Double-sided estimates of hitting probabilities in discrete planar domains

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based on *"Robust discrete complex analysis: a toolbox"*,
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in Probability Theory and Mathematical Statistics

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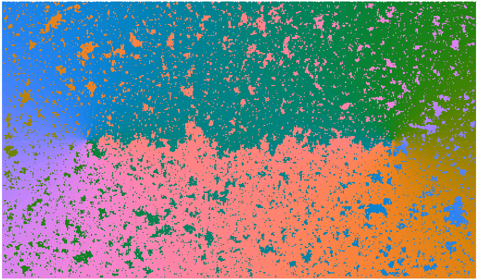
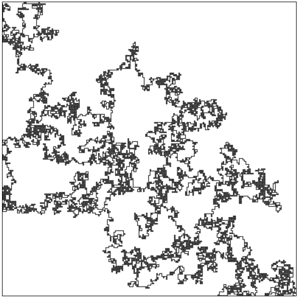


(spin representation, © C. Hongler)



(random cluster representation,

© S. Smirnov)

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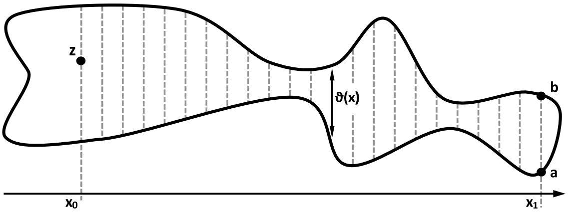

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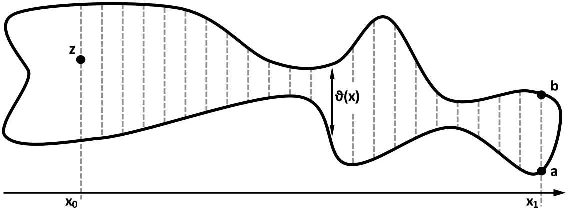

Theorem: (Ahlfors, Beurling, (Carleman))

$$\omega_\Omega(z;(ab)) \leqslant \frac{8}{\pi} \exp\left[-\pi \int_{x_0}^{x_1} \frac{dx}{\vartheta(x)}\right].$$

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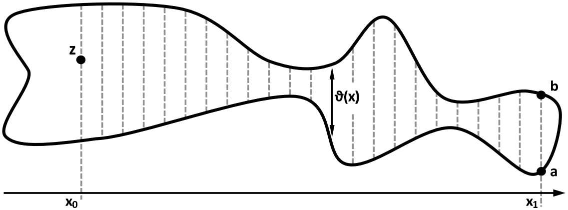

Theorem: (Ahlfors, Beurling, (Carleman))

$$\omega_\Omega(z;(ab)) \asymp \exp[-\pi \mathrm{L}_\Omega(z;(ab))], \quad \mathrm{L}_\Omega(z;(ab)) \geqslant \int_{x_0}^{x_1} (\vartheta(x))^{-1} dx.$$

Remark: $\Uparrow$ conformal invariance of $\omega_\Omega(z;(ab))$ and $L_\Omega(z;(ab))$.

Notation:

Let $(\Gamma; \mathrm{E}^\Gamma)$ be an inifinite planar graph embedded into $\mathbb{C}$ so that all its edges $(uv) \in \mathrm{E}^\Gamma$ are straight segments, $\mathrm{w}_{uv} = \mathrm{w}_{vu} > 0$ be some fixed edge weights (conductances), and $\mu_u := \sum_{v \sim u} \mathrm{w}_{uv}$ for $u \in \Gamma$.

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Discrete domains:

Let $(V^\Omega;\mathrm{E}^\Omega_{\mathrm{int}})$ be a *bounded connected subgraph* of $(\Gamma;\mathrm{E}^\Gamma)$. Denote by E^Ω_{bd} the set of all (oriented) edges $(a_{\mathrm{int}}a)\notin E^\Omega_{\mathrm{int}}$ such that $a_{\mathrm{int}}\in V^\Omega$ and $a\notin V^\Omega$. We set $\Omega:=\mathrm{Int}\,\Omega\cup\partial\Omega$,

$$\mathrm{Int}\,\Omega:=V^\Omega,\quad \partial\Omega:=\{(a\,;(a_{\mathrm{int}}a)):\ (a_{\mathrm{int}}a)\in E^\Omega_{\mathrm{bd}}\}.$$

Formally, the boundary $\partial\Omega$ of a discrete domain Ω should be treated as the set of oriented edges $(a_{\mathrm{int}}a)$, but we usually identify it with the set of corresponding vertices a, and think about $\mathrm{Int}\,\Omega$ and $\partial\Omega$ as subsets of Γ, if no confusion arises.

