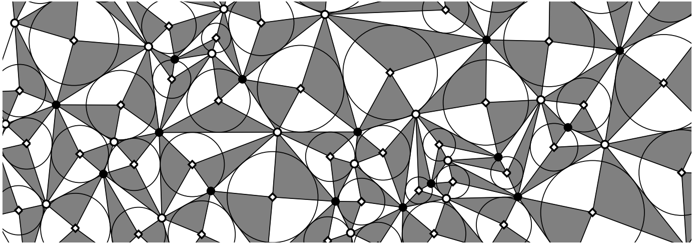
Planar Ising model:

convergence results on regular grids and s-embeddings of irregular graphs into $\mathbb{R}^{2,1}$



Dmitry Chelkak, ÉNS Paris

UMichigan, Ann Arbor, March 16, 2022

Ferromagnetic, w/o external field 2d nearest-neighbor Lenz–Ising model (1920)

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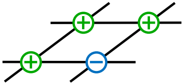

Boltzmann–Gibbs:

▷ energy [external field $h=0$]

$$H = -\sum_{u\sim u'} \sigma_u \sigma_{u'} - h \sum \sigma_u$$

▷ probability of a configuration (σ_u) is proportional to

$$\exp\left(-H[(\sigma_u)]/kT\right),$$

where T is the temperature

▷ $\sigma_u \sigma_{u'} = \pm 1 \rightsquigarrow x = e^{-2/kT}$.

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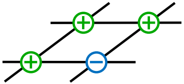
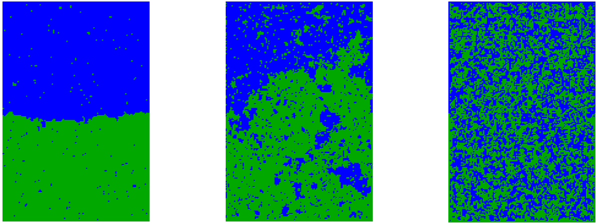


Archetypical example of a phase transition:



$x < x_{\text{crit}}$ $\qquad$ $x_{\text{crit}} = \tan \frac{\pi}{8}$ $\qquad$ $x > x_{\text{crit}}$

[samples with $+1/-1$ (Dobrushin) boundary conditions]

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Ernst Ising

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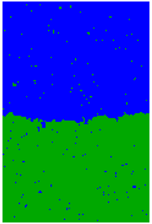

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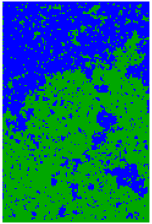

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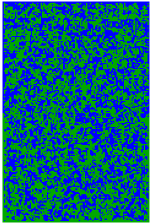

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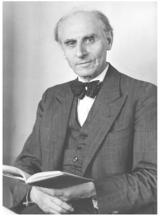

Pierre Curie (1895): metals lost ferromagnetic properties if $T \geqslant T_{\mathrm{crit}}$ [$T_{\mathrm{crit}} = 1043\mathrm{K}$ for iron]

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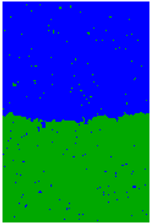

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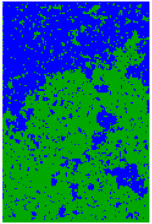

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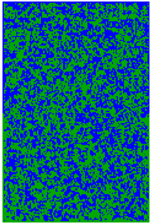

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[Peierls'36] $\exists$ phase transition; [Kramers–Wannier'41] x_{crit}

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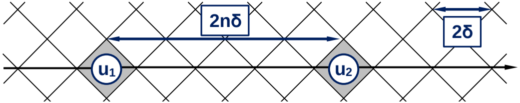
Non-trivial power laws at and near x_{crit}

[Kaufman–Onsager'49, Yang'52, McCoy–Wu'66+ ,...]



(i) If $x < x_{\mathrm{crit}}$, then $M(x) = \lim_{n\to\infty}(\mathbb{E}[\sigma_{u_1}\sigma_{u_2}])^{\frac{1}{2}} > 0$;

moreover, $M(x) \sim \mathrm{cst} \cdot (x_{\mathrm{crit}} - x)^{\frac{1}{8}}$ as $x \uparrow x_{\mathrm{crit}}$;

(ii) If $x = x_{\mathrm{crit}}$, then $\delta^{-\frac{1}{4}} \cdot \mathbb{E}[\sigma_{u_1}\sigma_{u_2}] \sim \mathrm{C}_\sigma^2 \cdot (2n\delta)^{-\frac{1}{4}}$.

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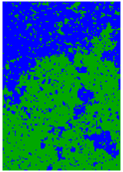

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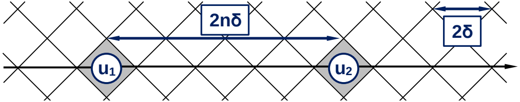

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▷ Since then, many lattice models of phase transition were suggested and studied.

