Conformal invariance of spin correlations in the planar Ising model

Dmitry Chelkak (Steklov Institute &
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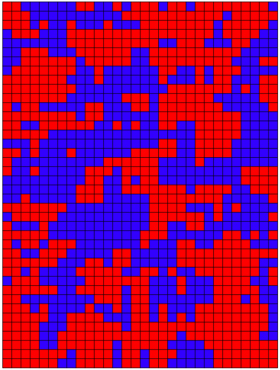
joint project with *Clément Hongler* and
Konstantin Izyurov **(arXiv:1202:2838)**,

"Conformal Invariance,
Discrete Holomorphicity
and Integrability"

Helsinki, June 13, 2012

2D Ising model:
(square grid)



Spins $\sigma_i = +1$ or -1.

Hamiltonian:

$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \,.$$

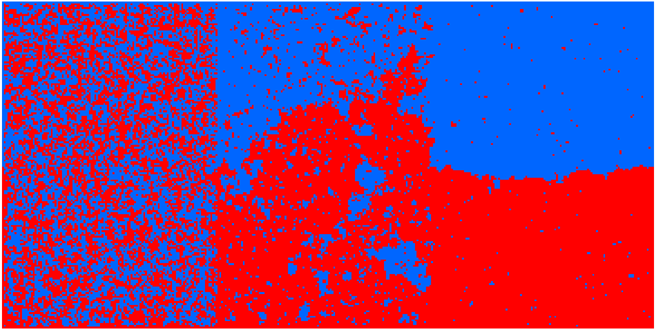
Partition function:

$$\mathbb{P}(conf.) \sim e^{-\beta H} \sim x^{\#\,\langle +- \rangle},$$

where

$$x = e^{-2\beta} \in [0,1]\,.$$

Phase transition, criticality:



$x > x_{\mathrm{crit}}$ $\quad$ $x = x_{\mathrm{crit}}$ $\quad$ $x < x_{\mathrm{crit}}$

(Dobrushin boundary values: two marked points a, b on the boundary; $+1$ on the arc (ab), -1 on the opposite arc (ba))

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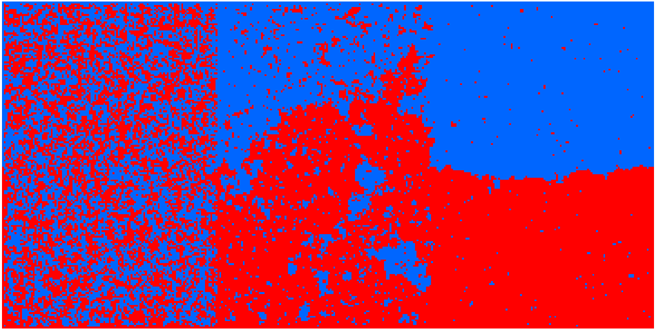

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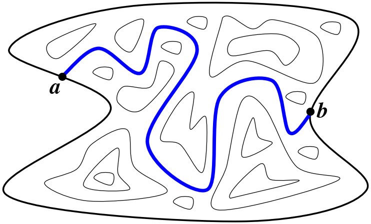
[Kramers-Wannier ∼41]: $x_{\mathrm{crit}} = \frac{1}{\sqrt{2}+1}$

Conformal invariance
(in the scaling limit):

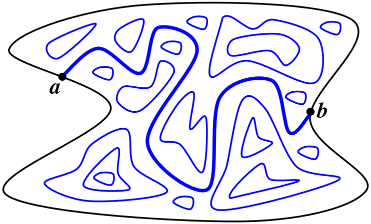
Geometry: single interface



Conformal invariance (in the scaling limit):

Geometry: single interface, or even the full loop ensemble

[cf. Hongler talk]



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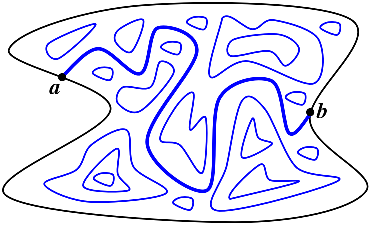
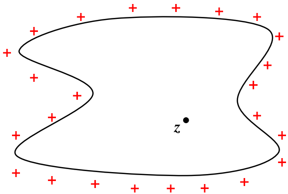


Correlations:
spin correlations, “boundary change operators”, energy density, fermionic observables



Question I:

$$\langle \sigma(z) \rangle_+^{\Omega} := \lim_{\delta \to 0} \quad \mathbb{E}_+^{\Omega^\delta}[\sigma(z^\delta)]$$

