqrt{z-u}} + \ldots$
Conformal covariance of spin correlations at criticality

**Example:** to handle $\mathbb{E}_{\Omega_{\delta}}^{+}[\sigma_{u}]$, one should consider the following b.v.p.:

- $f(z^\sharp) \equiv -f(z^\flat)$, branches over $u$;
- $\text{Im} \left[ f(\zeta) \sqrt{n(\zeta)} \right] = 0$ for $\zeta \in \partial \Omega$;
- $f(z) = \frac{1}{\sqrt{z-u}} + 2A_{\Omega}(u) \cdot \sqrt{z-u} + \ldots$

**Claim:** If $\Omega_{\delta}$ converges to $\Omega$ as $\delta \to 0$, then

- $(2\delta)^{-1} \log \left[ \frac{\mathbb{E}_{\Omega_{\delta}}^{+}[\sigma_{u_{\delta}}+2\delta]}{\mathbb{E}_{\Omega_{\delta}}^{+}[\sigma_{u_{\delta}}]} \right] \to \text{Re} \left[ A_{\Omega}(u) \right]$;
- $(2\delta)^{-1} \log \left[ \frac{\mathbb{E}_{\Omega_{\delta}}^{+}[\sigma_{u_{\delta}}+2i\delta]}{\mathbb{E}_{\Omega_{\delta}}^{+}[\sigma_{u_{\delta}}]} \right] \to -\text{Im} \left[ A_{\Omega}(u) \right]$.
Conformal covariance of spin correlations at criticality

Example: to handle $E^+_{\Omega_\delta}[\sigma_u]$, one should consider the following b.v.p.:

- $f(z^\#) \equiv -f(z^\flat)$, branches over $u$;
- $\text{Im} \left[ f(\zeta) \sqrt{n(\zeta)} \right] = 0$ for $\zeta \in \partial \Omega$;
- $f(z) = \frac{1}{\sqrt{z-u}} + 2A_{\Omega}(u) \cdot \sqrt{z-u} + \ldots$

Claim: If $\Omega_\delta$ converges to $\Omega$ as $\delta \to 0$, then

- $(2\delta)^{-1} \log \left[ \frac{E^+_{\Omega_\delta}[\sigma_{u_\delta}+2\delta]}{E^+_{\Omega_\delta}[\sigma_{u_\delta}]} \right] \to \text{Re} [ A_{\Omega}(u) ]$;
- $(2\delta)^{-1} \log \left[ \frac{E^+_{\Omega_\delta}[\sigma_{u_\delta}+2i\delta]}{E^+_{\Omega_\delta}[\sigma_{u_\delta}]} \right] \to - \text{Im} [ A_{\Omega}(u) ]$.

Conformal covariance $\frac{1}{8}$: for any conformal map $\phi : \Omega \to \Omega'$,

- $f_{[\Omega,a]}(w) = f_{[\Omega',\phi(a)]}(\phi(w)) \cdot (\phi'(w))^{1/2}$;
- $A_{\Omega}(z) = A_{\Omega'}(\phi(z)) \cdot \phi'(z) + \frac{1}{8} \cdot \phi''(z)/\phi'(z)$.
Conformal covariance of spin correlations at criticality

**Example:** to handle $\mathbb{E}_{\Omega_\delta}^+ [\sigma_u]$, one should consider the following b.v.p.:

- $f(z^\#) \equiv -f(z^\flat)$, branches over $u$;
- $\DeclareMathOperator{Im}{Im} \Im \left[ f(\zeta) \sqrt{n(\zeta)} \right] = 0$ for $\zeta \in \partial \Omega$;
- $f(z) = \frac{1}{\sqrt{z-u}} + 2A_{\Omega}(u) \cdot \sqrt{z-u} + \ldots$

**Claim:** If $\Omega_\delta$ converges to $\Omega$ as $\delta \to 0$, then

- $\left(2\delta\right)^{-1} \log \left[ \frac{\mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_\delta+2\delta}]}{\mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_\delta}]} \right] \to \Re \left[ A_{\Omega}(u) \right]$;
- $\left(2\delta\right)^{-1} \log \left[ \frac{\mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_\delta+2i\delta}]}{\mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_\delta}]} \right] \to -\Im \left[ A_{\Omega}(u) \right]$.

