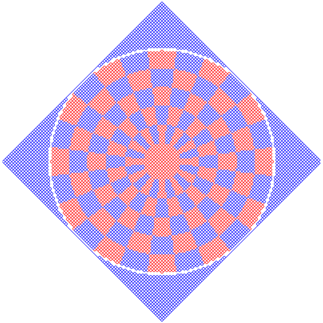
Bipartite dimer model:

t-emebeddings of graphs and GFF

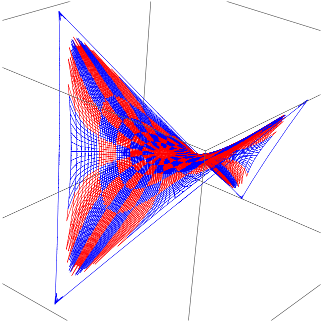
on surfaces in the Minkowski space



Dmitry Chelkak (ENS)

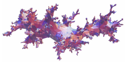
[recent/in progress joint works w/

Benoît Laslier,
Sanjay Ramassamy,
Marianna Russkikh]



FARF–IV, 15.06.2020

Outline:

▷ Long[!]-term motivation: 

▷ Intro: Thurston's height functions, conv. to GFF in a *non-trivial metric*.

▷ *T-embeddings:* basic concepts and *a priori regularity estimates* (w/ Laslier and Russkikh, `arXiv:2001.11871`).

▷ *Perfect t-embeddings* and *Lorentz-minimal surfaces*. *Main theorem* (w/ Laslier and Russkikh, `arXiv:20**.**`).

▷ (Some) open questions/perspectives.

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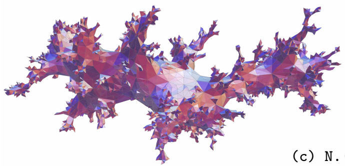
correlation functions/loop ensembles on *random maps* carrying the bipartite dimer [or the *critical Ising*] model by *embedding* them into $\mathbb{C}$ in a special way.



(c) N. Curien

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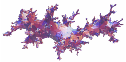
▷ Long[!]-term motivation: 

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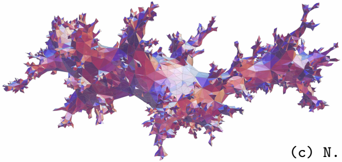

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'Bosonization' (Ising → dimers): [Dubédat'11,...]

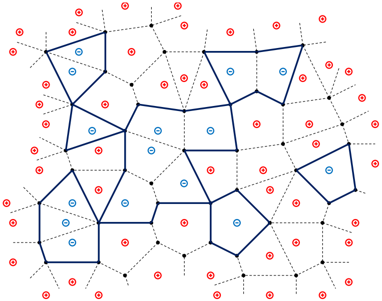
- **Detour: planar Ising model** [Lenz, 1920: centenary!]

- **Lenz-Ising model** on a planar graph G^* (dual to G) is a random assignment of $+/-$ spins to vertices of G^* (=faces of G) according to

$$\mathbb{P}\left[\text{conf. } \sigma \in \{\pm 1\}^{V(G^*)}\right] \propto \exp\left[\beta \sum_{e=\langle uv\rangle} J_{uv}\sigma_u\sigma_v\right] = \mathcal{Z}^{-1} \cdot \prod_{e=\langle uv\rangle:\sigma_u\neq\sigma_v} x_{uv}$$

where $J_{uv} > 0$ are interaction constants preassigned to edges $\langle uv\rangle$, $\beta = 1/kT$, and $x_{uv} = \exp[-2\beta J_{uv}]$.

- **Remark:** w/o magnetic field $\Rightarrow$ 'free fermion'.