▷ Theoretical physics [Belavin–Polyakov–Zamolodchikov'84+]:

the small mesh size limit $\delta \to 0$ at the critical point $x = x_{\mathrm{crit}}$ should be

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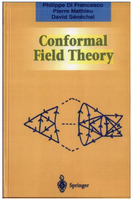

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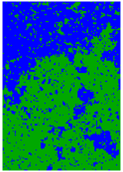
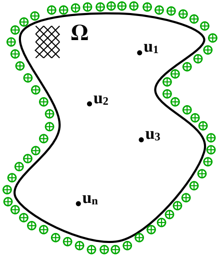


- **Theorem:** [w/ Hongler & Izyurov, *Ann. Math.'15*]

Let $x = x_{\mathrm{crit}}$ and Ω^δ approximate a domain $\Omega \subset \mathbb{C}$ on the square grids $\delta\mathbb{Z}^2$ with $\delta \to 0$. Then,

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}] \to \mathcal{C}^n_\sigma \cdot \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^+_\Omega .$$



If $\varphi : \Omega \to \Omega'$ is conformal, then

$$\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^+_\Omega = \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle^+_{\Omega'} \cdot \prod_{s=1}^n |\varphi'(u_s)|^{\frac{1}{8}} .$$

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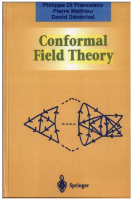

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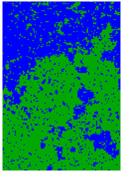

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Q: What makes the planar Ising model so special?

A: an important structure behind: 'discrete free fermions'

⇝ many intrinsic links with various subjects from orthogonal polynomials to cluster algebras

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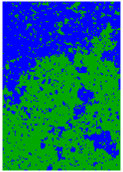

Outline:

▷ background:

- ▷ 'free fermions' and the propagation equation ;
- ▷ discrete holomorphic functions on $\mathbb{Z}^2$ at x_{crit} ;

▷ CFT description at x_{crit} on regular grids as $\delta \to 0$;

▷ universality in the bi-periodic case and beyond ;

▷ embeddings of (irregular) planar graphs into $\mathbb{R}^{2,1}$.

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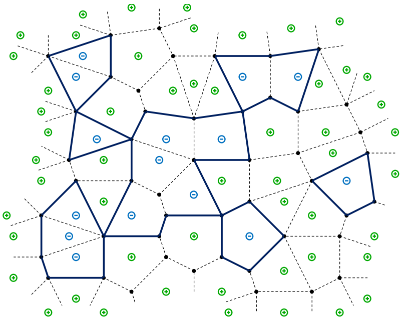
Free fermions in the Ising model on a planar graph [with '+' boundary conditions]

▷ **Boltzmann–Gibbs:** given a weighted graph (G°, J) one assigns ± 1 spins to its vertices ($\Leftrightarrow$ faces of the dual graph $G^\bullet$) so that the probability of (σ_u)

is proportional to $\exp\left[-\frac{1}{kT}\sum_{e=\langle uu'\rangle}(-J_e\sigma_u\sigma_{u'})\right]$,

where $J_e > 0$ are called interaction constants.



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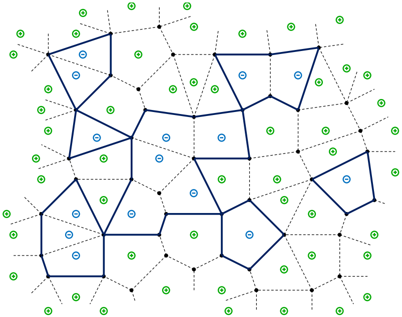

▷ This can be written as

$$\mathbb{P}\left[\text{sample } (\sigma_u)\right] = \mathcal{Z}^{-1}\prod_{e=\langle uu'\rangle:\,\sigma_u\neq\sigma_{u'}} x_e\,,$$

where $x_e := \exp[-2J_e/kT] \in (0,1)$. The normalizing factor $\mathcal{Z} = \mathcal{Z}(G,x)$ is called the partition function.

▷ Equivalently: choose an even subgraph $\mathcal{C}$ of $G^\bullet$

$$\mathcal{Z} = \sum_{\mathcal{C}\in\mathcal{E}(G^\bullet)}\prod_{e\in\mathcal{C}} x_e$$

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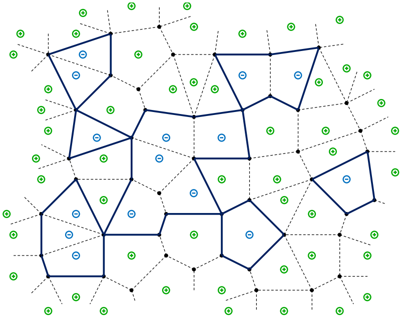
Proof: mappings onto dimer models, classical refs: [Kasteleyn'60s, Fisher'60s, ...]



