2D Ising model: s-holomorphicity and correlation functions

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[ Sample of a critical 2D Ising configuration (with two disorders), © Clément Hongler (EPFL) ]

Charles River Lectures
Boston, October 2, 2015
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Outline:

• Nearest-neighbor Ising model in 2D:
  o definition, phase transition
  o fermionic observables
  o local relations: s-holomorphicity
  o dimers and Kac–Ward matrices

• Conformal invariance at criticality:
  o s-holomorphic observables
  o spin correlations and other fields
  o interfaces and loop ensembles

• Research routes

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Nearest-neighbor Ising or Lenz-Ising model in 2D

**Definition:** *Lenz-Ising model* on a planar graph $G^*$ (dual to $G$) is a random assignment of $+/-$ spins to vertices of $G^*$ (faces of $G$)

Q: I heard this is called a (site) percolation?
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[sample of a honeycomb percolation]
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A: .. according to the following probabilities:

\[
\mathbb{P}[\text{conf. } \sigma \in \{\pm 1\}^V(G^*)] \propto \exp \left[ \beta \sum_{e=\langle uv \rangle} J_{uv} \sigma_u \sigma_v \right] \\
\propto \prod_{e=\langle uv \rangle: \sigma_u \neq \sigma_v} x_{uv} ,
\]

where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv \rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.  

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- It is also convenient to use the parametrization $x_{uv} = \tan(\frac{1}{2}\theta_{uv})$.
- Working with subgraphs of *regular lattices*, one can consider the *homogeneous model* in which all $x_{uv}$ are equal to each other.
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**Disclaimer:**
no external magnetic field.

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\mathbb{P} \left[ \text{conf. } \sigma \in \{\pm1\}^{V(G^*)} \right] \propto \exp \left[ \frac{\beta}{2} \sum_{\langle uv \rangle} J_{uv} \sigma_u \sigma_v \right] \\
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Phase transition (e.g., on $\mathbb{Z}^2$)

- Dobrushin boundary conditions: $+1$ on $(ab)$ and $-1$ on $(ba)$

\[ x < x_{\text{crit}} \quad x = x_{\text{crit}} \quad x > x_{\text{crit}} \]

- Ising (1925): no phase transition in 1D $\rightarrow$ doubts about 2+D;
- Peierls (1936): existence of the phase transition in 2D;
- Kramers-Wannier (1941): $x_{\text{self-dual}} = \sqrt{2} - 1 = \tan\left(\frac{1}{2} \cdot \frac{\pi}{4}\right)$;
- Onsager (1944): sharp phase transition at $x = \sqrt{2} - 1$. 
At criticality (e.g., on $\mathbb{Z}^2$):

- Kaufman-Onsager (1948-49), Yang (1952): scaling exponent $\frac{1}{8}$ for the magnetization (some spin correlations in $\mathbb{Z}^2$ at $x \uparrow x_{\text{crit}}$).
- In particular, for $\Omega_\delta \rightarrow \Omega$ and $u_\delta \rightarrow u \in \Omega$, it should be $E_{\Omega_\delta}[\sigma_{u_\delta}] \lesssim \delta^{\frac{1}{8}}$ as $\delta \rightarrow 0$. 

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Questions for the part #2:

- Convergence of correlations, e.g.
  \[ \delta^{-\frac{n}{8}} \mathbb{E}_{\Omega_\delta}[\sigma_{u_1,\delta} \ldots \sigma_{u_n,\delta}] \xrightarrow{\delta \to 0} \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_{\Omega} \]
- Convergence of curves: interfaces (e.g. generated by Dobrushin boundary conditions) to $\text{SLE}_3$'s, loop ensembles to $\text{CLE}_3$'s?
At criticality (e.g., on $\mathbb{Z}^2$):

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- Convergence of curves: interfaces (e.g. generated by Dobrushin boundary conditions) to SLE$_3$’s, loop ensembles to CLE$_3$’s?

Q: Why these limits are conformally invariant (covariant)?
Fermionic observables: combinatorial definition [Smirnov '00s]

For an oriented edge $a$ of $G$ and a midpoint $z_e$ of another edge $e$,

$$F_G(a, z_e) := \bar{\eta}_a \sum_{\omega \in \text{Conf}_G(a, z_e)} \left[ e^{-\frac{i}{2} \text{wind}(a \rightsquigarrow z_e)} \prod_{\langle uv \rangle \in \omega} x_{uv} \right],$$

where $\eta_a$ denotes the (once and forever fixed) square root of the direction of $a$.

