Planar Ising model at criticality:

state-of-the-art and perspectives

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Planar Ising model at criticality: outline

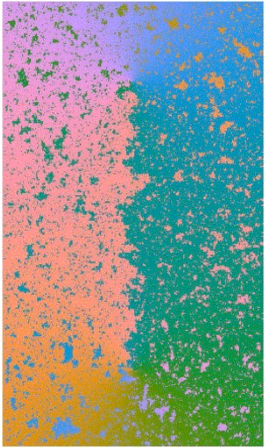
- **Combinatorics**

  Definition, phase transition
  Dimers and fermionic observables
  Spin correlations and fermions on double-covers
  Kadanoff–Ceva's disorders and propagation equation
  Diagonal correlations and orthogonal polynomials

- **Conformal invariance at criticality**

  S-holomorphic functions and Smirnov's s-harmonicity
  Spin correlations: convergence to tau-functions
  More fields and CFT on the lattice
  Convergence of interfaces and loop ensembles
  Tightness of interfaces and 'strong' RSW

- **Beyond regular lattices: s-embeddings [2017+]**
- **Perspectives and open questions**



[two disorders inserted]

(c) Clément Hongler (EPFL)

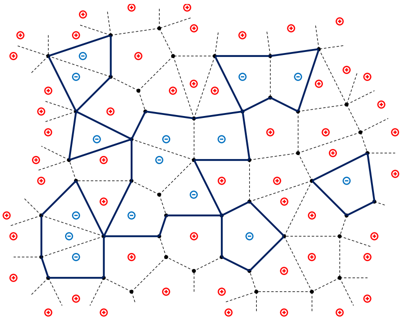
Planar Ising model: definition [Lenz, 1920] **[Centenary soon!]**

- **Lenz-Ising model** on a planar graph G^* (dual to G) is a random assignment of $+/-$ spins to vertices of G^* (=faces of G) according to

$$\mathbb{P}\left[\text{conf. } \sigma \in \{\pm 1\}^{V(G^*)}\right] \;\propto\; \exp\left[\beta \textstyle\sum_{e=\langle uv\rangle} J_{uv}\sigma_u\sigma_v\right] \;=\; \mathcal{Z}^{-1}\cdot \textstyle\prod_{e=\langle uv\rangle:\sigma_u\neq\sigma_v} x_{uv}\,,$$

where $J_{uv} > 0$ are interaction constants preassigned to edges $\langle uv\rangle$, $\beta = 1/kT$, and $x_{uv} = \exp[-2\beta J_{uv}]$.

- **Remark: w/o magnetic field ⇒ 'free fermion'.**



Ensemble of domain walls between '$+$' and '$-$' spins.

- **'$+$' boundary conditions ⇒ collection of loops.**

- **Example:** homogeneous model ($x_{uv} = x$) on $\mathbb{Z}^2$.

◦ Ising'25: no phase transition in 1D ⇝ doubts;
◦ Peierls'36: existence of the phase transition in 2(+)D;
◦ Kramers-Wannier'41: $x_{\text{self-dual}} = \sqrt{2} - 1$;
◦ **Onsager'44:** sharp phase transition at $x_{\text{crit}} = x_{\text{self-dual}}$.

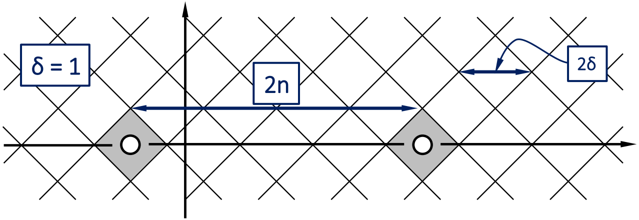
Planar Ising model: phase transition [Kramers–Wannier'41: $x_{\text{crit}} = \sqrt{2} - 1$ on $\mathbb{Z}^2$]

- **Spin-spin correlations:** e.g., two spins at distance $\mathbf{2n} \to \infty$ along a diagonal.

$x < x_{\text{crit}}$: does not vanish;

$\boldsymbol{x = x_{\text{crit}}}$: **power-law** decay;

$x > x_{\text{crit}}$: exponential decay.



Theorem ["diagonal correlations", Kaufman–Onsager'49, Yang'52, McCoy–Wu'66+]**:**

(i) For $\boldsymbol{x = \tan \tfrac{1}{2}\theta} < x_{\text{crit}}$, one has $\boldsymbol{\lim_{n\to\infty} \mathbb{E}^{x}_{\mathbb{C}^\diamond}[\sigma_0\sigma_{2n}] = (1 - \tan^4\theta)^{1/4}} > 0$.