[an example with '+' boundary conditions]

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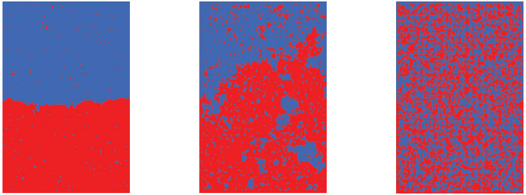
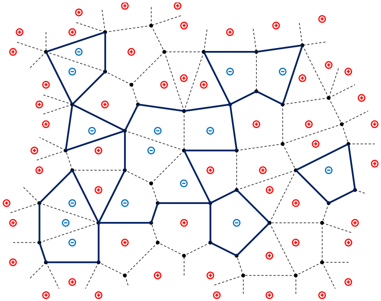
- **Example:** square grid [$x_{\mathrm{crit}} = \sqrt{2}-1$]



$x < x_{\mathrm{crit}}$ $\quad$ $x = x_{\mathrm{crit}}$ $\quad$ $x > x_{\mathrm{crit}}$

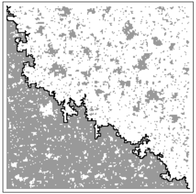
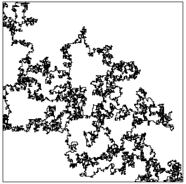


[an example with '+' boundary conditions]

Two descriptions as $\delta \to 0$:

- correlation functions (CFT);
- loop ensembles (SLE/CLE).

- **Known results on regular lattices:**
- *Critical Ising model:* [Smirnov'06 ⇝ ...]
  - *correlations* (fermions, spins, ...) converge to the Ising CFT ($c = \frac{1}{2}$);
  - *interfaces/loop ensembles* converge to $\mathrm{SLE}/\mathrm{CLE}(\kappa)$, $\kappa = 3, \frac{16}{3}$.

[Interfaces on the square lattice. (c) Smirnov'06]

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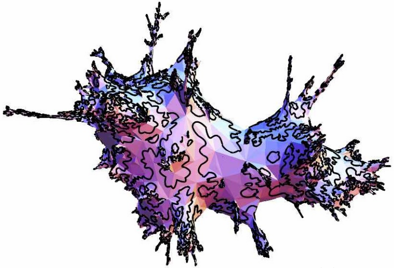


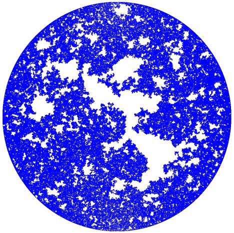
(c) N. Curien

- **Known results on regular lattices:**
- *Bipartite dimer model:* [Kenyon'00 $\rightsquigarrow$ ...]
  - *fluctuations of the height function* converge to the Gaussian Free Field *[to be discussed on the next slides]*
  - *double-dimers loop ensembles*

converge [??]

Kenyon'10,
Dubédat'14,
Basok–Chelkak'18,
... [still not quite] ...

to the nested CLE(4)



(c) D. Wilson

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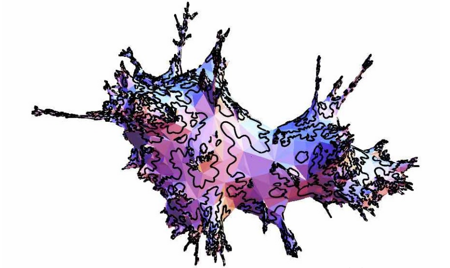


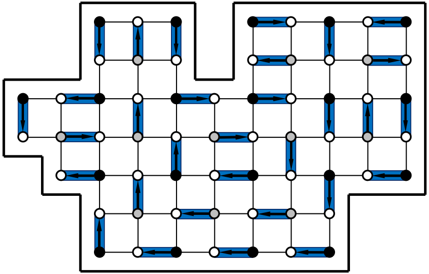
(c) N. Curien

Bipartite dimer model: basics

- $(\mathcal{G}, \nu_{bw})$ – finite weighted bipartite planar graph (w/ marked outer face);
- Dimer configuration = perfect matching $\mathcal{D} \subset E(\mathcal{G})$: subset of edges such that each vertex is covered exactly once;
- *Probability* $\mathbb{P}(\mathcal{D}) \propto \prod_{e \in \mathcal{D}} \nu_e$.