- The factor $e^{-\frac{i}{2} \text{wind}(a \rightsquigarrow z_e)}$ does not depend on the way how $\omega$ is split into non-intersecting loops and a path $a \rightsquigarrow z_e$. 

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- When both $a$ and $e$ are “boundary” edges, the factor $\overline{\eta}_a e^{-\frac{i}{2} \text{wind}(a \leadsto z_e)} = \pm \overline{\eta}_e$ is fixed and $F_G(a, z_e)$ becomes the partition function of the Ising model (on $G^*$) with Dobrushin boundary conditions.
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- Local relations: if we similarly define $F_G(a, \cdot)$ on “corners” of $G$, then for any $c \sim z_e \neq z_a$ one has

$$F_G(a, c) = e^{\pm \frac{i}{2} (\theta_e - \alpha(c, e))} \text{Proj}[ F_G(a, z_e); e^{\mp \frac{i}{2} \theta_e} \eta_e ].$$
Fermionic observables: local relations

- **Definition:**

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F_G(a, z_e) := \bar{\eta}_a \sum_{\omega \in \text{Conf}_G(a, z_e)} \left[ e^{-\frac{i}{2} \text{wind}(a \to z_e)} \prod_{\langle uv \rangle \in \omega} x_{uv} \right].
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- **Claim:** \( F_G(a, c) = e^{\pm \frac{i}{2} (\theta_e - \alpha(c, e))} \cdot \text{Proj}[F_G(a, z_e); e^{\mp \frac{i}{2} \theta_e \bar{\eta}_e}] \).

- **Proof:** a bijection between \( \text{Conf}_G(a, c) \) and \( \text{Conf}(a, z_e) \). 
  
  **Case A:**
  
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Fermionic observables: local relations

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\[ a \]

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Fermionic observables: local relations

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  F_G(a, z_e) := \bar{\eta}_a \sum_{\omega \in \text{Conf}_G(a, z_e)} \left[ e^{-\frac{i}{2} \text{wind}(a \leftrightarrow z_e)} \prod_{\langle uv \rangle \in \omega} x_{uv} \right].
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- **S-holomorphicity** (special self-dual weights on isoradial graphs):
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\[ F_G(a, c) = \text{Proj}[F_G(a, z_e); \overline{\eta}_c] \]

provided each edge \( e \) of \( G \) is a diagonal of a rhombic tile with half-angle \( \theta_e \) and the Ising model weights are given by \( x_e = \tan(\frac{1}{2} \theta_e). \)
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- \( \Rightarrow \) critical weights on regular grids:
  
  - square: \( x_{\text{crit}} = \tan \frac{\pi}{8} = \sqrt{2} - 1 \),
  - honeycomb: \( x_{\text{crit}} = \tan \frac{\pi}{6} = 1/\sqrt{3}, \ldots \)
2D Ising model as a dimer model on a non-bipartite graph (..., Fisher, Kasteleyn, ..., Kenyon, Dubedat, ...)

- There exist several representations of the 2D Ising model via dimers on an auxiliary graph
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• Definition of fermionic observables via dimers on $G_F$:

\[ F_G(a, c) = \bar{\eta}_c K_{c,a}^{-1} \quad \text{and} \quad F_G(a, z_e) = \bar{\eta}_e K_{e,a}^{-1} + \bar{\eta}_{\bar{e}} K_{\bar{e},a}^{-1}. \]

• Local relations: an equivalent form of the identity $K \cdot K^{-1} = \text{Id}$
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- **Kac–Ward formula (1952–..., 1999–...):** $\mathcal{Z}^2 = \det[\text{Id} - T],$
  
  \[
  T_{e,e'} = \begin{cases} 
    \exp\left[i\frac{\alpha(e,e')}{2}\right] \cdot (x_ex_{e'})^{1/2} & \text{if } e \neq e' \\
    0 & \text{if } e = e'
  \end{cases}
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  $$T_{e,e'} = \begin{cases} 
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  0 & \text{otherwise.} 
  \end{cases}$$
  [is equivalent to the Kasteleyn theorem for dimers on $G_F$]
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- **More information:** arXiv:1507.08242 [Ch., Cimasoni, Kassel]
Part II: conformal invariance at criticality [Smirnov '06]
[Ch., Duminil-Copin, Hongler, Izyurov, Kemppainen, Kytölä,... '09 – ...]