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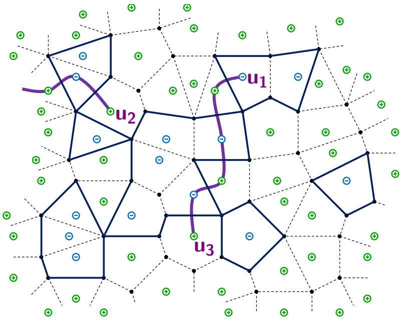

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where $\mathcal{A}_{[u_1,\ldots,u_n]}$ is constructed similarly to $\mathcal{A}$ but with $x_e \mapsto -x_e$ **along cuts**

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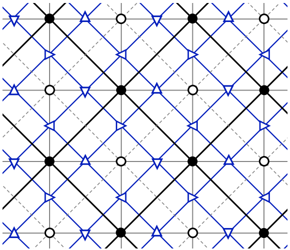
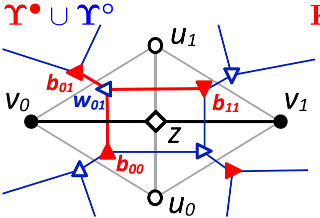
$\mathcal{A} : \mathbb{R}^{\Upsilon} \to \mathbb{R}^{\Upsilon}$, where

$\Upsilon :=$ the medial graph of $\Lambda := G^\circ \cup G^\bullet$



[if $G^\circ = \mathbb{Z}^2$, then there are four 'types' $\triangle, \triangleleft, \triangledown, \triangleright$ of vertices $c \in \Upsilon$]

Notation: $x_z = \tan\frac{1}{2}\theta_z$ with $\theta_z \in (0,\frac{\pi}{2})$ [recall that $x_{\mathrm{crit}} = \tan\frac{\pi}{8}$, i.e., $\theta_{\mathrm{crit}} = \frac{\pi}{4}$ for $\mathbb{Z}^2$]



Ker $\mathcal{A}$: functions on $\Upsilon^\bullet$ satisfying the equation

$$\mathbf{X}(\mathbf{b_{01}}) = \pm\mathbf{X}(\mathbf{b_{00}})\cos\theta_z \pm\mathbf{X}(\mathbf{b_{11}})\sin\theta_z$$

for $b_{00}, b_{01}, b_{11} \sim w_{01}$

▷ correspondence with a bipartite dimer model on $\Upsilon^\bullet \cup \Upsilon^\circ$: 'combinatorial bosonization' [Wu–Lin'75, Dubédat'11]

▷ the $\pm$ signs can be fixed via the Kasteleyn condition

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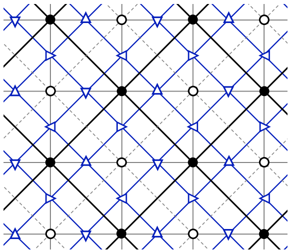

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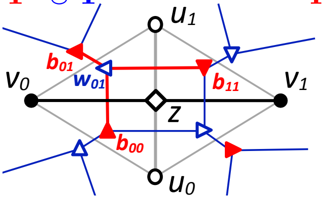

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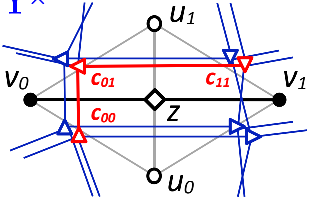

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$\triangleright$ $\Upsilon^\times$ branches over all $z \in \diamondsuit$, $v \in G^\bullet$ and $u \in G^\circ$

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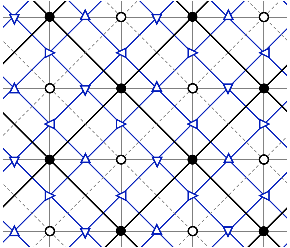

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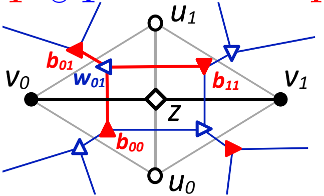
$\Upsilon^\bullet \cup \Upsilon^\circ$

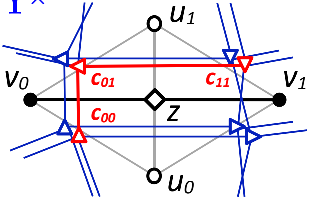


Ker $\mathcal{A}$: functions on $\Upsilon^\bullet$ satisfying the equation

$$\mathbf{X}(\mathbf{b_{01}}) = \pm\mathbf{X}(\mathbf{b_{00}})\cos\theta_z \pm\mathbf{X}(\mathbf{b_{11}})\sin\theta_z$$

for $b_{00}, b_{01}, b_{11} \sim w_{01}$

$\Upsilon^\times$



$\Longleftrightarrow$ spinors on $\Upsilon^\times$ satisfying the equation

$$\mathbf{X}(\mathbf{c_{01}}) = \mathbf{X}(\mathbf{c_{00}})\cos\theta_z + \mathbf{X}(\mathbf{c_{11}})\sin\theta_z$$