(ii) At criticality, $\mathbb{E}^{x=x_{\text{crit}}}_{\mathbb{C}^\diamond}[\sigma_0\sigma_{2n}] \; = \; \left(\frac{2}{\pi}\right)^n \cdot \prod_{k=1}^{n-1}\left(1 - \frac{1}{4k^2}\right)^{k-n} \; \sim \; \mathrm{C}_\sigma^2 \cdot (\mathbf{2n})^{-\frac{1}{4}}$.

Remark: Many highly nontrivial results on the *spin correlations in the infinite volume* are known. Reference: B.M.McCoy – T.T.Wu "The two-dimensional Ising model".

Planar Ising model: phase transition [Kramers–Wannier'41: $x_{crit} = \sqrt{2} - 1$ on $\mathbb{Z}^2$]

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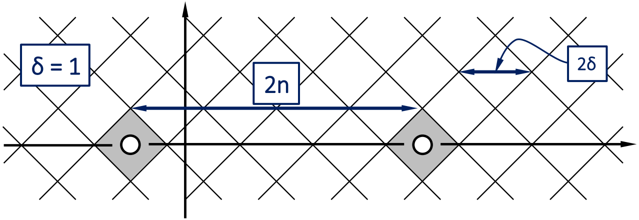


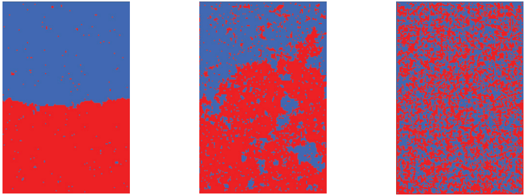
- **Domain walls structure:**

$x < x_{\text{crit}}$: "straight";

$x = x_{\text{crit}}$: SLE(3), CLE(3);

$x > x_{\text{crit}}$: SLE(6), CLE(6).

[this is not proved]



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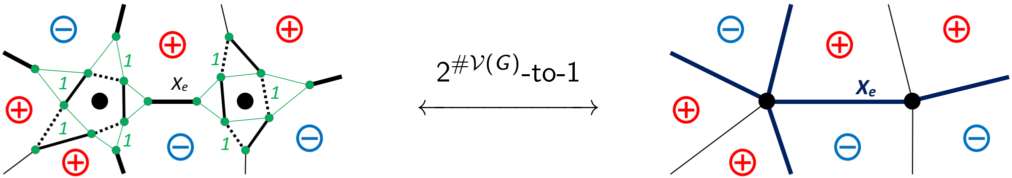
Combinatorics: planar Ising model via dimers ('60s) and fermionic observables



Fisher's graph $G^{\mathbf{F}}$: vertices are corners and oriented edges of G.

- **Kasteleyn's theory:** $\mathbf{F} = \overline{\mathbf{F}} = -\mathbf{F}^{\top}$, $\mathcal{Z} \cong \mathrm{Pf}[\mathbf{F}]$
- **Fermions:** $\langle \phi_c \phi_d \rangle := \mathbf{F}^{-1}(c, d) = -\langle \phi_d \phi_c \rangle$

Pfaffian (or Grassmann variables) formalism:

$$\langle \phi_{c_1} \ldots \phi_{c_{2k}} \rangle = \mathrm{Pf}[\langle \phi_{c_p} \phi_{c_q} \rangle]_{p,q=1}^{2k}$$

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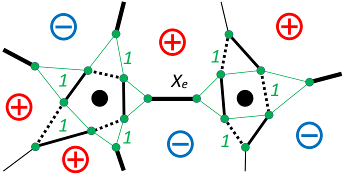
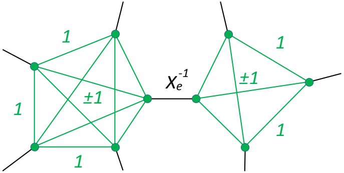


Fisher's graph G^{F}: vertices are corners and oriented edges of G.

There are other combinatorial correspondences of the same kind:

$$\mathcal{Z} \cong \mathrm{Pf}[\mathbf{F}] \cong \mathrm{Pf}[\mathbf{K}] \cong \mathrm{Pf}[\mathbf{C}]$$



Kasteleyn's terminal graph G^{K}, vertices = oriented edges of G.

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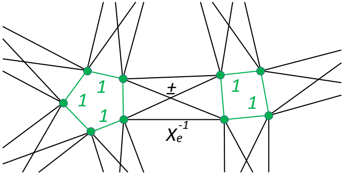

G^{C}: vertices = corners of G.