Quite a lot of technical work is needed, e.g.:

- to handle tricky boundary conditions [ Dirichlet for $\int \Re [f^2 \, dz]$ ];
- to prove convergence, incl. near singularities [ complex analysis ];
- to recover the normalization of $\mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_1} \ldots \sigma_{u_n}]$ [ probability ].
Explicit formulae for multi-point spin correlations

We define \( \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^+_{\Omega} := \exp[ \int \mathcal{L}(u_1, \ldots, u_n) ] \), where

\[
\mathcal{L}_\Omega(u_1, \ldots, u_n) := \sum_{s=1}^n \Re \left[ A_\Omega(u_s; u_1, \ldots, \hat{u_s}, \ldots, u_n) d\hat{u_s} \right],
\]

and the multiplicative normalization is chosen so that

\[
\langle \sigma_{u_1} \ldots \sigma_{u_{n-1}} \rangle^+ \sim \langle \sigma_{u_1} \ldots \sigma_{u_{n-1}} \rangle^+ \cdot \langle \sigma_{u_n} \rangle^+ \quad \text{as} \quad u_n \to \partial \Omega,
\]

\[
\langle \sigma_{u_1} \sigma_{u_2} \rangle^+ \sim |u_2 - u_1|^{-\frac{1}{4}} \quad \text{as} \quad u_2 \to u_1 \in \Omega.
\]

Coefficients \( A_\Omega(u_1; u_2, \ldots, u_n) \) are defined via the following b.v.p.:

1. \( f(z^\#) \equiv -f(z^\flat) \) is a holomorphic spinor on \([\Omega; u_1, \ldots, u_n]\);
2. \( \Im \left[ f(\zeta)(n(\zeta))^{\frac{1}{2}} \right] = 0 \) for \( \zeta \in \partial \Omega \);
3. \( f(z) = ic_s \cdot (z - u_s)^{-\frac{1}{2}} + \ldots \) for some (unknown) \( c_s \in \mathbb{R}, s \geq 2 \);
4. \( f(z) = (z - u_1)^{-\frac{1}{2}} + 2A_\Omega(u_1; u_2, \ldots, u_n) \cdot (z - u_1)^{\frac{1}{2}} + \ldots \).
Explicit formulae for multi-point spin correlations

We define \( \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^+_\Omega := \exp\left[ \int L(u_1, \ldots, u_n) \right] \), where

\[
L_\Omega(u_1, \ldots, u_n) := \sum_{s=1}^{n} \text{Re} \left[ A_\Omega(u_s; u_1, \ldots, \hat{u}_s, \ldots, u_n) du_s \right] ,
\]

and the multiplicative normalization is chosen so that

\[
\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^+_\Omega \sim \langle \sigma_{u_1} \ldots \sigma_{u_{n-1}} \rangle^+_\Omega \cdot \langle \sigma_{u_n} \rangle^+_\Omega \quad \text{as} \quad u_n \to \partial \Omega ,
\]

\[
\langle \sigma_{u_1} \sigma_{u_2} \rangle^+_\Omega \sim |u_2 - u_1|^{-\frac{1}{4}} \quad \text{as} \quad u_2 \to u_1 \in \Omega .
\]

Remarks: • The closedness of the differential form \( L_{\Omega,n} \) and the existence of an appropriate multiplicative normalization are not immediate (can be deduced along the proof of convergence);

• Similar techniques can be applied for more involved boundary conditions and/or in the multiply connected setup (when no explicit formulae are available), as well as to other fields...
Correlations at criticality: convergence to CFT predictions

• Three local primary fields: \( 1, \sigma \) (spin), \( \varepsilon \) (energy density);
  Scaling exponents: \( 0, \frac{1}{8}, 1 \).

• **Theorem**: [Ch.–Hongler–Izyurov]
  If \( \Omega_\delta \to \Omega \) and \( u_{k,\delta} \to u_k \) as \( \delta \to 0 \), then
  \[
  \delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}^{\oplus} [\sigma_{u_1,\delta} \cdots \sigma_{u_n,\delta}] \to C_\sigma \cdot \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle^+_{\Omega}
  \]
  where \( C_\sigma \) is a lattice-dependent constant,
  \[
  \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle^+_{\Omega} = \langle \sigma_{\varphi(u_1)} \cdots \sigma_{\varphi(u_n)} \rangle^+_{\Omega'} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}}
  \]
  for any conformal mapping \( \varphi : \Omega \to \Omega' \), and
  \[
  \left[ \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle^+_{\mathbb{H}} \right]^2 = \prod_{1 \leq s \leq n} (2 \operatorname{Im} u_s)^{-\frac{1}{4}} \times \sum_{\mu \in \{\pm 1\}^n} \prod_{s < m} \left| \frac{u_s - u_m}{u_s - \overline{u}_m} \right|^{\frac{\mu_s \mu_m}{2}}
  \]
Correlations at criticality: convergence to CFT predictions

- Three local primary fields: $1, \sigma$ (spin), $\varepsilon$ (energy density);
  Scaling exponents: $0, \frac{1}{8}, 1$.