(Very) particular example:

[Temperleyan domains $\mathcal{G}_{\mathrm{T}} \subset \mathbb{Z}^2$]



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- *In Temperleyan domains,* random walks and discrete harmonic functions with 'nice' boundary conditions naturally appear. This is a *very special case.*

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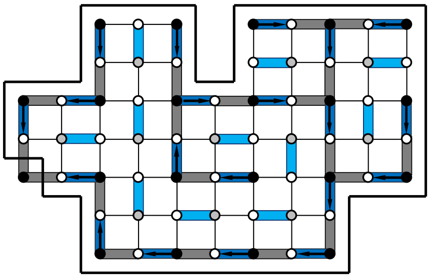
[Temperleyan domains $\mathcal{G}_{\mathrm{T}} \subset \mathbb{Z}^2$]



Temperley bijection: dimers on $\mathcal{G}_{\mathrm{T}}$ $\leftrightarrow$ *spanning trees* on another graph. This procedure is highly sensitive to the *microscopic structure* of the boundary.

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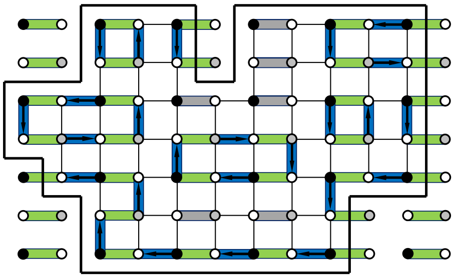

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- **Gaussian Free Field:** $\mathbb{E}[\hbar(z)] = 0$, $\mathbb{E}[\hbar(z)\hbar(w)] = G_\Omega(z,w) = -\Delta_\Omega^{-1}(z,w)$.

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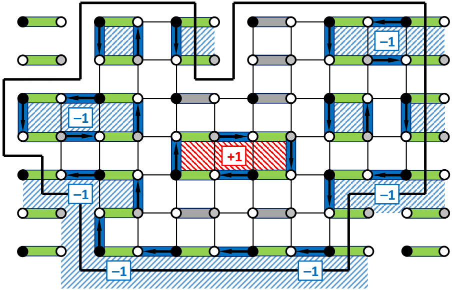
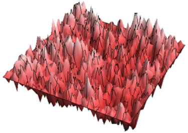


Theorem [Kenyon'00]**:**

$\delta\mathbb{Z}^2 \supset \mathcal{G}^\delta_{\mathrm{T}} \to \Omega \subset \mathbb{C}$

$\Rightarrow \hbar^\delta \to \pi^{-\frac{1}{2}}\mathrm{GFF}(\Omega)$



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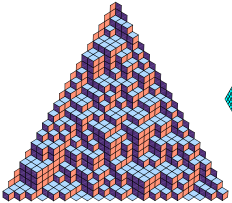
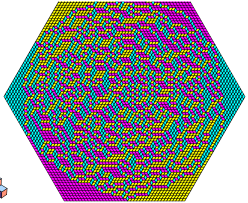
[!] Still, the limit of $\hbar^\delta$ as $\delta \to 0$ *heavily depends* on the limit of (deterministic) **boundary profiles of δh^δ.**

Examples (on $\mathbb{H}\mathrm{ex}^*$) [(c) Kenyon]:





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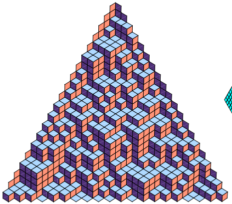
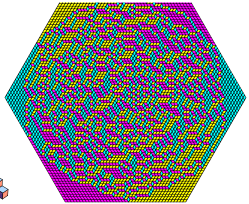




On periodic lattices:

- [Cohn–Kenyon–Propp'00] the random profile δh^δ concentrates near a surface maximizing certain *entropy functional.*
- **Prediction:** [Kenyon–Okounkov'06] $\hbar^\delta \to$ *GFF in a profile-dependent metric.*

[!] Problematic beyond periodic graphs.