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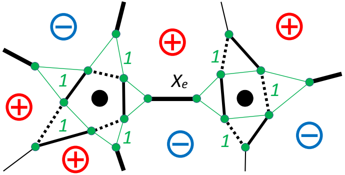

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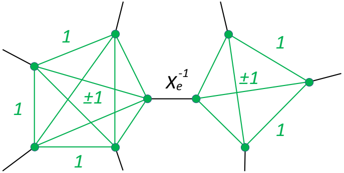

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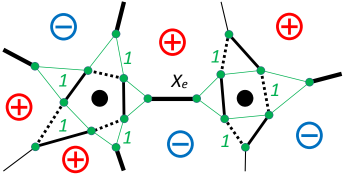
- **Two other useful techniques:**
  - **Kac–Ward matrix** is equivalent to $\mathbf{K}$;
  - **Smirnov's fermionic observables (2000s)** are combinatorial expansions of $\mathbf{Pf}[\mathbf{F}_{\mathcal{V}(G^{\mathbf{F}})\setminus\{c,d\}}]$.

Reference: arXiv:1507.08242 (w/ D. Cimasoni and A. Kassel)

"Revisiting the combinatorics of the 2D Ising model"

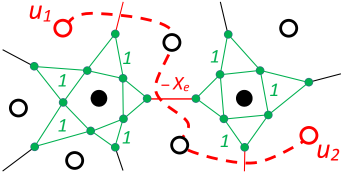
Combinatorics: spin correlations and fermions on double-covers



Fisher's graph G^{F}: vertices are corners and oriented edges of G.

Observation:

$$\mathbb{E}[\sigma_{u_1}\ldots\sigma_{u_n}] = \frac{\mathrm{Pf}[\,\mathrm{F}_{[u_1,..,u_n]}\,]}{\mathrm{Pf}[\,\mathrm{F}\,]}$$



One changes $x_e \mapsto -x_e$ along $\gamma_{[u_1,u_2]}$ to compute $\mathbb{E}[\sigma_{u_1}\sigma_{u_2}]$.

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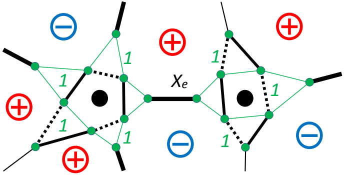

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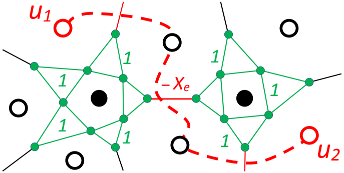

One changes $x_e \mapsto -x_e$ along $\gamma_{[u_1,u_2]}$ to compute $\mathbb{E}[\sigma_{u_1}\sigma_{u_2}]$.

Corollary: Let $w_1 \sim u_1$. The ratio $\dfrac{\mathbb{E}[\sigma_{w_1}\sigma_{u_2}\ldots\sigma_{u_n}]}{\mathbb{E}[\sigma_{u_1}\sigma_{u_2}\ldots\sigma_{u_n}]}$ can be expressed via $\mathrm{F}^{-1}_{[u_1,..,u_n]}$.

Remark: Instead of fixing cuts one can view $F^{-1}_{[u_1,..,u_n]}(c^\flat, d) = -F^{-1}_{[u_1,..,u_n]}(c^\sharp, d)$ as a spinor on the **double-cover** $G^{\mathrm{F}}_{[u_1,..,u_n]}$ of the graph G^{F} ramified over faces $u_1, .., u_n$.

Combinatorics: Kadanoff–Ceva('71) disorders and propagation equation

- Given (an even number of) *vertices* $v_1, \ldots, v_m$, consider the Ising model on (the faces of) the double-cover $G^{[v_1,..,v_m]}$ ramified over $v_1, \ldots, v_m$ with the *spin-flip symmetry constraint* $\sigma_{u^\flat} = -\sigma_{u^\sharp}$ provided that $u^\flat, u^\sharp$ lie over the same face u of G.

- Define $\langle \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle$

$$:= \mathbb{E}^{[v_1,..,v_m]}[\sigma_{u_1} \ldots \sigma_{u_n}] \cdot \mathcal{Z}^{[v_1,..,v_m]} / \mathcal{Z} \,.$$

[!] By definition, this (formal) correlator changes the sign when one of u_k goes around of one of v_s.

[two disorders inserted]

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- For a corner c of G, define $\chi_c := \mu_{v(c)} \sigma_{u(c)}$.
- **Proposition:** If all vertices $v(c_k)$ are distinct, then

$$\pm \langle \chi_{c_1} \ldots \chi_{c_{2k}} \rangle = \pm \langle \phi_{c_1} \ldots \phi_{c_{2k}} \rangle.$$

Proof: expand both sides combinatorially on G.