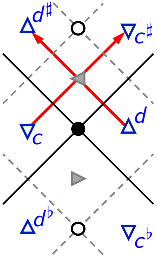
for $c_{00} \sim c_{01} \sim c_{11}$

▷ $\Upsilon^\times$ branches over all $z \in \diamondsuit$, $v \in G^\bullet$ and $u \in G^\circ$

$\mathcal{A} : \mathbb{R}^\Upsilon \to \mathbb{R}^\Upsilon$, where

Υ := the medial graph of $\Lambda := G^\circ \cup G^\bullet$

Critical model on $\mathbb{Z}^2$: $\theta = \frac{\pi}{4}$



$\rightsquigarrow$ **discrete CR equations**

$$X(d^\sharp) - X(d) = X(c^\sharp) - X(c)$$

$$X(d) - X(d^\flat) = X(c^\flat) - X(c)$$

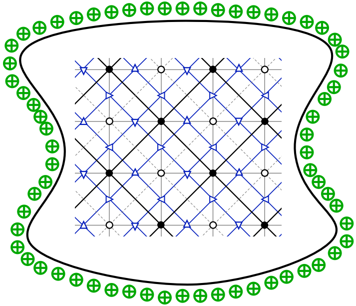
$\rightsquigarrow$ conformal invariance as $\delta \to 0$

Interpretation of $\mathcal{A}$ for the homogeneous model on $\delta\mathbb{Z}^2$ as $\delta \to 0$

the matrix $\mathcal{A} = -\mathcal{A}^\top : \mathbb{R}^\Upsilon \to \mathbb{R}^\Upsilon$ is a discretization of the (massive) Dirac operator $f \mapsto \partial\overline{f} + imf$,

$$m \asymp \delta^{-1} \cdot (x - x_{\mathrm{crit}}), \quad x_{\mathrm{crit}} = \tan \tfrac{\pi}{8}$$

$\rightsquigarrow$ **isomonodromic τ-functions** [Sato–Miwa–Jimbo'77, Wu–McCoy–Tracy–Barouch'76, ..., Palmer'07]



Spin correlations:

$$\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] = \frac{\mathrm{Pf}\, \mathcal{A}_{[u_1,\ldots,u_n]}}{\mathrm{Pf}\, \mathcal{A}}$$

$\mathcal{A}_{[u_1,\ldots,u_n]}$ acts similarly to $\mathcal{A}$ on functions/spinors that have (additional) **branchings over $u_1, \ldots, u_n$.**

In finite $\Omega \subset \mathbb{C}$: **Riemann-type boundary conditions $\overline{f} = \tau f$,**

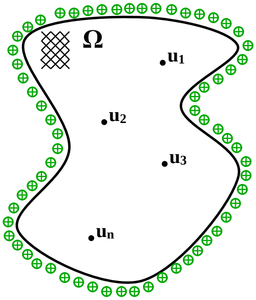
$\tau =$ 'unit tangent vector to $\partial\Omega$'

Convergence of correlations in discrete domains at criticality: Ising CFT

• **Theorem:** [Ch.–Hongler–Izyurov, *Ann. Math.'15*]

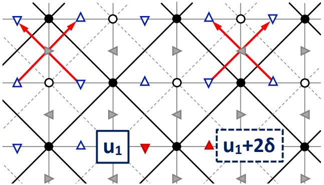
Let $x = x_{\mathrm{crit}}$, $\Omega \subset \mathbb{C}$ be a (bounded simply connected) domain and $\Omega^\delta \subset \delta\mathbb{Z}^2$ approximate Ω as $\delta \to 0$. Then,

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}] \;\to\; \mathcal{C}^n_\sigma \cdot \langle \boldsymbol{\sigma_{u_1} \ldots \sigma_{u_n}} \rangle^{+}_{\Omega}.$$



Idea: control $\dfrac{\mathbb{E}[\sigma_{u_1+2\delta} \ldots \sigma_{u_n}]}{\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}]} = \mathcal{A}^{-1}_{[u_1,\ldots,u_n]}(u_1 + \tfrac{1}{2}\delta, u_1 + \tfrac{3}{2}\delta)$

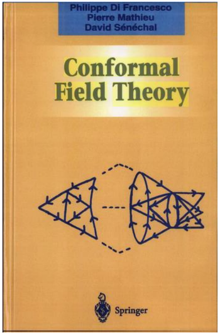
up to $o(\delta)$ by viewing the kernel $\mathcal{A}^{-1}_{[u_1,\ldots,u_n]}(u_1 + \tfrac{1}{2}\delta, \cdot)$ as a solution to an appropriate discrete Riemann-type b.v.p. Non-trivial technicalities at $\partial\Omega^\delta$ and near singularities.