Main tool: discrete (s-)holomorphic functions

- (A fair amount of) work is needed to understand how to use them for the rigorous analysis when \( \Omega_\delta \to \Omega \), especially in rough domains formed by fractal interfaces.
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- continuum–discrete: decipher the limit of discrete quantities from the convergence \( F^\delta \rightarrow f \) [e.g., coefficients at singularities].
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- basic observables: [Smirnov ’06], universality: [Ch., Smirnov ’09]
- energy density field: [Hongler, Smirnov ’10], [Hongler ’10]
  - spinor version, some ratios of spin correlations: [Ch., Izyurov ’11]
  - spin field: [Ch., Hongler, Izyurov ’12]
- mixed correlations in multiply-connected $\Omega$’s [on the way]
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Typical strategy to prove the convergence of interfaces:
• choose a family of martingales w. r. t. the growing interface $\gamma^\delta$
  [there are many, e.g., $E^{ab}_{\Omega^\delta}[\sigma_z]$ would do the job for $+1/−1$ b. c.];
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- prove uniform convergence of the (re-scaled) quantities as $\delta \rightarrow 0$ [the one above (done in 2012) is not an optimal choice, there are others that are easier to handle (first done in 2006–2009)];
- prove the convergence $\gamma^\delta \rightarrow \gamma$ and recover the law of $\gamma$ using this family of martingales [some probabilistic techniques are needed].
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- stress-energy tensor [Ch., Glazman, Smirnov, on the way]

Some papers/preprints (convergence of interfaces):
- $+/-$ b.c. (conv. to $\text{SLE}_3$ in a weak topology): [Ch., Smirnov ’09]
- $+$/free/$-$ b.c. (dipolar $\text{SLE}_3$): [Hongler, Kytölä ’11]
- multiply-connected setups: [Izyurov ’13]
- strong topology (tightness of curves): [Kemppainen, Smirnov ’12], [Ch. ’12], [Ch., Duminil-Copin, Hongler ’13], [Ch., D.-C., H., K., S. ’13]
- free b.c. (exploration tree): [Benoist, Duminil-Copin, Hongler ’14]
- [on the way by smb]: full loop ensemble (convergence to $\text{CLE}_3$)
Part II: conformal invariance at criticality [Smirnov '06]
[Ch., Duminil-Copin, Hongler, Izyurov, Kemppainen, Kytölä, ... '09 – ...]

Main tool: discrete (s-)holomorphic functions

- (A fair amount of) work is needed to understand how to use them for the rigorous analysis when $\Omega_\delta \to \Omega$, especially in rough domains formed by fractal interfaces.

Some papers/preprints (convergence of correlations):

- basic observables: [Smirnov '06], universality: [Ch., Smirnov '09]
- energy density field: [Hongler, Smirnov '10], [Hongler '10]
- spinor version, some ratios of spin correlations: [Ch., Izyurov '11]
- spin field: [Ch., Hongler, Izyurov '12]
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\cite{Ch,Duminil-Copin,Hongler,Izyurov,Kemppainen,Kytölä,...09–...}

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- multiply-connected setups: \cite{Izyurov13}
- strong topology (tightness of curves): \cite{Kemppainen,Smirnov12},
  \cite{Ch12}, \cite{Ch,Duminil-Copin,Hongler13}, \cite{Ch,D.-C.,H.,K.,S.13}
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Conformal covariance of correlation functions at criticality

- Three primary fields:
  - $1$, $\sigma$ (spin), $\varepsilon$ (energy density);
  - Scaling exponents: $0$, $\frac{1}{8}$, $1$.

- **Energy density**: for an edge $e$ of $\Omega$, let

  $$
  \varepsilon_e := \sigma_{e^\#} \sigma_{e^\flat} - \varepsilon_{\text{inf.vol.}}
  = (1 - \varepsilon_{\text{inf.vol.}}) - 2 \cdot \chi[e \in \omega]
  $$

where $e^\#$ and $e^\flat$ are two faces adjacent to $e$.