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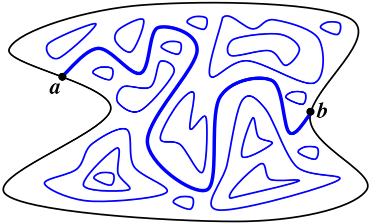

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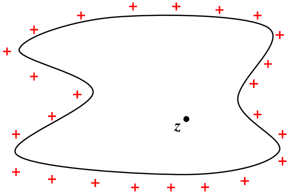

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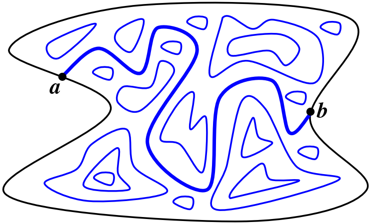

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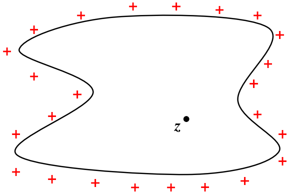

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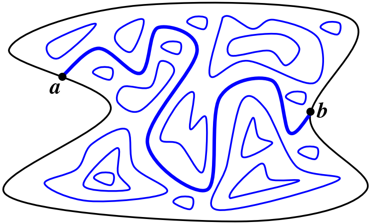

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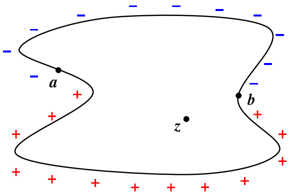

Question II:

$$\frac{\langle \sigma(z) \rangle_{ab}}{\langle \sigma(z) \rangle_{+}} := \lim_{\delta \to 0} \frac{\mathbb{E}_{ab}[\sigma(z^{\delta})]}{\mathbb{E}_{+}[\sigma(z^{\delta})]}$$

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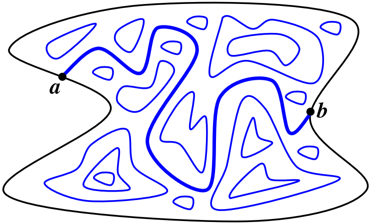

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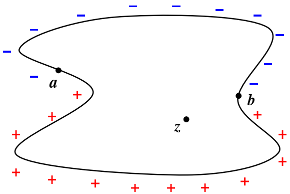

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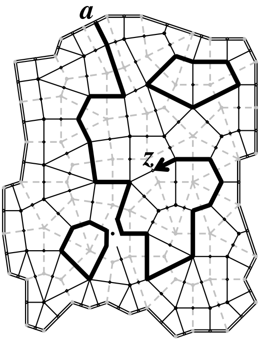
(same for several bulk $z_0,\ldots,z_k$ and boundary $a_1,\ldots,a_{2n}$ points)

Basic fermionic observable and its discrete holomorphicity.

The function F^δ is discrete holomorphic



Basic fermionic observable: *[cf. Cardy talk]*

$$F^\delta(z) := \frac{Z_{config.:a\leadsto z}\left[e^{-\frac{i}{2}\mathrm{winding}(a\leadsto z)}\right]}{Z_{config.:a\leadsto b}\left[e^{-\frac{i}{2}\mathrm{winding}(a\leadsto b)}\right]}, \quad z \in \diamondsuit.$$

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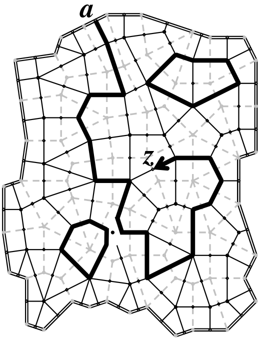
The function F^δ is discrete holomorphic, i.e., satisfies some discrete version of the Cauchy-Riemann identities.

Proof: Natural combinatorial bijection between the two sets of configurations involved into $F^\delta(z_1)$, $F^\delta(z_2)$ gives one *real* equation for any neighbors $z_{1,2}$.