[two disorders inserted]

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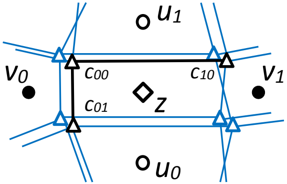

Parameterization:

$$x_e = \tan \tfrac{1}{2}\theta_e$$

- **Propagation equation:** Let $X(c) := \langle \chi_c \mathcal{O}[\mu,\sigma] \rangle$.
  Then $\boldsymbol{X(c_{00}) = X(c_{01}) \cos\theta_e + X(c_{10}) \sin\theta_e}$.
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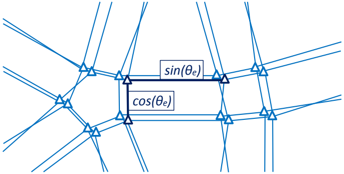

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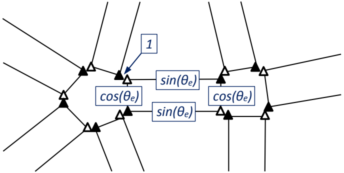


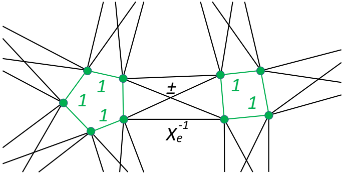
G^{D} : bipartite (Wu–Lin'75).

Fact: $\mathrm{D}^{-1} = \mathrm{C}^{-1} + \mathrm{local}$.

- **Propagation equation:** Let $X(c) := \langle \chi_c \mathcal{O}[\mu,\sigma] \rangle$.
  Then $X(c_{00}) = X(c_{01}) \cos\theta_e + X(c_{10}) \sin\theta_e$.

  [Perk'80, Dotsenko–Dotsenko'83, . . . , Mercat'01]

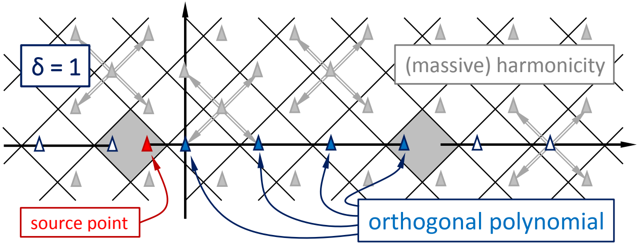
- **Bosonization:** To obtain a combinatorial representation of the model via *dimers on* G^{D} one should start with *two Ising configurations*
  [e.g., see Dubédat'11, Boutillier–de Tilière'14]



G^{C} : vertices = corners of G.

Infinite-volume limit on $\mathbb{Z}^2$: diagonal correlations and orthogonal polynomials

- The propagation equation implies the (massive) harmonicity of spinors on each type of the corners.
- Fourier transform allows to construct such a spinor explicitly.
- Its values on $\mathbb{R}$ must be coefficients of an *orthogonal polynomial*



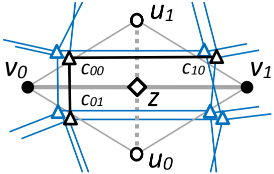
Theorem ["diagonal correlations", Kaufman–Onsager'49, Yang'52, McCoy–Wu'66+]:

(i) For $x = \tan\frac{1}{2}\theta < x_{\mathrm{crit}}$, one has $\lim_{n\to\infty}\mathbb{E}^{x}_{\mathbb{C}^\diamond}[\sigma_0\sigma_{2n}] = (1-\tan^4\theta)^{1/4} > 0$.

(ii) At criticality, $\mathbb{E}^{x=x_{\mathrm{crit}}}_{\mathbb{C}^\diamond}[\sigma_0\sigma_{2n}] = \left(\frac{2}{\pi}\right)^n \cdot \prod_{k=1}^{n-1}\left(1-\frac{1}{4k^2}\right)^{k-n} \sim \mathcal{C}_\sigma^2 \cdot (2n)^{-\frac{1}{4}}$.

Remark: Originally considered as a very involved derivation, nowadays it can be done in two pages (see arXiv:1605:09035), based on the strong Szegö theorem for simple *real weights on* $\mathbb{T}$.

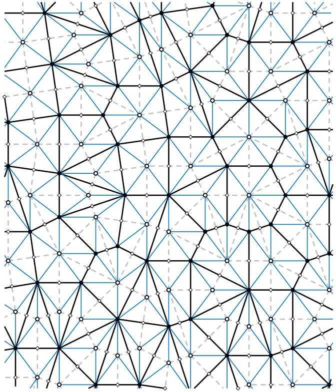
Conformal invariance at x_{crit}: s-holomorphicity



Assume that each $(v_0 u_0 v_1 u_1)$ is drawn as a *rhombus* with an *angle* $2\theta_{v_0 v_1}$ and

$$\boldsymbol{x_e = \tan \tfrac{1}{2}\theta_e}$$

- **Critical Z-invariant model** [Baxter'86] **on isoradial graphs:** [...,Boutillier–deTilière–Raschel'16]



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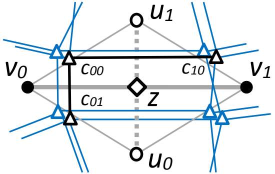
  Then $\boldsymbol{X(c_{00}) = X(c_{01}) \cos \theta_e + X(c_{10}) \sin \theta_e}$.