[!] Warning: the link with discrete holomorphicity is very subtle: it does not work neither for general planar graphs nor for $\mathbb{Z}^2$ with inhomogeneous weights x_e.

Convergence of correlations in discrete domains at criticality: Ising CFT

$\langle \ldots \rangle$ [arXiv:2103.10263 w/ Hongler and Izyurov]



▷ convergence of mixed correlations: spins $\delta^{-\frac{1}{8}}\sigma_u$, fermions $\delta^{-\frac{1}{2}}\psi$, energy densities $\delta^{-1}(\sigma_u\sigma_{u'}-\sqrt{2}/2)$, $u\sim u'$, etc

in multiply connected domains, with mixed boundary conditions. No explicit formulae are available; the limits are defined via appropriate Riemann-type b.v.p.

▷ *consistent definition* of Ising CFT correlations $\langle\mathcal{O}\rangle_\Omega^{\mathfrak{b}}$ in multiply connected domains + *fusion rules:* e.g.,

as $w\to z$ one has *both* $\langle\psi_w\psi_z^\star\mathcal{O}\rangle_\Omega^{\mathfrak{b}} = \frac{i}{2}\langle\varepsilon_z\mathcal{O}\rangle_\Omega^{\mathfrak{b}}+\ldots$

$$\langle\sigma_w\sigma_z\mathcal{O}\rangle_\Omega^{\mathfrak{b}} = |w-z|^{-\frac{1}{4}}\langle\mathcal{O}\rangle_\Omega^{\mathfrak{b}}+\tfrac{1}{2}|w-z|^{\frac{3}{4}}\langle\varepsilon_z\mathcal{O}\rangle_\Omega^{\mathfrak{b}}+\ldots$$

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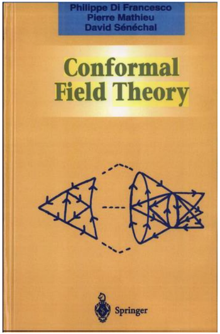
▷ *consistent definition* of Ising CFT correlations $\langle \mathcal{O} \rangle_\Omega^\flat$ in multiply connected domains + *fusion rules*.



Moreover, similar results are now available for the **near-critical model** $x = x_{\mathrm{crit}} + m\delta$
The limits of correlation functions are **not conformally covariant** and defined via solutions of appropriate Riemann-type b.v.p.'s for $\partial_{\overline{z}} f + imf = 0$ ('massive' fermions).

[SC Park arXiv:1811.06636, 2103.04649 ; Ch.–Izyurov–Mahfouf arXiv:2104.12858 ; ...]

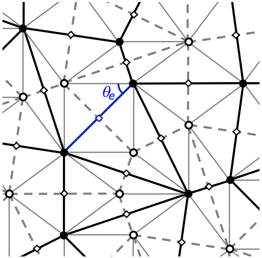
Universality on isoradial grids/rhombic tilings (Baxter's Z-invariant Ising model)

G°: each face is inscribed into a circle of common radius δ;
[equivalently, $\Lambda = G^\circ \cup G^\bullet$ form a tiling of the plane by rhombi]

special interaction parameters: $x_e = \tan \frac{1}{2}\theta_e$.

All the convergence results available on $\mathbb{Z}^2$ (correlations, interfaces, loop ensembles) hold within this class of models.

[w/ Smirnov, *Inv. Math.*'12] **"Universality[!?]** in the 2D Ising model and conformal invariance of fermionic observables"



"Proof": This setup still leads to a 'nice' notion of discrete holomorphic functions [Duffin'68], more-or-less the same ideas/techniques as for $\mathbb{Z}^2$ can be applied.

Particular cases:

triangular $x_{\mathrm{crit}} = \tan \frac{\pi}{6}$

hexagonal $x_{\mathrm{crit}} = \tan \frac{\pi}{12}$

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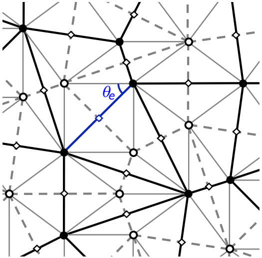

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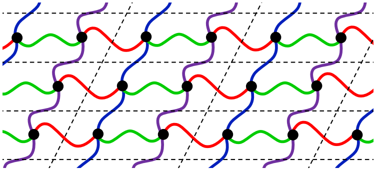
[w/ Smirnov, *Inv. Math.*'12] **"Universality[!?]** in the 2D Ising model and conformal invariance of fermionic observables"

Problem: this framework is too rigid

e.g., consider a *'generic' bi-periodic Ising model:* the criticality condition is known [$x(\mathcal{E}_{00}) = x(\mathcal{E} \smallsetminus \mathcal{E}_{00})$] but such models do not admit isoradial embeddings ...



criticality condition: $1 + x_3x_4 = x_3 + x_4 + x_1x_2(1+x_3)(1+x_4)$

Wanted: to draw (G°, x) so that the matrix $\mathcal{A}$ admits a 'discrete-complex-analysis' interpretation.