Known results: $\delta\mathbb{Z}^2 \supset \mathcal{G}^\delta_{\mathrm{T}} \to \Omega \subset \mathbb{C}$

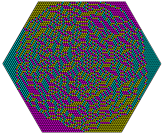
- $\hbar^\delta \to \pi^{-1/2} \cdot \mathrm{GFF}(\Omega)$ [Kenyon'00]

- **Non-flat case: $\mathbf{GFF}_\mu(\boldsymbol{\Omega})$**

▷ *Temperleyan-type domains* $\subset \mathrm{Hex}^*$ coming from T-graphs [Kenyon'04]

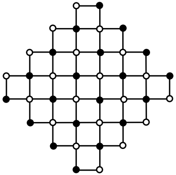
▷ *'polygons' via 'integrable probability'* and (rather hard) asymptotic analysis [Petrov, Bufetov–Gorin, ... '12+]

▷ thorough analysis of *concrete setups (e.g., Aztec diamonds)* w/ interesting behavior [Chhita–Johansson–Young, ... '12+]

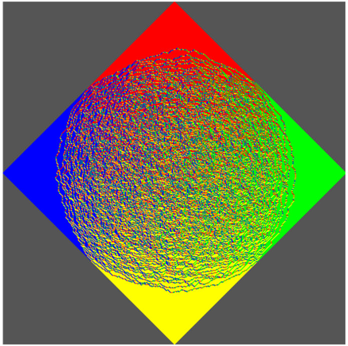


Aztec diamonds
$A_n \subset n^{-1}\mathbb{Z}^2$:

[Elkies – Kuperberg – Larsen – Propp '92, ...]



[(c) A. & M. Borodin, S. Chhita]



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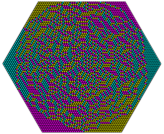
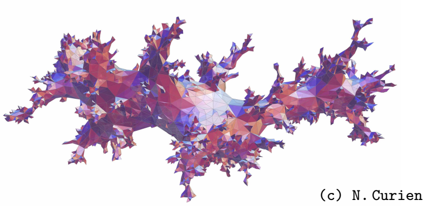


- **Known tools: problematic to apply**
$\updownarrow$ [?] **to generic graphs $(\mathcal{G}, \nu)$**
- **Long[!]-term goal:**

attack *random maps* carrying the bipartite dimer [or the critical Ising] model.



- **Wanted:** *special embeddings* of abstract weighted bipartite planar graphs + *'discrete complex analysis' techniques* on such embeddings

$\rightsquigarrow$ *complex structure in the limit.*

Theorem: [Ch. – Laslier – Russkikh] [arXiv:2001.11871 + 20**.**]

Let $\mathcal{G}^\delta$, $\delta \to 0$, be finite weighted bipartite planar graphs. Assume that

- $\mathcal{T}^\delta$ are *perfect t-embeddings* of $(\mathcal{G}^\delta)^*$ [satisfying assumption $\textsc{Exp-Fat}(\delta)$];
- as $\delta \to 0$, the images of $\mathcal{T}^\delta$ converge to a domain D_ξ [$\xi \in \mathrm{Lip}_1(\mathbb{T})$, $|\xi| < \frac{\pi}{2}$];
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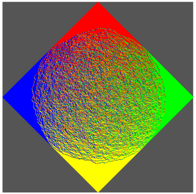
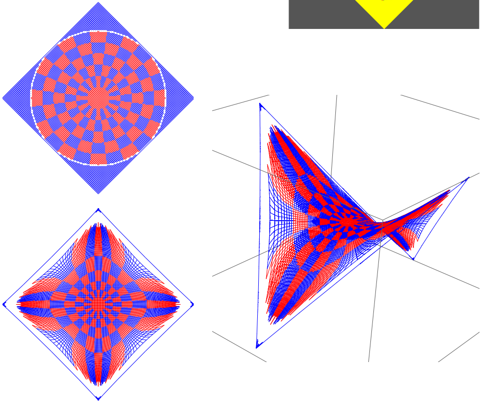
Then, *height functions fluctuations* in the dimer models on $\mathcal{T}^\delta$ converge to the *standard Gaussian Free Field* in the *intrinsic metric of* $\mathrm{S}_\xi \subset \mathbb{R}^{2+1} \subset \mathbb{R}^{2+2}$.