Remark: In particular, this setup includes

- square ($x_{\mathrm{crit}} = \sqrt{2} - 1 = \tan \frac{\pi}{8}$),
- honeycomb ($x_{\mathrm{crit}} = 1/\sqrt{3} = \tan \frac{\pi}{6}$),
- triangular ($x_{\mathrm{crit}} = 2 - \sqrt{3} = \tan \frac{\pi}{12}$) and
- rectangular ($2x_{\mathrm{h}}/(1-x_{\mathrm{h}}^2) \cdot 2x_{\mathrm{v}}/(1-x_{\mathrm{v}}^2) = 1$) grids.

Conformal invariance at x_{crit}: s-holomorphicity



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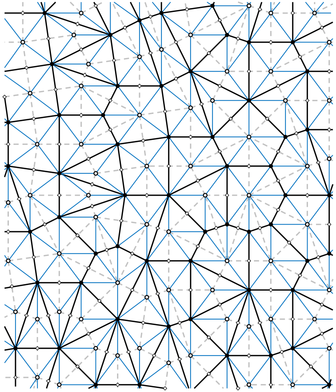

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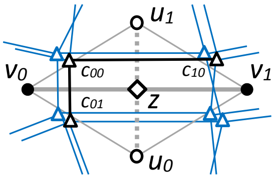
- **S-holomorphicity:** Let $F(c) := \eta_c \delta^{-1/2} X(c)$,
  where $\eta_c := e^{i\frac{\pi}{4}} \exp[-\frac{i}{2} \arg(v(c) - u(c))]$.

  Then $F(c) = \mathrm{Pr}[F(z); \eta_c] = \frac{1}{2}[F(z) + \eta_c^2 \overline{F(z)}]$
  for some $F(z) \in \mathbb{C}$ and all corners $c \sim z$.

Conformal invariance at $x_{\rm crit}$: s-holomorphicity



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- **A priori regularity theory for s-holomorphic functions** [Ch.–Smirnov'09] is based on the following miraculous fact:

- **Smirnov's s-harmonicity:**

  Let F be s-holomorphic. Then

  $$\Delta^\bullet H_F \geqslant 0, \quad \Delta^\circ H_F \leqslant 0,$$

  where

  the function H_F is defined by

  $$H_F(v) - H_F(u) := (X(c))^2$$

  and can/should be viewed as

  $$H_F = \int \mathrm{Im}[F(z)^2 dz].$$

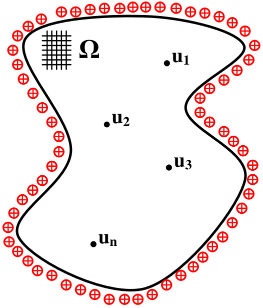
Conformal invariance at x_{crit}: spin correlations ['12, w/ C. Hongler & K. Izyurov]

- **Theorem:** Let $\Omega \subset \mathbb{C}$ be a (bounded) simply connected domain and $\Omega_\delta \to \Omega$ as $\delta \to 0$. Then

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}] \underset{\delta \to 0}{\to} \mathrm{C}^n_\sigma \cdot \langle \boldsymbol{\sigma_{u_1} \ldots \sigma_{u_n}} \rangle^{+}_{\Omega},$$

where $\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^{+}_{\Omega} = \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle^{+}_{\Omega'} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}}$ for conformal mappings $\varphi : \Omega \to \Omega'$ and

$$\left[\langle \boldsymbol{\sigma_{u_1} \ldots \sigma_{u_n}} \rangle^{+}_{\mathbb{H}} \right]^2 = \prod_{1 \leqslant s \leqslant n} (2\,\mathrm{Im}\ u_s)^{-\frac{1}{4}} \cdot \sum_{\beta \in \{\pm 1\}^n} \prod_{s<m} \left| \frac{u_s - u_m}{u_s - \overline{u}_m} \right|^{\frac{\beta_s \beta_m}{2}}.$$