S-embeddings [Proc.ICM2018, arXiv:2006.14559, ...]

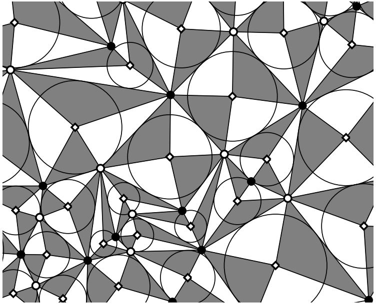
The framework of rhombic tilings is too rigid:

- ▷ it is even not general enough to be applied to 'generic' critical bi-periodic models

Not to mention really interesting setups:

- ▷ e.g., $\mathbb{Z}^2$ with random interaction constants x_e
- ▷ random planar maps carrying the Ising model

[?] 'discrete-complex-analysis' interpretation of $\mathcal{A}$



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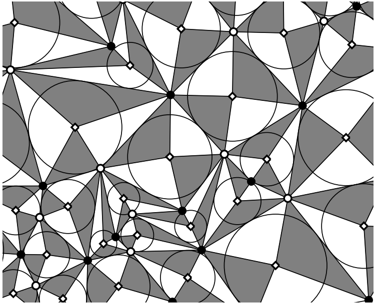

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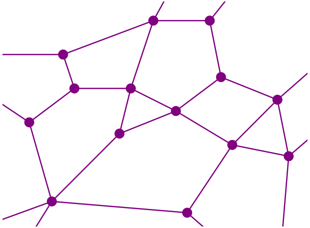
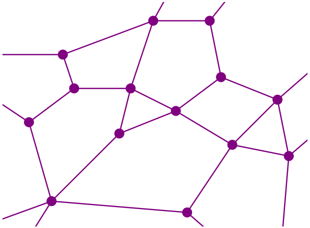


Analogy: Tutte's harmonic embedding $\mathcal{H} : G \to \mathbb{C}$ is a complex-valued (local) solution of $\Delta\mathcal{H} = 0$:

the position of each vertex is the (weighted) barycenter of the positions of its neighbors

[$\Rightarrow$ the random walk on $\mathcal{H}(G)$ is a martingale $\Rightarrow$ ...]

S-embeddings [Proc.ICM2018, arXiv:2006.14559, ...]



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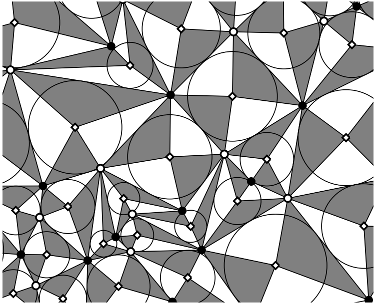

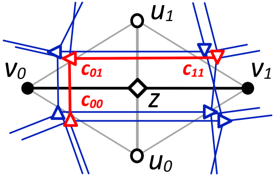

[!] S-embeddings into $\mathbb{R}^{2,1} \cong \mathbb{C}\times\mathbb{R}$: [tilings of the plane by tangential quads]

(local) $\mathbb{C}$-solution $\mathcal{X}(c_{01}) = \mathcal{X}(c_{00})\cos\theta_z + \mathcal{X}(c_{11})\sin\theta_z$

$\Updownarrow$

s-embedding $\quad \mathcal{S}_\mathcal{X}(v_p^\bullet) - \mathcal{S}_\mathcal{X}(u_q^\circ) := (\mathcal{X}(c_{pq}))^2$

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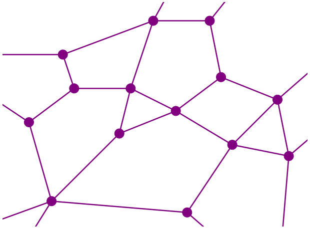
S-embeddings [Proc.ICM2018, arXiv:2006.14559, ...]



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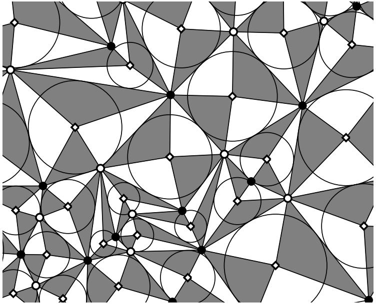

Particular case: for rhombic tilings of mesh size δ one has

$$\mathcal{Q}_{\mathcal{X}} = \pm\tfrac{1}{2}\delta.$$

The third coordinate disappears as $\delta \to 0$.

