<u>Informal</u> [!!!] discussion of the critical planar (= 2D n.n.) Ising model

[Dmitry Chelkak, Zoom @ University of Helsinki, May 21, 2020]

Rough plan:

- <u>Correlations</u> in the critical Z-invariant model on *isoradial graphs;*
- *Criticality* $\leftrightsquigarrow$ kernel ("$0 \in \mathrm{spec}(D)$") of the 3-terms propagation equation
  [!] already here there are questions to investigate;
- *S-embeddings*
  - ▷ what is understood as of now [spoiler: RSW+SLE(16/3) under $\mathcal{Q} = O(\delta)$, δ – *uniform scale* + *bounded angles*]
  - ▷ precise links with *t-embeddings (dimers)* and the [Ch-Laslier-Russkikh] project; ($\mathcal{Q} \rightsquigarrow$ non-flat conformal structures; *regularity assumption* $\textsc{Exp-Fat}$)
  - ▷ how this should look like eventually (wishful thinking);
- **[!]** Random maps: widely open, the ultimate goal is to develop a proper framework.

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[!] originally, convergence heavily relied on *Smirnov's sub-/super-harmonicity on* Γ/Γ^*.

- ▷ analysis of fermionic observables: `arXiv:0910.2045` [ChSmi12]
- ▷ energy density: [Hongler–Smirnov] $\varepsilon_e := (\sin\theta_e)^{-1}\big[\sigma_{u^-(e)}\sigma_{u^+(e)} - \frac{\pi-2\theta_e}{\pi\cos\theta_e}\big]$

  [formally written on $\mathbb{Z}^2$ but works on isoradial graphs more-or-less ad verbum]
- ▷ spin correlations <u>on $\mathbb{Z}^2$</u>: arXiv:1202.2838 [CHI15]
- ▷ <u>on</u> <u>isoradial</u> <u>graphs</u>: Ch.–Izyurov–Mahfouf'20; the normalization C_σ is universal.

- To some extent this program is summarized in `arXiv:1605.09035, 1712.04192`.

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▷ "Mixed correlations" project [CHI, 2016–2020]: *among other things, the goal of the current version is to make the proofs free from the direct comparison with harmonic functions on* Γ/Γ^* *(e.g., another strategy for proving convergence statements).*

- *Criticality* $\leftrightsquigarrow$ kernel of the propagation equation ("$0 \in \mathrm{spec}(D)$").

▷ Kadanoff-Ceva fermions $\chi_c = \sigma_{u(c)}\mu_{v(c)}$ satisfy the 3-terms propagation equation.
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▷ In the periodic setup, the criticality (see Cimasoni–Duminil-Copin or Z.Li) is equivalent to the existence of two periodic solutions $\rightsquigarrow$ s-embeddings.

[Slides-ICM2018] (available at http://www.pdmi.ras.ru/~dchelkak),

[arXiv:1712.04192, Section 6] (very dense, early developments w/o proofs),

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- Question: To understand (beyond the periodic setup) the existence of a complex-valued solution to the propagation equation in the infinite-volume (as a spectral property of the propagator, in the random maps setup, etc)

- *Critical model on s-embeddings, state-of-the-art.*
- [in preparation Ch.'20]: RSW + convergence of the FK-Dobrushin observables ($\Rightarrow$ convergence to SLE(16/3)) under the following (restrictive!) assumptions

[though this covers all periodic graphs and Marcin Lis's circle patterns with radii $\asymp \delta$]:

  - ▷ all quads are of size $O(\delta)$ + uniformly bounded angles
  - ▷ $\mathcal{Q} = O(\delta)$ *(where $\mathcal{Q}$ is "the origami map" = "the function L").*

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- Main tool: s-holomorphic functions on s-embeddings
  $\subset$ t-holomorphic functions on t-embeddings [CLR1, `arXiv:2001.11871`]

- **Open questions (as of now), how all this should look like eventually:**

  RSW under the assumption $\textsc{Lip}(\kappa, \delta)$ (+ maybe smth like $\textsc{Exp-Fat}(\delta)$ as in [CLR1])

  Convergence to a *conformal structure defined by (the graph in $\mathbb{R}^{2+1}$ of) $\mathcal{Q}$*
  (as in [CLR1, Section 6.4], [Ch.–Ramassamy, `arXiv:2002.07540`] and [CLR2])

• [!] Random maps (weighted by Ising [or bipartite dimers]): a theory to develop

Good points:

▷ proper embeddings based on *infinite-volume solutions to the propagation equation* (+ discrete complex analysis on them allowing to analyze Ising correlations);

▷ though this generality still requires a lot of work for Ising, the basic assumptions LIP & EXP-FAT from [CLR1] could be OK for embeddings of random maps.

Wishful thinking/speculations: how could we observe links with the LCFT?

▷ though the convergence of *interfaces* to SLEs is supposed to hold quenched

– *the conformal structure comes from a (random!) $\mathcal{Q}$ in a non-trivial way;*

– limits of *correlations* must include additional non-trivial factors coming from "local scales" (replacement of $\delta^{-1/2}$ and $\delta^{-1/8}C_\sigma$, not universal anymore).

A firm conjecture on how random surfaces in $\mathbb{R}^{2+1}$ should be involved into the overall picture is extremely wanted! (Should it come in conjunction with the relevant QLE?)