[$\varepsilon_{\text{inf.vol.}}$ is lattice-dependent: $= 2^{-\frac{1}{2}}$ (square), $= \frac{2}{3}$ (honeycomb), ...]
Conformal covariance of correlation functions at criticality

• Three primary fields:
  \(1, \sigma\) (spin), \(\varepsilon\) (energy density);
  Scaling exponents: 0, \(\frac{1}{8}\), 1.

• **CFT prediction:**
  If \(\Omega_\delta \to \Omega\) and \(e_{k,\delta} \to e_k\) as \(\delta \to 0\), then
  \[
  \delta^{-n} \cdot \mathbb{E}_{\Omega_\delta}^+ [\varepsilon_{u_1,\delta} \cdots \varepsilon_{u_n,\delta}] \to C_\varepsilon^n \cdot \langle \varepsilon_{e_1} \cdots \varepsilon_{e_n} \rangle_{\Omega}^+,
  \]
  where \(C_\varepsilon^n\) is a lattice-dependent constant,
  \[
  \langle \varepsilon_{u_1} \cdots \varepsilon_{u_n} \rangle_{\Omega}^+ = \langle \varepsilon_{\varphi(u_1)} \cdots \varepsilon_{\varphi(u_n)} \rangle_{\Omega'}^+ \cdot \prod_{s=1}^n |\varphi'(u_s)|
  \]
  for any conformal mapping \(\varphi : \Omega \to \Omega'\), and
  \[
  \langle \varepsilon_{z_1} \cdots \varepsilon_{z_n} \rangle_{\mathbb{H}}^+ = (\pi i)^{-n} \cdot \text{Pf} \left[(z_s - z_m)^{-1}\right]_{s,m=1}^{2n}, \quad z_s = \bar{z}_{2n+1-s}.
  \]
Conformal covariance of correlation functions at criticality

- Three primary fields: $1$, $\sigma$ (spin), $\varepsilon$ (energy density);
  Scaling exponents: $0$, $\frac{1}{8}$, $1$.

- **Theorem**: [Hongler–Smirnov, Hongler]

  If $\Omega_{\delta} \to \Omega$ and $e_{k,\delta} \to e_k$ as $\delta \to 0$, then
  $$\delta^{-n} \cdot \mathbb{E}_{\Omega_{\delta}}^+ [\varepsilon_{u_1,\delta} \cdots \varepsilon_{u_n,\delta}] \to C_\varepsilon \cdot \langle \varepsilon_{e_1} \cdots \varepsilon_{e_n} \rangle^+_{\Omega},$$

  where $C_\varepsilon$ is a lattice-dependent constant,
  $$\langle \varepsilon_{u_1} \cdots \varepsilon_{u_n} \rangle^+_{\Omega} = \langle \varepsilon_{\varphi(u_1)} \cdots \varepsilon_{\varphi(u_n)} \rangle^+_{\Omega'} \cdot \prod_{s=1}^n |\varphi'(u_s)|$$

  for any conformal mapping $\varphi : \Omega \to \Omega'$, and
  $$\langle \varepsilon_{z_1} \cdots \varepsilon_{z_n} \rangle^+_{\H} = (\pi i)^{-n} \cdot \text{Pf} \left[ (z_s - z_m)^{-1} \right]^{2n}_{s,m=1}, \quad z_s = \overline{z}_{2n+1-s}.$$

- **Ingredients**: convergence of $K_{e,a}^{-1}$ and Pfaffian formalism
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  If \( \Omega_\delta \to \Omega \) and \( u_{k,\delta} \to u_k \) as \( \delta \to 0 \), then
  \[
  \delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}^+ [\sigma_{u_1,\delta} \cdots \sigma_{u_n,\delta}] \xrightarrow[\delta \to 0]{} C^n_{\sigma} \cdot \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle^+_{\Omega},
  \]
  where \( C_{\sigma} \) is a lattice-dependent constant,

  \[
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  \]
  for any conformal mapping \( \varphi : \Omega \to \Omega' \), and

  \[
  \left[ \langle \sigma_{z_1} \cdots \sigma_{z_n} \rangle^+_{\mathbb{H}} \right]^2 = \prod_{1 \leq s \leq n} (2 \Im z_s)^{-\frac{1}{4}} \times \sum_{\mu \in \{\pm 1\}^n} \prod_{s < m} \frac{Z_s - Z_m}{Z_s - \overline{Z_m}} \left( \frac{\mu_s \mu_m}{2} \right)^{\mu_s \mu_m}
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- **Theorem**: [Ch.–Hongler–Izyurov]
  