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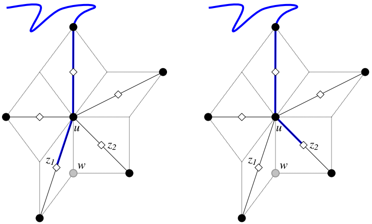

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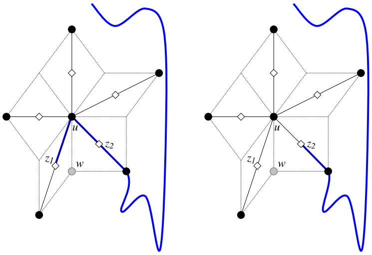

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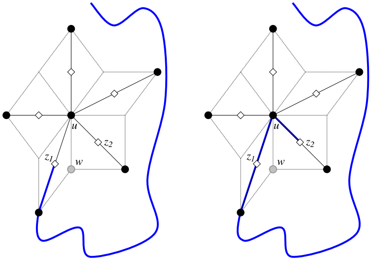

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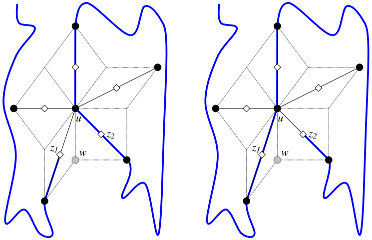

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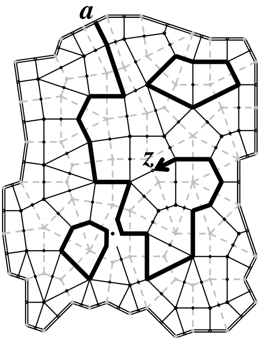

Remarks: (i) there is a strong *physical motivation* for this definition (coming from the "order and disorder operators technique"),

but one can easily define the observable and derive holomorphicity using simple combinatorial arguments ("local rearrangements");

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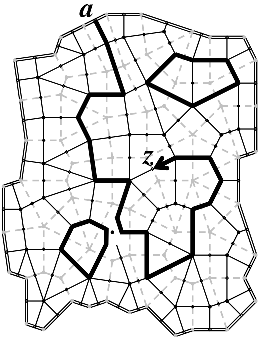

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(ii) *this observable was suggested by Smirnov (∼06) as a tool* for the rigorous proof of the Ising model conformal invariance;

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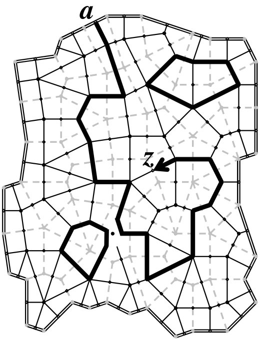

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(ii) *this observable was suggested by Smirnov (∼06) as a tool* for the rigorous proof of the Ising model conformal invariance;
(iii) (hard) *technical problems arises when passing to the limit* (Riemann-type boundary conditions etc).

Conformal invariance (in the scaling limit):

• *Basic fermionic observables: done (Smirnov-Ch., ∼09).*

Theorem: As $\delta \to 0$, properly normalized (at the point b) discrete holomorphic observables $\delta^{-1/2}F^{\delta}$ converge to holomorphic functions $\Psi_{(\Omega;a,b)}$ such that

$$\Psi_{(\Omega;a,b)}(z) = (\phi'(z))^{1/2} \cdot \Psi_{(\phi\Omega;\phi a,\phi b)}(\phi z)$$

for any conformal mapping $\phi : \Omega \to \phi\Omega$.

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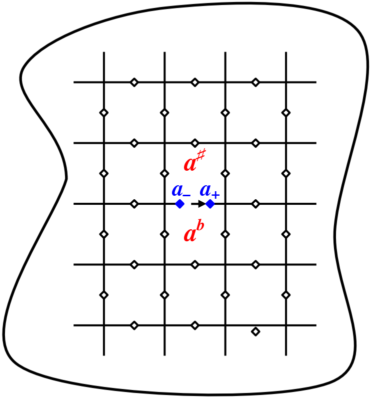
Corollary: *[Smirnov et al, ∼09-11]*
Convergence of Dobrushin interfaces to SLE_3 curves.

Conformal invariance (in the scaling limit):

- *Basic fermionic observables: done (Smirnov-Ch., ∼09).*
- *Energy density field: done (Hongler-Smirnov, Hongler, ∼10).*

Definition: For an edge a in Ω^δ, denote

$$\varepsilon_+^\delta(a) := \mathbb{E}_+[\sigma(a^\sharp)\sigma(a^\flat)]$$



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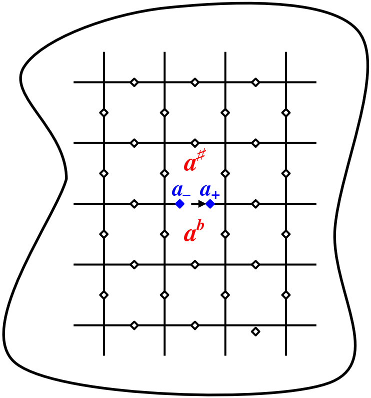
Theorem: As $\delta \to 0$, properly renormalized discrete energy densities $\delta^{-1} \cdot (\varepsilon_{+}^{\delta}(a) - \sqrt{2}/2)$ converge to the continuum limit $\mathcal{E}_{\Omega}$ having the following covariance under conformal mappings:

$$\mathcal{E}_{\Omega}(a) = |\phi'(z)| \cdot \mathcal{E}_{\phi\Omega}(\phi a).$$



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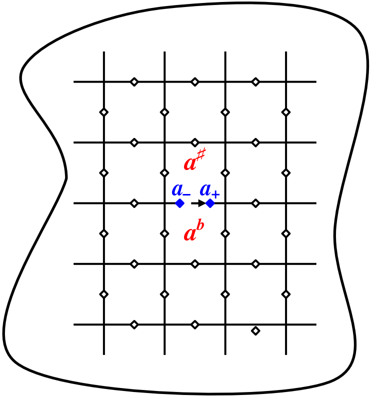
Moreover, all correlations of the renormalized discrete energy densities

$$\delta^{-1} \cdot (\varepsilon_+^\delta(a_j) - \sqrt{2}/2)$$

converge to the continuum limits, and this result extends to any number of boundary points $b_1, \ldots, b_{2n}$, where the boundary conditions change from "$+$" to "$-$".



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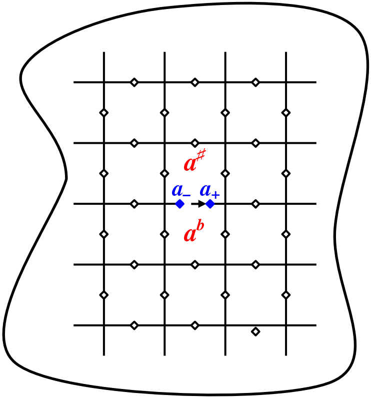
Main idea: Consider the similar observable with a "source point" a_+. Then $\mathbf{F(a_+)}$ counts configurations *without* a, while $-\mathbf{F(a_-)}$ counts configurations *with* a:

$$\varepsilon(a) = \frac{F(a_+) - (-F(a_-))}{F(a_+) + (-F(a_-))}\,.$$



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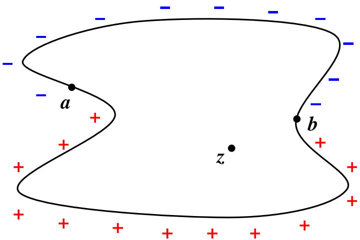
- *Ratios of spin correlations ("+−"/"+"): done (Izyurov-Ch., ∼11).*

Theorem: As $\delta \to 0$, the ratio

$$\frac{\mathbb{E}_{ab}[\sigma(z^\delta)]}{\mathbb{E}_{+}[\sigma(z^\delta)]}$$

tends to the conformally invariant limit (namely, $\cos[\pi \mathrm{hm}_\Omega(z,(ba))]$).



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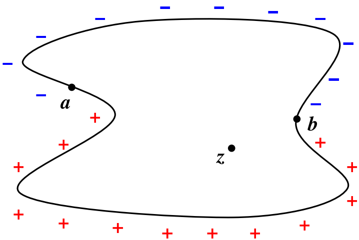

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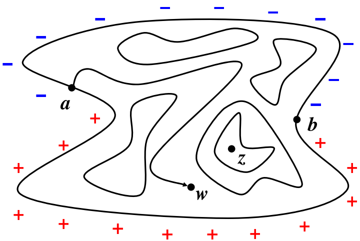
$$\widetilde{F}^{\delta}(w) := Z_{config.:a\rightsquigarrow w}\Big[\, e^{-\frac{i}{2}\mathrm{winding}(a\rightsquigarrow w)} \times (-1)^{\#[\text{loops around } z]} \times \text{sign } \pm 1 \text{ depending on the sheet of } \widetilde{\Omega}^{\delta}\,\Big]$$

$\widetilde{F}^{\delta}$ is a *spinor holomorphic observable* defined on a *double-cover* $\widetilde{\Omega}^{\delta}$ of Ω^{δ}.