Illustration: Aztec diamonds

[Ch. – Ramassamy] [arXiv:2002.07540]





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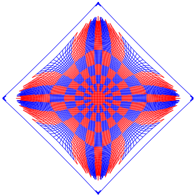
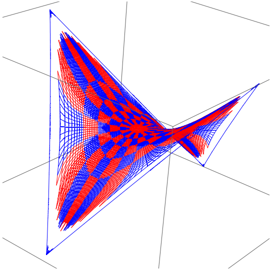
- **Domains D_ξ, surfaces S_ξ:**
- 1-Lipschitz function $|\xi(\phi)| < \frac{\pi}{2}$ on $\mathbb{T}$;
- D_ξ: inside of $z(\phi) = e^{i\phi}/\cos(\xi(\phi))$;
- S_ξ spans $\mathrm{L}_\xi := (z(\phi), \tan(\xi(\phi)))_{\phi \in \mathbb{T}}$

$\mathrm{L}_\xi \subset \{x \in \mathbb{R}^{2+1} : \|x\|^2 = x_1^2 + x_2^2 - x_3^2 = 1\}$.

Aztec case $(\mathrm{D}_\xi, \mathrm{S}_\xi)$:





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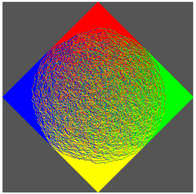
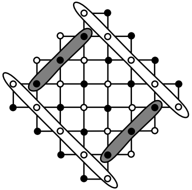
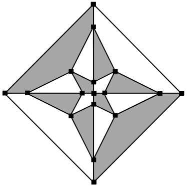
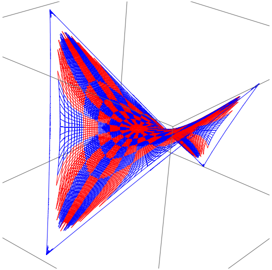








Embeddings of weighted bipartite planar graphs carrying the dimer model

[and admitting reasonable notions of discrete complex analysis]

Coulomb gauges [Kenyon – Lam – Ramassamy – Russkikh, `arXiv:1810.05616`]

⇕

t-embeddings [Ch. – Laslier – Russkikh, `arXiv:2001.11871`, `arXiv:20**.**`]

Particular cases: harmonic/ *Tutte's embeddings* [via the Temperley bijection]

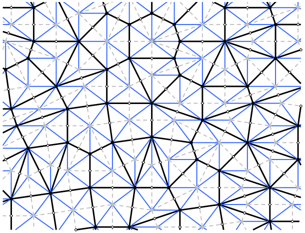
Ising model *s-embeddings* [`arXiv:1712.04192`, via the bosonization]

∪

Extremely particular case:

Baxter's critical Z-invariant Ising model on *rhombic lattices/isoradial graphs*

[Ch. – Smirnov, `arXiv:0910.2045`
"*Universality in the 2D Ising model and conformal invariance of fermionic observables*"]



Embeddings of weighted bipartite planar graphs carrying the dimer model

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- *t-embeddings = Coulomb gauges:* given $(\mathcal{G},\nu)$, find $\mathcal{T}:\mathcal{G}^*\to\mathbb{C}$ $[\mathcal{G}^*$ – *augmented dual*$]$ s.t.