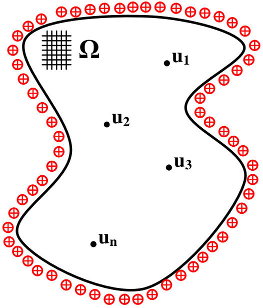


- **Techniques:** Analysis of the kernel $\mathbf{D}^{-1}_{[u_1,..,u_n]}$ viewed as the s-holomorphic solution to a discrete Riemann-type boundary value problem. Applying Smirnov's trick, boundary conditions $\mathbf{Im}[F(\zeta)\tau(\zeta)^{1/2}] = \mathbf{0}$ become $\int^{\zeta} \mathrm{Im}[F(z)^2 dz] = H_F(\zeta) = \mathbf{0},\ \zeta \in \partial\Omega.$

Conformal invariance at x_{crit}: spin correlations ['12, w/ C. Hongler & K. Izyurov]

As $\delta \to 0$, one gets the **isomonodromic τ-function**

$$:\det D_{[\Omega;u_1,\dots,u_n]}: , \quad \text{where} \quad D_{[\Omega;u_1,\dots,u_n]}f := \partial\overline{f}$$



is an anti-Hermitian operator acting in (originally) the *real Hilbert space* of spinors $f : \Omega_{[u_1,\dots,u_n]} \to \mathbb{C}$ satisfying Riemann-type b.c. $\overline{f} = \tau f$ *on* $\partial\Omega$.

[Kyoto school (Jimbo, Miwa, Sato, Ueno)'70s ; . . . ; Palmer'07 "Planar Ising correlations"; Dubédat'11]

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- **Remark:** Passing to the *complex Hilbert space* one gets the (massless) Dirac operator

$$\begin{pmatrix} 0 & \partial \\ \overline{\partial} & 0 \end{pmatrix} \begin{pmatrix} f \\ \widetilde{f} \end{pmatrix} = \begin{pmatrix} \partial \widetilde{f} \\ \overline{\partial} f \end{pmatrix}$$

with *b.c.* $\widetilde{f} = \tau f$. For $\Omega = \mathbb{H}$ this operator boils down to

$$f \mapsto \overline{\partial} f \text{ on } \mathbb{C}_{[u_1,\dots,u_n,\overline{u}_1,\dots,\overline{u}_n]}.$$

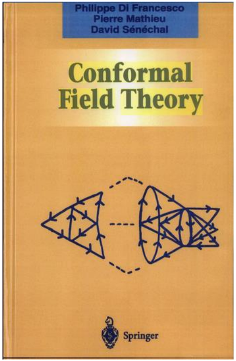
- **Convergence of random distributions:** Basing on the convergence of multi-point spin correlations, one can study the convergence of *random fields* $(\delta^{-\frac{1}{8}} \sigma_u)_{u \in \Omega_\delta}$ to a (non-Gaussian!) random Schwartz distribution on Ω [Camia–Garban–Newman '13, Furlan–Mourrat '16] (see also [Caravenna–Sun–Zygouras '15] for disorder relevance results).

Conformal invariance at x_{crit}: more fields and CFT on the lattice

From the CFT perspective, the 2D critical Ising model is

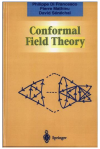
- **FFF (= Fermionic Free Field):** $\mathcal{Z} = \mathrm{Pf}[\mathrm{D}]$.
- Minimal model with central charge $c = \frac{1}{2}$ and three primary fields $\mathbf{1}, \boldsymbol{\sigma}, \varepsilon$ with scaling exponents $\mathbf{0}, \frac{\mathbf{1}}{\mathbf{8}}, \mathbf{1}$.
- **Convergence results:**

  Fermions: [Smirnov'06 ($\mathbb{Z}^2$), Ch.–Smirnov'09 (isoradial)];
  Energy densities: $\varepsilon := \sqrt{2} \cdot \sigma_{e^-}\sigma_{e^+} - 1 = \frac{i}{2}\psi_e\psi_e^\star$
  [Hongler–Smirnov'10, Hongler'10];
  Spins: [Ch.–Hongler–Izyurov'12];



- **Mixed correlations:** [Ch.–Hongler–Izyurov, '16-'18]
  spins (σ), disorders (μ), fermions ($\psi, \psi^\star$), energy densities (ε) in multiply connected domains Ω, with mixed fixed/free boundary conditions. The limits of correlations are defined via solutions to appropriate Riemann-type boundary value problems in Ω.

Conformal invariance at x_{crit}: more fields and CFT on the lattice



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- **And more** [Hongler–Kytölä–Viklund'17, ...]: E.g., one can define an action of the *Virasoro algebra* on *local lattice fields* via the Sugawara construction applied to lattice fermions.