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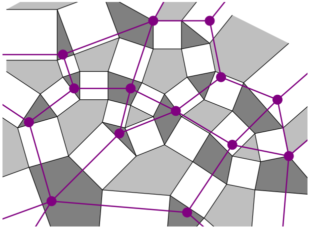

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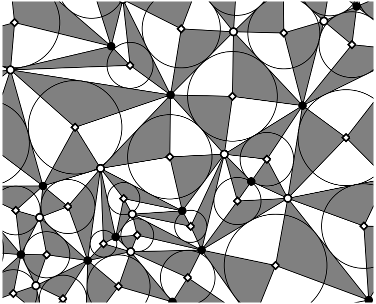


Remark: There also exists a unifying framework: t-embeddings of the bipartite dimer model into $\mathbb{R}^{2,2}$ (aka Coloumb gauges) [2001.11871, 2002.07540, 2109.06272, ... w/ Laslier, Ramassamy & Russkikh]

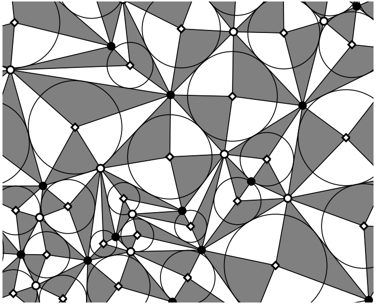
T-embeddings: bipartite planar tilings such that the black/white angles are balanced ($\sum = \pi$) at each vertex

▷ 'discrete complex analysis techniques' (a priori regularity of discrete holomorphic functions under very mild assumptions: e.g., harmonic functions on Tutte's embeddings are Lipschitz)

▷ links with cluster algebras etc (notably in the periodic setup) [arXiv:1810.05616 Kenyon–Lam–Ramassamy–Russkikh]

S-embeddings [Proc.ICM2018, arXiv:2006.14559, ...]

'discrete-complex-analysis' interpretation of $\mathcal{A}$:



▷ s-holomorphic functions

$$\Pr[\, F(z); \overline{\mathcal{X}(c)}\,\mathbb{R}\,] \;=\; X(c)/\mathcal{X}(c)\,.$$

($X \in \mathbb{R}$ satisfies the 3-terms equation $\Leftrightarrow$ $F \in \mathbb{C}$ exists)

▷ $F(z)d\mathcal{S}_{\mathcal{X}} + \overline{F(z)}d\mathcal{Q}_{\mathcal{X}}$ is a closed form

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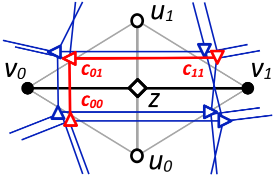

[!] S-embeddings [tilings of the plane by tangential quads]
into $\mathbb{R}^{2,1} \cong \mathbb{C}\times\mathbb{R}$:

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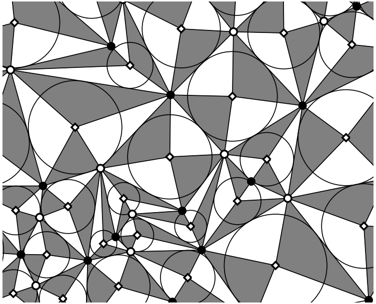

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[cf. Ch.–Smirnov'12]:

for rhombic tilings one has $\mathcal{Q}_{\mathcal{X}}=\pm\frac{1}{2}\delta$

$\Rightarrow$ the third coordinate disappears as $\delta\to 0$ $\Rightarrow$

conformal invariance

[!] S-embeddings [tilings of the plane by tangential quads]
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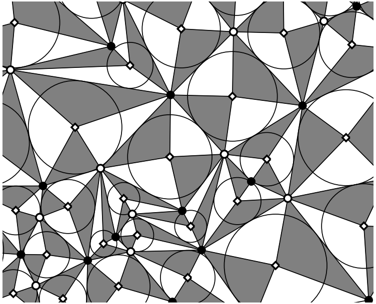

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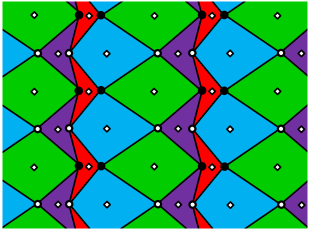
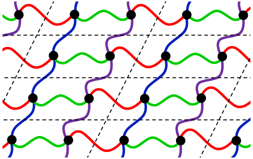


Theorem: conformal invariance and universality of the limit (of interfaces) for all critical bi-periodic models.