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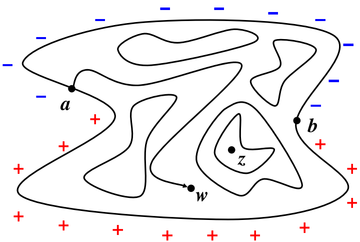

Then

$$\frac{\mathbb{E}_{ab}[\sigma(z^\delta)]}{\mathbb{E}_+[\sigma(z^\delta)]} = \frac{\widetilde{F}^\delta(b)F^\delta(a)}{F^\delta(b)\widetilde{F}^\delta(a)}.$$

Theorem *(Izyurov-Ch., arXiv:1105.5709)*: Let $\Omega \subset \mathbb{C}$ be a bounded multiple connected domain with two marked points a, b on the outer boundary γ_0, and $\gamma_1, \ldots, \gamma_m$ be some of the inner components of $\partial\Omega$. If $\Omega^\delta \to \Omega$ as $\delta \to 0$, then

$$\frac{\mathbb{E}_{a^\delta b^\delta}[\,\sigma(\gamma_1^\delta)\sigma(\gamma_2^\delta)\ldots\sigma(\gamma_m^\delta)\,]}{\mathbb{E}_+[\,\sigma(\gamma_1^\delta)\sigma(\gamma_2^\delta)\ldots\sigma(\gamma_m^\delta)\,]} \;\to\; \vartheta_{ab}^{(\Omega)}(\gamma_1,\ldots,\gamma_m)\,,$$

where the limit is a conformal invariant of $(\Omega; a, b)$ which can be written *explicitly* for $\Omega = \mathbb{C}_+ \setminus \{z_1, \ldots, z_m\}$.

Remark: For multiply connected Ω, we consider *monochromatic* (constant, but unknown) boundary conditions on the inner components of $\partial\Omega$.

Theorem *(Izyurov-Ch., arXiv:1105.5709)*: Let $\Omega \subset \mathbb{C}$ be a bounded multiple connected domain with two marked points a, b on the outer boundary γ_0, and $\gamma_1, \ldots, \gamma_m$ be some of the inner components of $\partial\Omega$. If $\Omega^\delta \to \Omega$ as $\delta \to 0$, then

$$\frac{\mathbb{E}_{a^\delta b^\delta}[\,\sigma(\gamma_1^\delta)\sigma(\gamma_2^\delta)\ldots\sigma(\gamma_m^\delta)\,]}{\mathbb{E}_+[\,\sigma(\gamma_1^\delta)\sigma(\gamma_2^\delta)\ldots\sigma(\gamma_m^\delta)\,]} \;\to\; \vartheta_{ab}^{(\Omega)}(\gamma_1,\ldots,\gamma_m)\,,$$

where the limit is a conformal invariant of $(\Omega; a, b)$ which can be written *explicitly* for $\Omega = \mathbb{C}_+ \setminus \{z_1, \ldots, z_m\}$.

Corollary: For $2n+2$ boundary points the following is fulfilled:

$$\frac{\mathbb{E}_{a_0^\delta\ldots a_{2n+1}^\delta}[\,\sigma(\gamma_1^\delta)\ldots\sigma(\gamma_m^\delta)\,]}{\mathbb{E}_+[\sigma(\gamma_1^\delta)\ldots\sigma(\gamma_m^\delta)\,]} \to \frac{\mathrm{Pf}\,[\,\zeta_{a_ja_k}^{-1}\vartheta_{a_ja_k}^{(\Omega)}(\gamma_1,\ldots,\gamma_m)\,]_{j<k}}{\mathrm{Pf}\,[\,\zeta_{a_ja_k}^{-1}\,]_{0\leqslant j<k\leqslant 2n+1}}\,,$$

where $\zeta_{ab}^{\Omega} = \zeta_{ab}^{\Omega}$ are conformal invariants of $(\Omega; a, b)$ independent of single-point inner components. In particular, $\zeta_{ab}^{\mathbb{C}_+\setminus\{z_1,..,z_m\}} = |b-a|$.

Conformal invariance (in the scaling limit):

- *Basic fermionic observables: done (Smirnov-Ch., ∼09).*
- *Energy density field: done (Hongler-Smirnov, Hongler, ∼10).*
- *Ratios of spin correlations (“+−”/“+”): done (Izyurov-Ch., ∼11).*
- *Spin correlations with “+” boundary conditions:* **done**
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$$\varrho(\delta)^{-\frac{k+1}{2}} \cdot \mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a_0}\sigma_{a_1}\dots\sigma_{a_k}\right] \underset{\delta\to 0}{\longrightarrow} \langle\sigma_{a_0}\sigma_{a_1}\dots\sigma_{a_k}\rangle^{+}_{\Omega},$$

where the *functions* $\langle\sigma_{a_0}\sigma_{a_1}\dots\sigma_{a_k}\rangle^{+}_{\Omega}$ have the covariance

$$\langle\sigma_{a_0}\sigma_{a_1}\dots\sigma_{a_k}\rangle^{+}_{\Omega} = \prod_{j=0}^{k}|\varphi'(a_j)|^{\frac{1}{8}} \cdot \langle\sigma_{\phi a_0}\sigma_{\phi a_1}\dots\sigma_{\phi a_k}\rangle^{+}_{\phi\Omega}.$$

under conformal mappings $\phi:\Omega\to\phi\Omega$.