▷ weights ν_e are *gauge equivalent* to $\chi_{(vv')^*} := |\mathcal{T}(v')-\mathcal{T}(v)|$ (i.e., $\nu_{bw} = g_b\chi_{bw}g_w$ for some $g: B\cup W\to\mathbb{R}_+$) and

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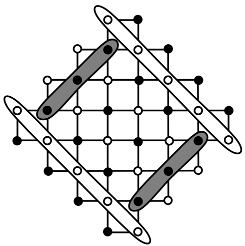

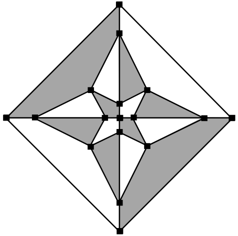

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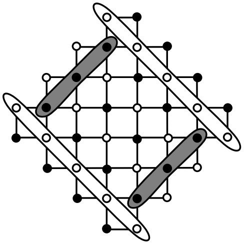
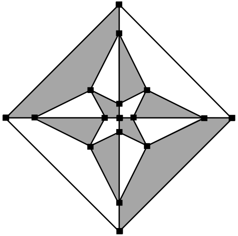
- **Warning:** for general $(\mathcal{G}, \nu)$, the *existence* of perfect t-embeddings is not known though they do exist in particular cases + the count of #(degrees of freedom) matches.





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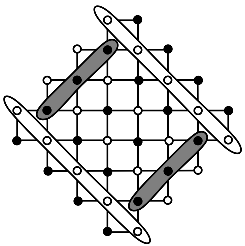
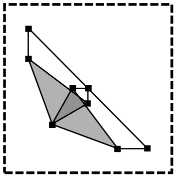
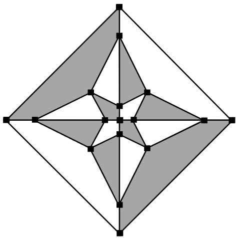


- *origami maps $\mathcal{O}$:* $\mathcal{G}^* \to \mathbb{C}$ ["fold $\mathbb{C}$ along segments of $\mathcal{T}$"]



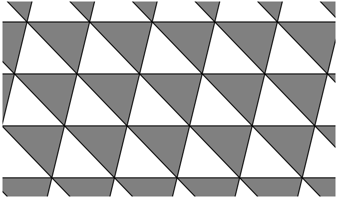
- *T-graphs* $\mathcal{T} + \alpha^2 \mathcal{O}$, $|\alpha| = 1$: **[GeoGebra]**

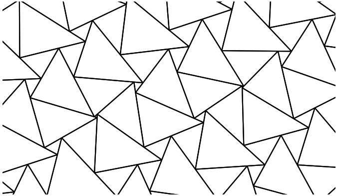


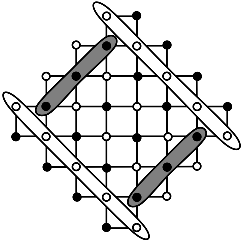
Embeddings of weighted bipartite planar graphs carrying the dimer model

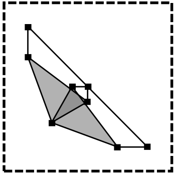
[and admitting reasonable notions of discrete complex analysis]

- **"Regular" case:** triangular grids [Kenyon'04 + Laslier'13]







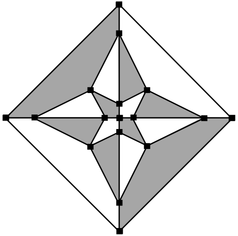


- *T-graphs* $\mathcal{T}+\alpha^2\mathcal{O}$, $|\alpha|=1$: **[GeoGebra]**

- *t-holomorphic functions* $F^\circ : W \to \mathbb{C}$

  $\overline{\alpha}\cdot\{$ gradients of harmonic on $\mathcal{T}+\alpha^2\mathcal{O}$ $\}$

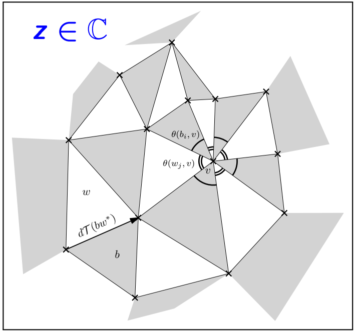
  [this notion does not depend on α]



Embeddings of weighted bipartite planar graphs carrying the dimer model

[and admitting reasonable notions of discrete complex analysis]

A priori regularity theory [arXiv:2001.11871]