"Proof:" there exists a canonical $\mathcal{S}$ with bi-periodic $\mathcal{Q}=\mathcal{Q}^{\delta}=O(\delta)$

$\rightsquigarrow$ holomorphic functions as $\delta\to 0$



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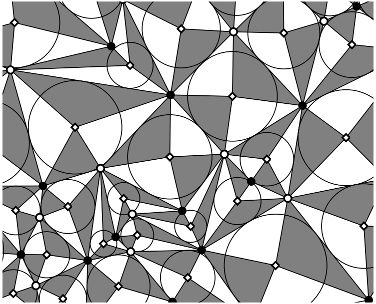

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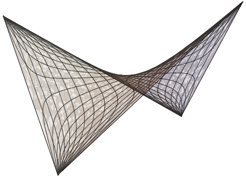
---

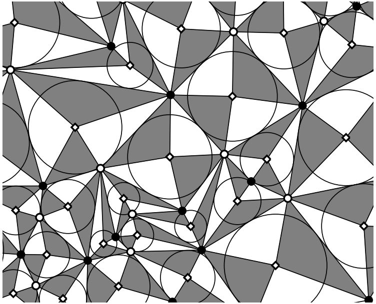
If $(\mathcal{S}^\delta,\mathcal{Q}^\delta)\to(z,t(z))=:\mathrm{S}\subset\mathbb{R}^{2,1}$ $\Rightarrow$ subseq. limits of fermionic observables satisfy the condition $f(z)dz+\overline{f(z)}dt$ – closed form, which can be written as the conjugate Beltrami equation



$$\partial_\zeta\overline{f}=\overline{\nu}\cdot\partial_\zeta f \qquad \nu=-\partial_\zeta t/\partial_\zeta z$$

in the **conformal parametrization** **ζ of the surface $\mathrm{S}\subset\mathbb{R}^{2,1}$**

S-embeddings [Proc.ICM2018, arXiv:2006.14559, ...]

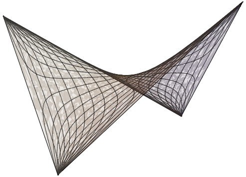


▷ **Assume that $(\mathcal{S}^\delta, \mathcal{Q}^\delta) \to$ smooth $\mathrm{S} \subset \mathbb{R}^{2,1}$.**

Then, the functions $\phi := z_\zeta^{1/2} \cdot f + \overline{z}_\zeta^{1/2} \cdot \overline{f}$ satisfy the equation $\partial_\zeta \overline{\phi} + im\phi = 0$, where

- ▷ ζ is a conformal parametrization of $\mathrm{S} \subset \mathbb{R}^{2,1}$,
- ▷ m is the **mean curvature of S** multiplied by its metric element in the parametrization ζ.

If $(\mathcal{S}^\delta, \mathcal{Q}^\delta) \to (z, t(z)) =: \mathrm{S} \subset \mathbb{R}^{2,1}$ $\Rightarrow$ subseq. limits of fermionic observables satisfy the condition $f(z)dz + \overline{f(z)}dt$ – closed form, which can be written as the conjugate Beltrami equation



$$\partial_\zeta \overline{f} = \overline{\nu} \cdot \partial_\zeta f$$
$$\nu = -\partial_\zeta t / \partial_\zeta z$$

in the **conformal parametrization ζ of the surface $\mathrm{S} \subset \mathbb{R}^{2,1}$**

S-embeddings [Proc.ICM2018, arXiv:2006.14559, ...]

▷ **Assume that $(\mathcal{S}^\delta, \mathcal{Q}^\delta) \to$ smooth $\mathrm{S} \subset \mathbb{R}^{2,1}$.**

Then, the functions $\phi := z_\zeta^{1/2} \cdot f + \overline{z}_\zeta^{1/2} \cdot \overline{f}$ satisfy the equation $\boldsymbol{\partial_\zeta \overline{\phi} + im\phi = 0}$, where

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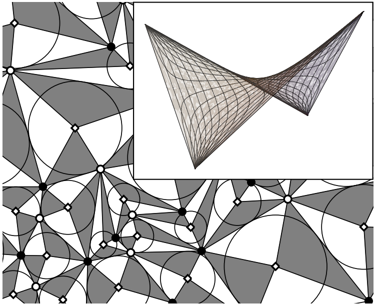

Important open questions/research directions:

▷ to understand how these embeddings/surfaces behave in various setups of interest:

- – random media (e.g., random interaction constants x_e on $\mathbb{Z}^2$);
- – critical random planar maps weighted by the Ising model [? $\leftrightsquigarrow$ Liouville CFT ?]: sounds like 'canonical fluctuations' near Lorentz-minimal surfaces should arise.

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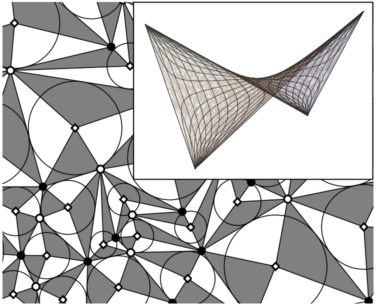

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Thank you!