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Remark: It is known (T.T.Wu, ∼73) that $\varrho(\delta) \sim \mathcal{C}\cdot\delta^{\frac{1}{4}}$ as $\delta \to 0$ *[can be re-derived using our methods, Hongler-Ch., ∼12]*.

Explicit formulae for $\langle \sigma_{a_0} \sigma_{a_1} \ldots \sigma_{a_k} \rangle_\Omega^+$:

Predicted by CFT methods *[Cardy, ∼84]*:

$$\langle \sigma_a \rangle_{\mathbb{C}_+}^+ = \frac{2^{\frac{1}{4}}}{(2\,\mathrm{Im}\ a)^{\frac{1}{8}}} = 2^{\frac{1}{4}} \cdot (\mathrm{rad}_\Omega^{\mathrm{conf}}(a))^{-\frac{1}{8}}$$

$$\langle \sigma_a \sigma_b \rangle_{\mathbb{C}_+}^+ = \frac{\sqrt{\xi_{ab} + \xi_{ab}^{-1}}}{(2\,\mathrm{Im}\ a)^{\frac{1}{8}} (2\,\mathrm{Im}\ b)^{\frac{1}{8}}}, \quad \xi_{ab} := \left| \frac{b-a}{b-\overline{a}} \right|^{\frac{1}{2}}$$

$$= \frac{\langle \sigma_a \rangle_\Omega^+ \langle \sigma_b \rangle_\Omega^+}{(1 - \exp[-2\mathrm{d}_\Omega^{\mathrm{hyp}}(a,b)])^{1/4}}$$

$$[\ \langle \sigma_a \sigma_b \sigma_c \rangle_{\mathbb{C}_+}^+ = \ldots (\textit{explicit}) \ldots\ , \textit{etc}\ \ldots]$$

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For $k \geqslant 2$, we *define* $\langle \sigma_{a_0}\ldots\sigma_{a_k}\rangle_\Omega^+ := \exp[\,\int \mathcal{L}(a_0,\ldots,a_k)\,]$, where

$$\mathcal{L}_\Omega(a_0,\ldots,a_k) := \sum_{j=0}^{k} \mathrm{Re}\,[\,\mathcal{A}_\Omega(a_j; a_0, ..., \hat{a}_j, ..., a_k)da_j\,],$$

coefficients $\mathcal{A}_\Omega(a; a_1, ..., a_k) = (\frac{\partial}{\partial\,\mathrm{Re}\,a} - i\frac{\partial}{\partial\,\mathrm{Im}\,a})\log\langle \sigma_a\sigma_{a_1}\ldots\sigma_{a_k}\rangle_\Omega^+$ are given explicitly (see below) and the primitive is chosen so that

$$\langle \sigma_a\sigma_{a_1}\ldots\sigma_{a_k}\rangle_\Omega^+ \sim \langle \sigma_a\rangle_\Omega^+ \cdot \langle \sigma_{a_1}\ldots\sigma_{a_k}\rangle_\Omega^+ \quad \text{as} \quad a \to \partial\Omega.$$

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Remark: (i) $\mathcal{A}_\Omega(a;a_1,\ldots,a_k)$ can be found as a solution to some $k\times k$ linear system with explicit coefficients;

(ii) both *existence* of the *primitive* $\int\,\mathcal{L}_\Omega(a_0,\ldots,a_k)$ and *consistent multiplicative normalizations* for different k resemble properties of the lattice spin correlations and are proven along the way.

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CFT prediction: *[k ⩾ 2: Burkhardt, Guim, ∼93]*

$$\langle \sigma_{a_0}\ldots\sigma_{a_k}\rangle_{\mathbb{C}_+}^+ = \prod_{m=0}^{k}\frac{1}{(2\,\mathrm{Im}\ a_m)^{\frac{1}{8}}}\left[2^{-\frac{k+1}{2}}\sum_{\mu_0,\ldots,\mu_k=\pm1}\prod_{s<m}(\xi_{a_s a_m})^{\frac{\mu_s\mu_m}{2}}\right]^{\frac{1}{2}}$$