  If $\Omega_{\delta} \to \Omega$ and $u_{k,\delta} \to u_k$ as $\delta \to 0$, then
  
  $$
  \delta^{-\frac{n}{8}} \cdot E_{\Omega_{\delta}}^+ \left[ \sigma_{u_1,\delta} \cdots \sigma_{u_n,\delta} \right] \to \delta \to 0 C_{\sigma}^n \cdot \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle_{\Omega}^+,
  $$

  where $C_{\sigma}$ is a lattice-dependent constant,

  $$
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  $$
Conformal covariance of spin correlations at criticality

- spin configurations on $G^*$
  $\iff$ domain walls on $G$
  $\iff$ dimers on $G_F$

- Kasteleyn’s theory: $Z = \text{Pf}[K]$

$[K = -K^T$ is a weighted adjacency matrix of $G_F]$
Conformal covariance of spin correlations at criticality

- spin configurations on $G^*$
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- **Kasteleyn’s theory:** $\mathcal{Z} = \text{Pf}[K]$
  $[K = -K^\top$ is a weighted adjacency matrix of $G_F$]

- **Claim:**
  $$\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] = \frac{\text{Pf}[K[u_1,\ldots,u_n]]}{\text{Pf}[K]},$$
  where $K[u_1,\ldots,u_n]$ is obtained from $K$ by changing the sign of its entries on slits linking $u_1,\ldots,u_n$ (and, possibly, $u_{\text{out}}$) pairwise.
Conformal covariance of spin correlations at criticality

- spin configurations on $G^*$
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- Kasteleyn's theory: $\mathcal{Z} = \text{Pf}[K]$
  \[ K = -K^\top \] is a weighted adjacency matrix of $G_F$
- Claim:
  \[ \mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] = \frac{\text{Pf}[K_{[u_1,\ldots,u_n]]]}{\text{Pf}[K]}, \]
  where $K_{[u_1,\ldots,u_n]}$ is obtained from $K$ by changing the sign of its entries on slits linking $u_1, \ldots, u_n$ (and, possibly, $u_{\text{out}}$) pairwise.
- More invariant way to think about entries of $K_{[u_1,\ldots,u_n]}^{-1}$:
  double-covers of $G$ branching over $u_1, \ldots, u_n$
Conformal covariance of spin correlations at criticality

**Main tool:** spinors on the double cover \([\Omega_\delta; u_1, \ldots, u_n]\).

\[
F_{\Omega_\delta} (z) := \left[ \mathcal{Z}_{\Omega_\delta}^+ [\sigma_{u_1} \cdots \sigma_{u_n}] \right]^{-1} \cdot \sum_{\omega \in \text{Conf}_{\Omega_\delta} (u_1 \rightarrow, z)} \phi_{u_1,\ldots,u_n} (\omega, z) \cdot x^\text{#edges(\omega)},
\]

\[
\phi_{u_1,\ldots,u_n} (\omega, z) := e^{-i \frac{1}{2} \text{wind}(p(\omega))} \cdot (-1)^{\text{#loops}(\omega \setminus p(\omega))} \cdot \text{sheet}(p(\omega), z).
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F_{Ω_δ}(z) := \left[ Z^{+}_{Ω_δ} \left[ \sigma_{u_1} \cdots \sigma_{u_n} \right] \right]^{-1} \cdot \sum_{\omega \in \text{Conf}_{Ω_δ}(u_1 \rightarrow, z)} \phi_{u_1,\ldots,u_n}(\omega, z) \cdot x_{\text{crit}}^{\#\text{edges}(\omega)},
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- \(\text{wind}(p(\gamma))\) is the winding of the path \(p(\gamma) : u_1 \rightarrow = u_1 + \frac{\delta}{2} \sim z\);
- \(\#\text{loops}\) – those containing an odd number of \(u_1, \ldots, u_n\) inside;
- \(\text{sheet}(p(\gamma), z) = +1\), if \(p(\gamma)\) defines \(z\), and \(-1\) otherwise.
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\[\text{Claim: } F_{\Omega_\delta}(u_1 + \frac{3\delta}{2}) = \frac{E_{\Omega_\delta}^+ [\sigma_{u_1+2\delta} \cdots \sigma_{u_n}]}{E_{\Omega_\delta}^+ [\sigma_{u_1} \cdots \sigma_{u_n}]}\]
Conformal covariance of spin correlations at criticality