- **Mixed correlations:** [Ch.–Hongler–Izyurov, '16 -'18] spins (σ), disorders (μ), fermions ($\psi, \psi^\star$), energy densities (ε) in multiply connected domains Ω, with mixed fixed/free boundary conditions. The limits of correlations are defined via solutions to appropriate Riemann-type boundary value problems in Ω.

Conformal invariance at x_{crit}: interfaces and loop ensembles

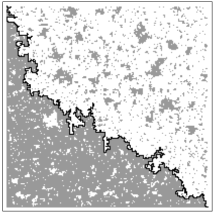
– Dobrushin b.c., weak topology:
[Smirnov'06], [Ch.–Smirnov'09]

– Dipolar SLE(3) $(+/\mathrm{free}/-$ b.c.):
[Hongler–Kytölä'11], [Izyurov'14]

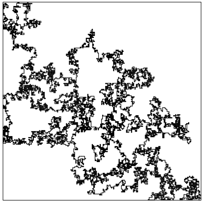
– Strong topology (tightness of curves):
[Kemppainen–Smirnov'12]

– Brief summary up to that date:
[Ch–DC–H–K–S, arXiv:1312.0533]

• **Theorem [Smirnov'06]:**



Ising interfaces $\to$ SLE(3)



FK-Ising ones $\to$ SLE(16/3)

– Spin-Ising boundary arc ensemble for free b.c.: [Benoist–Duminil-Copin–Hongler'14]

– Convergence of the full spin-Ising loop ensemble to **CLE(3):** [Benoist–Hongler'16]

– Exploration of FK boundary loops: [Kemppainen–Smirnov'15], see also [Garban–Wu'18]

– Convergence of the FK loop ensemble to **CLE(16/3):** [Kemppainen–Smirnov'16]

– "CLE percolations" [Miller–Sheffield–Werner'16]: FK-Ising $\rightsquigarrow$ CLE(16/3) $\rightsquigarrow$ CLE(3)

Conformal invariance at x_{crit}: interfaces and loop ensembles

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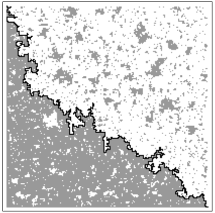
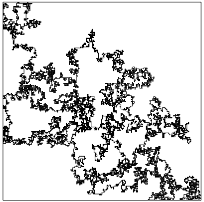
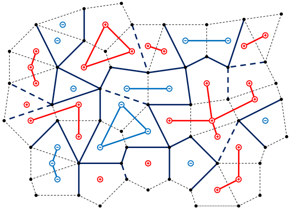


Ising interfaces → SLE(3)



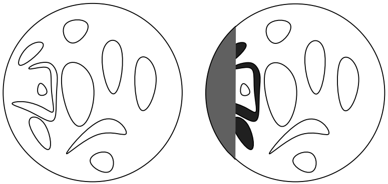
FK-Ising ones → SLE(16/3)

- **Fortuin–Kasteleyn** (=random cluster) expansion of the spin-Ising model [Edwards–Sokol coupling]:
  spins ⇝ FK: $p_e := 1 - x_e$ percolation on spin clusters;
  FK ⇝ spins: toss a fair coin for each of the FK clusters.

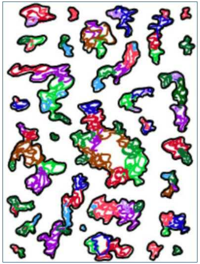


Conformal invariance at x_{crit}: CLE(3) = ? [Sheffield–Werner, arXiv:1006.2374]

- **Question:** What could be a good candidate for the *scaling limit of the outermost domain walls* surrounding '$-$' clusters in Ω_δ (with '$+$' b.c.)?



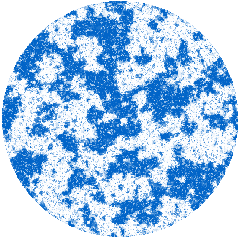
- **Intuition:** This random loop ensemble should
  (a) be *conformally invariant*;
  (b) satisfy the *domain Markov property:* given the loops intersecting $D_1 \setminus D_2$, the remaining ones form the same CLEs in the complement.



- **Theorem:** Provided that its loops do not touch each other, a CLE must have the following law for some intensity $c \in (0, 1]$:
  (i) sample a (countable) set of *Brownian loops* using the natural conformally-friendly *Poisson process* of intensity c;
  (ii) fill the *outermost clusters.*

- **Nesting:** Iterate the construction inside all the *first-level loops.*

Conformal invariance at x_{crit}: convergence of loop ensembles



Sample with free b.c.
(c) C. Hongler (EPFL)

- **Subtlety in the passage from SLEs to CLEs:**

  To prove the convergence to a CLE, one uses an iterative *exploration procedure* (e.g., [B–H'16] alternate between exploring boundary arc ensembles for free b.c. and FK-Ising clusters touching the boundary).