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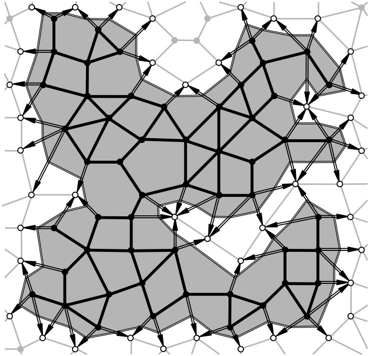

$(V^\Omega; \mathrm{E}^\Omega_{\mathrm{int}})$ – bounded and connected,

$E^\Omega_{\mathrm{bd}} := \{(a_{\mathrm{int}}a) \notin \mathrm{E}^\Omega_{\mathrm{int}} : a_{\mathrm{int}} \in V^\Omega, a \notin V^\Omega\}$,

$\Omega := \mathrm{Int}\,\Omega \cup \partial\Omega$, $\quad \mathrm{Int}\,\Omega := V^\Omega$,

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dashed – *polygonal representation*

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Partition function of the random walk:

For a bounded discrete domain $\Omega \subset \Gamma$ and $x, y \in \Omega$,

$$\mathrm{Z}_\Omega(x; y) := \sum_{\gamma \in S_\Omega(x;y)} \mathrm{w}(\gamma), \quad \mathrm{w}(\gamma) := \frac{\prod_{k=0}^{n(\gamma)-1} \mathrm{w}_{u_k u_{k+1}}}{\prod_{k=0}^{n(\gamma)} \mu_{u_k}},$$

where $S_\Omega(x; y) = \{\gamma = (x = u_0 \sim u_1 \sim \cdots \sim u_{n(\gamma)} = y)\}$ is the set *of all nearest-neighbor paths connecting* x *and* y *inside* Ω (i.e., $u_1, \ldots, u_{n(\gamma)-1} \in \mathrm{Int}\,\Omega$ while we admit $x, y \in \partial\Omega$).

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Further, for $A, B \subset \Omega$, let $\mathrm{Z}_\Omega(A; B) := \sum_{x \in A,\ y \in B} \mathrm{Z}_\Omega(x; y)$.

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Examples: $x, y \in \operatorname{Int}\Omega$: $G_\Omega(x; y)$ Green's function in Ω;
$x \in \operatorname{Int}\Omega, B \subset \partial\Omega$: $\omega_\Omega(x; B)$ hitting prob. (= harmonic measure).

Assumptions on the graph Γ and the edge weights w_{uv}:

- *uniformly bounded degrees:* there exists a constant $\nu_0 > 0$ such that, for all $u \in \Gamma$, $\mu_u := \sum_{(uv) \in \mathrm{E}^\Gamma} \mathrm{w}_{uv} \leqslant \nu_0$ and $\mathrm{w}_{uv} \geqslant \nu_0^{-1}$;

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- *no "flat" angles:* there exists a constant $\eta_0 > 0$ such that all angles between neighboring edges do not exceed $\pi - \eta_0$ (NB: $\Rightarrow$ all degrees of *faces* of Γ are bounded by $2\pi/\eta_0$);

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- *edge lengths are locally comparable:* there exists a constant $\rho_0 \geqslant 1$ such that, for any vertex $u \in \Gamma$, one has

$$\max_{(uv) \in \mathrm{E}^\Gamma} |v - u| \leqslant \rho_0 r_u, \quad \text{where} \quad r_u := \min_{(uv) \in \mathrm{E}^\Gamma} |v - u|;$$

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- *Γ is "quantitatively locally finite"*: for any $\rho \geqslant 1$ there exists some constant $\nu(\rho) > 0$ such that, uniformly over all $u \in \Gamma$,

$$\#\{v \in \Gamma : |v-u| \leqslant \rho r_u\} \leqslant \nu(\rho).$$

Assumptions on the graph Γ and the edge weights w_{uv}:

- **Assumption S ("space"):** There exist two positive constants $\eta_0, c_0 > 0$ such that, uniformly over all *discrete discs* $\mathrm{B}_r^\Gamma(u)$, $u \in \Gamma$, $r \geqslant r_u$, and $\theta \in [0, 2\pi]$, one has

$$\omega_{\mathrm{B}_r^\Gamma(u)}(\, u\, ; \{a \in \partial \mathrm{B}_r^\Gamma(v) : \arg(a-u) \in [\theta, \theta+(\pi-\eta_0)]\}) \geqslant c_0 .$$

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  In other words, there are no exceptional directions: the random walk started at the center of any discrete disc $\mathrm{B}_r^\Gamma(u)$ can exit this disc through any given boundary arc of the angle $\pi-\eta_0$ with probability uniformly bounded away from 0.

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- **Assumption T ("time"):** There exist two positive constants $c_0, C_0 > 0$ such that, uniformly over all $u \in \Gamma$ and $r \geqslant r_u$,

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  In other words, if one considers some time parametrization such that the (expected) time spent by the walk at a vertex v before it jumps is of order r_v^2, then the expected time spent in a discrete disc $\mathrm{B}_r^\Gamma(u)$ by the random walk started at u before it hits $\partial\mathrm{B}_r^\Gamma(u)$ should be of order r^2, uniformly over all discs.