- $\mathcal{T}^\delta$ satisfies $\textsc{Lip}(\kappa,\delta)$ for $\kappa < 1$ and $\delta > 0$ if

$$|z' - z| \geq \delta \quad \Rightarrow \quad |\mathcal{O}^\delta(z') - \mathcal{O}^\delta(z)| \leq \kappa \cdot |z' - z|.$$

- (triangulations) $\mathcal{T}^\delta$ satisfy $\textsc{Exp-Fat}(\delta)$ as $\delta \to 0$ if

for each $\beta > 0$, if one removes all '$\exp(-\beta\delta^{-1})$-fat' triangles from $\mathcal{T}^\delta$, then the size of remaining vertex-connected components tends to zero as $\delta \to 0$.

Results: • *Hölder* regularity of *t-holomorphic* functions,

• *Lipschitz* regularity of *harmonic* functions on $\mathcal{T}^\delta + \alpha^2\mathcal{O}^\delta$.

• *What can be said on subsequential limits?*

Embeddings of weighted bipartite planar graphs carrying the dimer model

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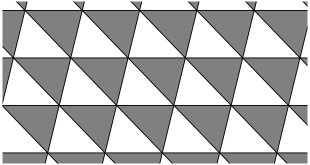
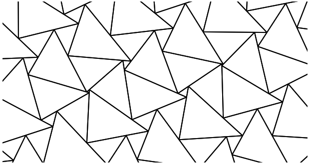
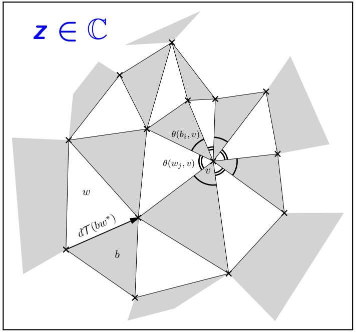
A priori regularity theory [arXiv:2001.11871]

- Assume that $\mathcal{O}^\delta(z) \to \vartheta(z)$, $\delta \to 0$. Then, limits of harmonic functions on $\mathcal{T}^\delta + \alpha^2\mathcal{O}^\delta$ are martingales wrt to a *certain diffusion* whose coefficients *depend on* ϑ, α.







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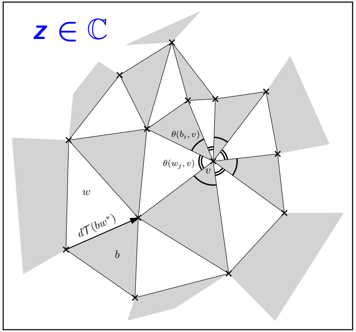


- Assume that $\mathcal{O}^\delta(z) \to \vartheta(z),\ z \in \mathrm{D},\ \delta \to 0$ and that
- $\{(z, \vartheta(z))\}_{z\in\mathrm{D}} \subset \mathbb{R}^{2+2}$ is a *Lorentz-minimal* surface.
- Let a *parametrization* ζ be *conformal* $z_\zeta \overline{z}_\zeta = \vartheta_\zeta \overline{\vartheta}_\zeta$ and *harmonic* $z_{\zeta\bar\zeta} = \vartheta_{\zeta\bar\zeta} = 0$.
- Then, subsequential limits of harmonic functions on all T-graphs $\mathcal{T}^\delta + \alpha^2\mathcal{O}^\delta$, $|\alpha| = 1$, and, moreover, all limits of dimer height functions *correlations are harmonic in* ζ.