**Example:** to handle $\mathbb{E}^+_{\Omega_\delta}[\sigma_u]$, one should consider the following b.v.p.:

- $f(z^*) \equiv -f(z)$, branches around $u$;
- $\text{Im} \left[ f(\zeta) \sqrt{n(\zeta)} \right] = 0$ for $\zeta \in \partial \Omega$;
- $f(z) = \frac{1}{\sqrt{z-u}} + \ldots$
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Claim: For $\Omega_\delta \to \Omega$ as $\delta \to 0$,

- $(2\delta)^{-1} \log \left[ \mathbb{E}^{+}_{\Omega_\delta} [\sigma_{u_\delta+2\delta}] / \mathbb{E}^{+}_{\Omega_\delta} [\sigma_{u_\delta}] \right] \to \text{Re} [A_{\Omega}(u)]$;
- $(2\delta)^{-1} \log \left[ \mathbb{E}^{+}_{\Omega_\delta} [\sigma_{u_\delta+2i\delta}] / \mathbb{E}^{+}_{\Omega_\delta} [\sigma_{u_\delta}] \right] \to -\text{Im} [A_{\Omega}(u)]$. 
Conformal covariance of spin correlations at criticality

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Conformal covariance $\frac{1}{8}$: for any conformal map $\phi : \Omega \to \Omega'$,

- $f_{[\Omega,a]}(w) = f_{[\Omega',\phi(a)]}(\phi(w)) \cdot (\phi'(w))^{1/2}$;
- $A_{\Omega}(z) = A_{\Omega'}(\phi(z)) \cdot \phi'(z) + \frac{1}{8} \cdot \phi''(z)/\phi'(z)$. 
Conformal covariance of spin correlations at criticality

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Claim: For $\Omega_\delta \to \Omega$ as $\delta \to 0$,

- $(2\delta)^{-1} \log \left[ \frac{E_{\Omega_\delta}^+[\sigma_u+2\delta]}{E_{\Omega_\delta}^+[\sigma_u]} \right] \to \text{Re} [A_{\Omega}(u)]$;
- $(2\delta)^{-1} \log \left[ \frac{E_{\Omega_\delta}^+[\sigma_u+2i\delta]}{E_{\Omega_\delta}^+[\sigma_u]} \right] \to -\text{Im} [A_{\Omega}(u)]$.

Work to be done:

- to handle tricky boundary conditions (Dirichlet for $\int \text{Re}[f^2dz]$);
- to prove convergence, incl. near singularities [complex analysis];
- to recover the normalization of $E_{\Omega_\delta}^+[\sigma_u]$ [probabilistic techniques].
Some research routes

- Better understanding of the CFT description at criticality: more fields, Virasoro algebra at the lattice level, “geometric” observables, height functions, Riemann surfaces etc.
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- Near-critical (massive) regime \( x - x_{\text{crit}} = m \cdot \delta \): convergence of correlations, massive \( \text{SLE}_3 \) curves and loop ensembles etc.
- Super-critical regime: e.g., convergence of interfaces to \( \text{SLE}_6 \) curves for any fixed \( x > x_{\text{crit}} \) [known only for \( x = 1 \) (percolation)]

\[ x = x_{\text{crit}} \quad \text{and} \quad x = 1 \]
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\[ x = x_{\text{crit}} \]

\[ (x - x_{\text{crit}}) \cdot \delta^{-1} \to \infty \]

\[ x = 1 \]
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**Tool**: local relations and spinor observables are always there!
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- Not only nearest-neighbor interactions
  [recent progress for the energy density field due to Giuliani, Greenblatt and Mastropietro, arXiv:1204.4040]
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Thank you!