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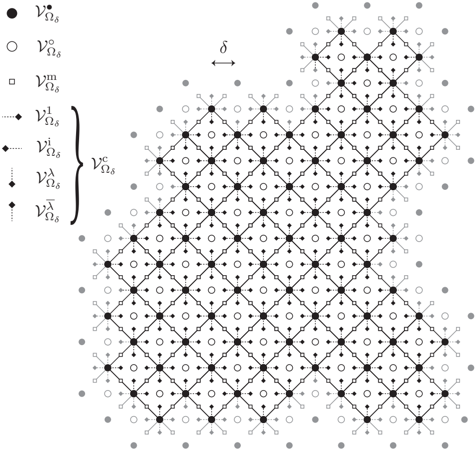
Remark: Formulae agree (i) for small k; (ii) if all $a_0,\ldots,a_k\in i\mathbb{R}_+$.
Open question: to check in full generality.

Two parts of the proof:



Notation:

We work on the square grid rotated by 45° of diagonal mesh sizes 2δ (thus, the distance between adjacent spins is $\sqrt{2}\delta$), and define *s-holomorphic* observables at both *"midedges"* $\mathcal{V}^{\mathrm{m}}_{\Omega_\delta}$ and (four types of) *"corners"* $\mathcal{V}^{\mathrm{c}}_{\Omega_\delta} = \mathcal{V}^{1}_{\Omega_\delta} \cup \mathcal{V}^{\mathrm{i}}_{\Omega_\delta} \cup \mathcal{V}^{\lambda}_{\Omega_\delta} \cup \mathcal{V}^{\bar{\lambda}}_{\Omega_\delta}$, so that the value at the corner is a common projection of the values at nearby midedges.

Two parts of the proof:

I. Convergence of logarithmic derivatives:

Theorem 1:

$$\frac{1}{2\delta}\left(\frac{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a+2\delta}\sigma_{a_1}\dots\sigma_{a_k}\right]}{\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a}\sigma_{a_1}\dots\sigma_{a_k}\right]}-1\right) \underset{\delta\to 0}{\longrightarrow} \mathrm{Re}\,\mathcal{A}_\Omega(a;a_1,\dots,a_k),$$

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Corollary:

$$\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{a}\sigma_{a_1}\dots\sigma_{a_k}\right] \sim \varrho_{k+1}(\delta,\Omega_\delta)\cdot\langle\sigma_a\sigma_{a_1}\dots\sigma_{a_k}\rangle^{+}_{\Omega}$$

for some normalizing factors $\varrho_{k+1}(\delta,\Omega_\delta)$ that might depend on Ω and the number of points $a, a_1, \dots, a_k$ but not on their positions.

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II. Matching the normalizations $\varrho_{k+1}(\delta, \Omega_\delta)$:

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$$\frac{\mathbb{E}^{\mathrm{free}}_{\Omega^{\bullet}_{\delta}}[\sigma_{a+\delta}\sigma_{b+\delta}]}{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a\sigma_b]} \underset{\delta\to 0}{\longrightarrow} \mathcal{B}_\Omega(a;b) = \exp[-\tfrac{1}{2}\mathrm{d}^{\mathrm{hyp}}_{\Omega}(a,b)]$$

(in particular, along the way we also prove *convergence of two-point correlations with free boundary conditions*).

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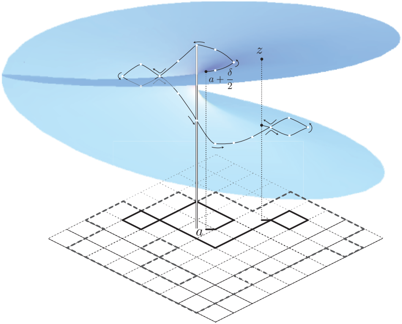
(where $\varrho(\delta) := \mathbb{E}_{\mathbb{C}_\delta}[\sigma_0\sigma_1]$). Further, asymptotic decorrelation as one of the points $a, a_1, \ldots, a_k$ approaches the boundary $\partial\Omega$ gives

$$\varrho_{k+1}(\delta,\Omega_\delta) \sim \varrho_1(\delta,\Omega_\delta)\varrho_k(\delta,\Omega_\delta) \;\Rightarrow\; \varrho_{k+1}(\delta,\Omega_\delta) \sim \varrho(\delta)^{\frac{k+1}{2}}.$$

Main tool: observable branching at the source $a \in \Omega$.