Theorem: [Ch. – Laslier – Russkikh] [`arXiv:2001.11871 + 20**.**`]

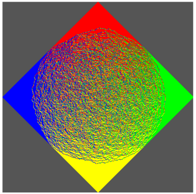
Let $\mathcal{G}^\delta$, $\delta \to 0$, be finite weighted bipartite planar graphs. Assume that

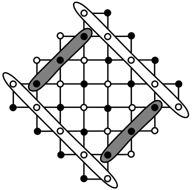
- $\mathcal{T}^\delta$ are *perfect t-embeddings* of $(\mathcal{G}^\delta)^*$ [satisfying assumption $\textsc{Exp-Fat}(\delta)$];
- as $\delta \to 0$, the images of $\mathcal{T}^\delta$ converge to a domain D_ξ [$\xi \in \mathrm{Lip}_1(\mathbb{T})$, $|\xi| < \frac{\pi}{2}$];
- *origami maps* $(\mathcal{T}^\delta, \mathcal{O}^\delta)$ converge to a *Lorentz-minimal surface* $\mathrm{S}_\xi \subset \mathrm{D}_\xi \times \mathbb{R}$.

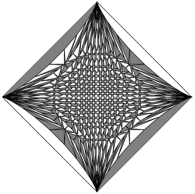
Then, *height functions fluctuations* in the dimer models on $\mathcal{T}^\delta$ converge to the *standard Gaussian Free Field* in the *intrinsic metric of* $\mathrm{S}_\xi \subset \mathbb{R}^{2+1} \subset \mathbb{R}^{2+2}$.

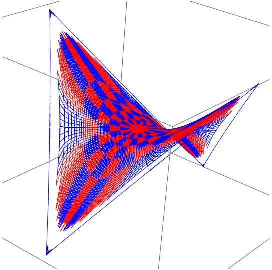
Illustration: Aztec diamonds

[Ch. – Ramassamy] [`arXiv:2002.07540`]









Open questions, perspectives *[general $(\mathcal{G},\nu)$]*

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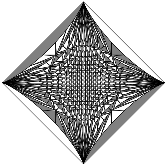

▷ $\deg f_{\mathrm{out}} = 4$: OK [KLRR]

▷ #(degrees of freedom): OK

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- **Why does Lorentz geometry appear?**

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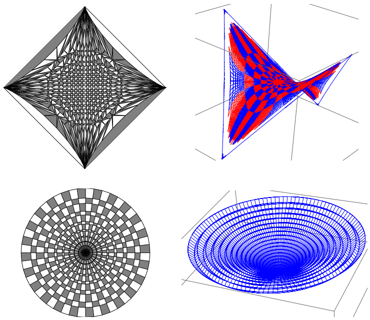
  $\rightsquigarrow$ *Lorentz-minimal cusp* $(z, \operatorname{arcsinh}|z|)$.



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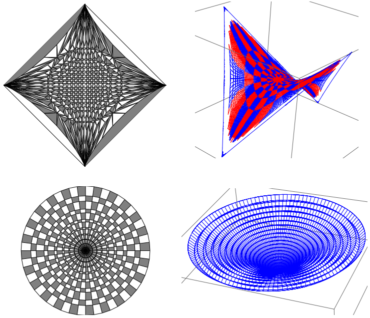

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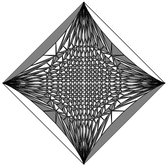

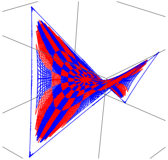

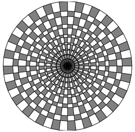

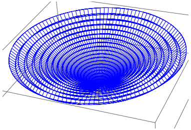

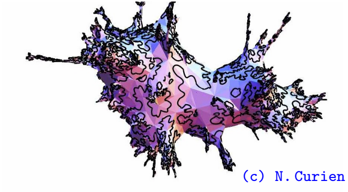

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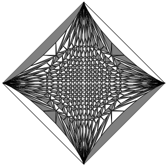

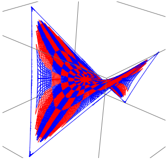

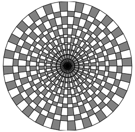

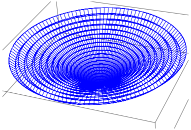

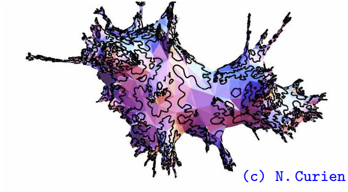

THANK YOU!