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(Closed) answer: (A. Nachmias, private communication): YES.

Uniform estimates of $\omega_\Omega(z;(ab))$ in simply connected Ω's:

Let Ω be a simply connected discrete domain, $u \in \operatorname{Int}\Omega$, and $(ab) \subset \partial\Omega$ be a (far from z) boundary arc. Let $\mathrm{C}_\Omega(z)$ be the boundary of a discrete disc $\mathrm{B}_r^\Gamma(z)$, $r = \frac{1}{4}\operatorname{dist}(z;\partial\Omega)$.

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Theorem: For some *constants* $\beta_{1,2}, C_{1,2} > 0$, the following estimates are fulfilled *uniformly for all configurations* (Ω, z, a, b):

$$C_1 \exp[-\beta_1 \mathrm{L}_{(\Omega,z,a,b)}] \leqslant \omega_\Omega(z;(ab)) \leqslant C_2 \exp[-\beta_2 \mathrm{L}_{(\Omega,z,a,b)}],$$

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Corollary: Uniformly for all discrete domains (Ω, z, a, b), one has

$$\log(1+\omega_{\mathrm{disc}}^{-1}) \asymp \mathrm{L}_{\mathrm{disc}} \asymp \mathrm{L}_{\mathrm{cont}} \asymp \log(1+\omega_{\mathrm{cont}}^{-1})$$

(hardly available by any coupling arguments, if ω's are exp. small)

Extremal Length $\mathrm{L}_\Omega(\mathrm{C}_\Omega(z);(ab))$:

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In particular, any function $g:\mathrm{E}^\Omega\to\mathbb{R}_+$ gives a lower bound for $\mathrm{L}_\Omega(\mathrm{C}_\Omega(z);(ab))$.

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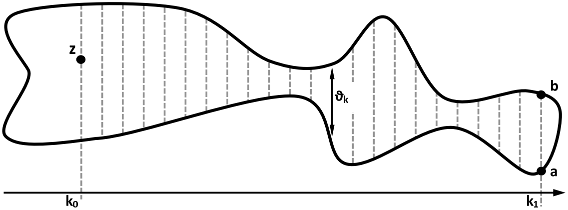

Corollary: *For any $\Omega \subset \mathbb{Z}^2$ and some absolute constants $\beta, C > 0$,*

$$\omega_\Omega(z;(ab)) \leqslant C\exp[-\beta \textstyle\sum_{k=k_0}^{k_1} \vartheta_k^{-1}].$$

Proof: take $g := \vartheta_k^{-1}$ on horizontal edges.

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Let Ω be a simply connected discrete domain, $u \in \mathrm{Int}\,\Omega$, and $(ab) \subset \partial\Omega$ be a (far from z) boundary arc. Let $\mathrm{C}_\Omega(z)$ be the boundary of a discrete disc $\mathrm{B}^\Gamma_r(z)$, $r = \frac{1}{4}\,\mathrm{dist}(z;\partial\Omega)$.

Theorem: For some *constants* $\beta_{1,2}, C_{1,2} > 0$, the following estimates are fulfilled *uniformly for all configurations* (Ω, z, a, b):

$$C_1 \exp[-\beta_1 \mathrm{L}_{(\Omega,z,a,b)}] \leqslant \omega_\Omega(z;(ab)) \leqslant C_2 \exp[-\beta_2 \mathrm{L}_{(\Omega,z,a,b)}],$$

where $\mathrm{L}_{(\Omega,z,a,b)} = \mathrm{L}_\Omega(\mathrm{C}_\Omega(z);(ab))$ denotes the extremal length (aka effective resistance) between $\mathrm{C}_\Omega(z)$ and (ab) in Ω.

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THANK YOU!

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If time permits ... some ideas of the proof on the next slides

$\rightarrow$

Some ideas of the proof:

- Work with discrete *quadrilaterals* $(\Omega; a, b, c, d)$: simply connected domains with four marked boundary points

(then use some additional reduction to handle $\omega_\Omega(z; (ab))$, RW partition functions in annuli, and corresponding extremal lengths $\mathrm{L}_{(\Omega,z,a,b)}$).

Some ideas of the proof:

- Work with discrete *quadrilaterals* $(\Omega; a, b, c, d)$: simply connected domains with four marked boundary points;
- Discrete *cross-ratios* Y_Ω:

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- $\mathrm{Y}_\Omega\widetilde{\mathrm{Y}}_\Omega = 1$, $\mathrm{L}_\Omega\widetilde{\mathrm{L}}_\Omega \asymp 1$. Moreover, $\widetilde{\mathrm{Z}}_\Omega \asymp \widetilde{\mathrm{L}}_\Omega^{-1}$, if $\geqslant \mathrm{const}$.

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THANK YOU ONCE MORE!