- Theorem: [Hongler–Smirnov, Hongler (2010)]
  If $\Omega_\delta \rightarrow \Omega$ and $e_k, \delta \rightarrow z_k$ as $\delta \rightarrow 0$, then
  \[
  \delta^{-n} \cdot \mathbb{E}_{e_{1,\delta}} \cdots e_{n,\delta} \rightarrow C_\varepsilon \cdot \langle \varepsilon_{z_1} \cdots \varepsilon_{z_n} \rangle^+_\Omega
  \]
  where $C_\varepsilon$ is a lattice-dependent constant,
  \[
  \langle \varepsilon_{z_1} \cdots \varepsilon_{z_n} \rangle^+_\Omega = \langle \varepsilon_{\varphi(z_1)} \cdots \varepsilon_{\varphi(z_n)} \rangle^+_{\Omega'} \cdot \prod_{s=1}^n |\varphi'(u_s)|
  \]
  for any conformal mapping $\varphi : \Omega \rightarrow \Omega'$, and
  \[
  \langle \varepsilon_{z_1} \cdots \varepsilon_{z_n} \rangle^+_{\mathbb{H}} = i^n \cdot \text{Pf} \left[ (z_s - z_m)^{-1} \right]_{s,m=1}^{2n}, \quad z_s = \overline{z}_{2n+1-s}.
  \]

- Ingredients: convergence of basic fermionic observables (via Riemann-type b.v.p.) and (built-in) Pfaffian formalism
Correlations at criticality: convergence to CFT predictions

[Ch.–Hongler–Izyurov (2016, in progress)]

- Convergence of **mixed correlations**: spins ($\sigma$), disorders ($\mu$), fermions ($\psi$), energy densities ($\varepsilon$) (in multiply connected domains $\Omega$, with mixed free/fixed boundary conditions) to conformally covariant limits that can be defined via solutions to appropriate Riemann-type boundary value problems in $\Omega$.

- Standard **CFT fusion rules**

  $\sigma\mu \rightsquigarrow \eta\psi + \overline{\eta\psi}, \quad \psi\sigma \rightsquigarrow \mu, \quad \psi\mu \rightsquigarrow \sigma,$
  
  $i\psi\overline{\psi} \rightsquigarrow \varepsilon, \quad \sigma\sigma \rightsquigarrow 1 + \varepsilon, \quad \mu\mu \rightsquigarrow 1 - \varepsilon$

  can be deduced from properties of solutions to Riemann-type b.v.p.

- **Stress-energy tensor**: [Ch.–Glazman–Smirnov (2016)]
Correlations at criticality: convergence to CFT predictions

[Ch.–Hongler–Izyurov (2016, in progress)]

- Convergence of **mixed correlations**: spins ($\sigma$), disorders ($\mu$), fermions ($\psi$), energy densities ($\varepsilon$) (in multiply connected domains $\Omega$, with mixed free/fixed boundary conditions) to conformally covariant limits that can be defined via solutions to appropriate Riemann-type boundary value problems in $\Omega$.

- Standard **CFT fusion rules**, e.g. $\sigma\sigma \sim 1 + \varepsilon$:

$$\langle \sigma_{u'}\sigma_{u} \ldots \rangle_{\Omega}^{b} = |u' - u|^{-\frac{1}{4}} \left[ \langle \ldots \rangle_{\Omega}^{b} + \frac{1}{2}|u' - u|\langle \varepsilon_{u} \ldots \rangle_{\Omega}^{b} + o(|u' - u|) \right],$$

can be deduced from properties of solutions to Riemann-type b.v.p.

Some research routes / open questions

- Better understanding of the CFT description at criticality: other fields, fusion rules, height functions, “geometric” observables (e.g., probabilities of concrete topologies of domain walls)
- Near-critical (massive) regime $x - x_{\text{crit}} = m \cdot \delta$: convergence of correlations, massive SLE$_3$ curves and loop ensembles etc.
- Super-critical regime: e.g., convergence of interfaces to SLE$_6$ curves for any fixed $x > x_{\text{crit}}$ [known only for $x = 1$ (percolation)]

- Renormalization

  - $x = x_{\text{crit}}$
  - $x = 1$
Some research routes / open questions

- Better understanding of the CFT description at criticality: other fields, fusion rules, height functions, “geometric” observables (e.g., probabilities of concrete topologies of domain walls)
- Near-critical (massive) regime $x - x_{\text{crit}} = m \cdot \delta$: convergence of correlations, massive $\text{SLE}_3$ curves and loop ensembles etc.
- Super-critical regime: e.g., convergence of interfaces to $\text{SLE}_6$ curves for any fixed $x > x_{\text{crit}}$ [known only for $x = 1$ (percolation)]
- Irregular graphs, random interactions etc: many questions...

Tool: local relations and spinor observables are always there!
**Some research routes / open questions**

- Better understanding of the CFT description at criticality: other fields, fusion rules, height functions, “geometric” observables (e.g., probabilities of concrete topologies of domain walls)
- **Near-critical (massive) regime** $x - x_{\text{crit}} = m \cdot \delta$: convergence of correlations, massive $\text{SLE}_3$ curves and loop ensembles etc.
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**Tool**: local relations and spinor observables are always there!

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**Extended version of this talk**: arXiv:1605.09035

**Thank you!**