$$F(z) := \frac{1}{\mathcal{Z}^{+}_{\Omega_\delta}[\sigma_a\sigma_{a_1}\ldots\sigma_{a_k}]} \sum_{\gamma\in\mathcal{C}_{\Omega_\delta}(a+\frac{\delta}{2},z)} x_{\mathrm{crit}}^{\#\mathrm{edges}(\gamma)} \cdot \phi_{a;a_1,\ldots,a_k}(\gamma,z),$$

$$\phi_{a;a_1,\ldots,a_k}(\gamma,z) := e^{-\frac{i}{2}\mathrm{wind}(\mathrm{p}(\gamma))} \cdot (-1)^{\#\mathrm{loops}(\gamma\setminus\mathrm{p}(\gamma))} \cdot \mathrm{sheet}(\mathrm{p}(\gamma),z).$$



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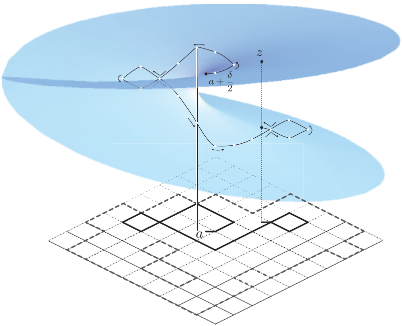

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Proposition 2: if $k = 1$, then, due to Kramers-Wannier duality,

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- convergence results for the s-hol observable (discrete integration of F^2, **technical issues** near $a, \ldots, a_k$)
- local analysis near $a, \ldots, a_k$ (**technical issues**, independent construction of the "full-plane observable")

$\Longrightarrow$ **Theorems 1,2**

Definition and conformal covariance of $\mathcal{A}_\Omega(a; a_1, \ldots, a_k)$:

Let $f = f_{[\Omega,a;a_1,\ldots,a_k]}$ be the (unique) holomorphic spinor in Ω, branching around each of $a, a_1, \ldots, a_k$ and satisfying the following:

$$\lim_{z\to a} \sqrt{z-a}\cdot f(z) = 1, \quad \lim_{z\to a_j} \sqrt{z-a_j}\cdot f(z) \in i\mathbb{R};$$

$$\text{and} \quad \operatorname{Im}\left[f(z)\sqrt{\nu_{\mathrm{out}}(z)}\right] = 0 \ \text{ for } z \in \partial\Omega.$$

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$$\text{and} \quad \operatorname{Im}\left[f(z)\sqrt{\nu_{\mathrm{out}}(z)}\right] = 0 \ \text{ for } z \in \partial\Omega.$$

Then, we define $\mathcal{A}_\Omega$ expanding f near a:

$$f_{[\Omega,a;a_1,\ldots,a_k]}(z) = \frac{1}{\sqrt{z-a}} + 2\mathcal{A}_\Omega(a; a_1, \ldots, a_k)\sqrt{z-a} + \ldots$$

Definition and conformal covariance of $\mathcal{A}_\Omega(a; a_1, \ldots, a_k)$:

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Conformal covariance: If $\phi : \Omega \to \phi\Omega$ is conformal, then

$$f_{[\Omega,a;a_1,\ldots,a_k]}(z) = (\phi'(z))^{1/2} \cdot f_{[\phi\Omega,\phi a;\phi a_1,\ldots,\phi a_k]}(\phi z) \quad \text{and}$$

$$\mathcal{A}_\Omega(a; a_1, \ldots, a_k) = \phi'(a) \cdot \mathcal{A}_{\phi\Omega}(\phi a; \phi a_1, \ldots, \phi a_k) + \frac{1}{8}\frac{\phi''(a)}{\phi'(a)}.$$

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Remark: This covariance property of logarithmic derivatives

$$\mathcal{A}_\Omega(a; a_1, \ldots, a_k) = \left(\tfrac{\partial}{\partial \operatorname{Re} a} - i\tfrac{\partial}{\partial \operatorname{Im} a}\right) \log\langle \sigma_a \sigma_{a_1} \ldots \sigma_{a_k}\rangle_\Omega^+$$

directly leads to conformal covariance of spin-spin correlations:

$$\langle \sigma_{a_0}\sigma_{a_1} \ldots \sigma_{a_k}\rangle_\Omega^+ = \prod_{j=0}^k |\varphi'(a_j)|^{\frac{1}{8}} \cdot \langle \sigma_{\phi a_0}\sigma_{\phi a_1} \ldots \sigma_{\phi a_k}\rangle_{\phi\Omega}^+.$$

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<u>**Addendum**</u>: The method allows one to treat **multiply connected** domains *[cf. Izyurov talk]* and **mixed correlations** (energies-spins) (w/o PDE analysis usual for CFT methods) – *[work in progress]*.

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THANK YOU!