  To ensure that discrete and continuous exploration processes do not deviate from each other (e.g., to control relevant *stopping times*), one needs uniform *crossing estimates* in rough domains **['strong' RSW]**

– Spin-Ising boundary arc ensemble for free b.c.: [Benoist–Duminil-Copin–Hongler'14]
– Convergence of the full spin-Ising loop ensemble to **CLE(3):** [Benoist–Hongler'16]
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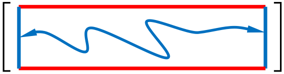
Conformal invariance at x_{crit}: tightness of interfaces

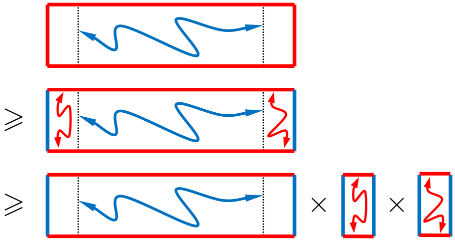
- **Crossing estimates (RSW):** due to the FKG inequality it is enough to prove that

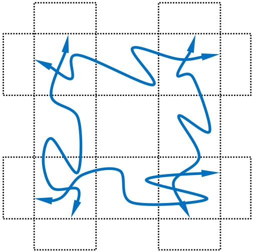
$\mathbb{P}$  $\geqslant \eta(k) > 0$

for rectangles of a given aspect ratio $k > \sqrt{3}+1$, uniformly over all scales.

$\Downarrow$ [Aizenman–Burchard'99]
$\Downarrow$ [Kemppainen–Smirnov'12]

Arm exponents $\Delta_n \geqslant \varepsilon n \Rightarrow$ tightness of curves and of the corresponding Loewner driving forces ξ_t^δ: $\mathbb{E}[\exp(\varepsilon|\xi_t^\delta|/\sqrt{t})] \leqslant C$.

- 

- $\Rightarrow$ 

Conformal invariance at x_{crit}: tightness of interfaces and 'strong' RSW

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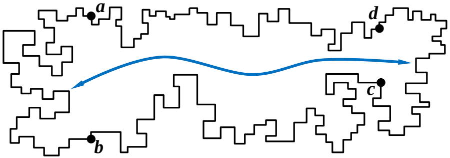


Theorem: [Ch.–Duminil-Copin–Hongler'13]
Uniformly w.r.t. $(\Omega_\delta; a, b, c, d)$ and b.c.,

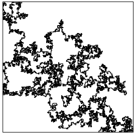
$$\mathbb{P}^{\mathrm{FK}}[(ab) \leftrightarrow (cd)] \geqslant \eta(\mathrm{L}_{\Omega;(ab),(cd)}) > 0,$$

where $\mathrm{L}_{\Omega;(ab),(cd)}$ is the discrete extremal length (= effective resistance) of the quad.

• **Remark:** Such a uniform lower bound is not straightforward even for the **random walk** partition functions [**'toolbox':** arXiv:1212.6205].

Beyond regular lattices or isoradial graphs: (periodic) s-embeddings

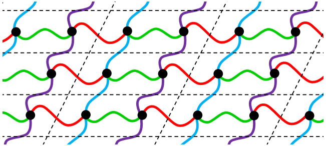
- **Question:** generalize convergence results from the very particular isoradial case to (as) general (as possible) weighted graphs.
- **A model question:** (reversible) random walks in a periodic (or in your favorite) environment.



[Smirnov'06]: $\mathbb{Z}^2$

[Ch.–Smirnov'09]: isoradial

- **Theorem [Ch., 2018]:** The convergence of critical FK-Ising interfaces to SLE(16/3) holds for all periodic weighted graphs.



horizontal: x_1, x_2;
vertical: x_3, x_4.

Beyond regular lattices or isoradial graphs: (periodic) s-embeddings

- **Question:** generalize convergence results from the very particular isoradial case to (as) general (as possible) weighted graphs.
- **A model question:** (reversible) random walks in a periodic (or in your favorite) environment.
- **But ... how should we draw a planar graph?**
  - Invariance under the star-triangle transform;
  - Compatibility with the isoradial setup.
- **Random walks:** Tutte's barycentric embeddings.

  [!] For periodic graphs, we also need to fix the conformal modulus of the fundamental domain.
- **Planar Ising model:** s-embeddings.

- **Criticality:** $x(\mathcal{E}_0) = x(\mathcal{E}_1)$

  [Cimasoni–Duminil-Copin'12]

$$1 + x_3x_4 = x_3 + x_4 + x_1x_2 + x_1x_2x_3 + x_2x_3x_4 + x_1x_2x_3x_4$$

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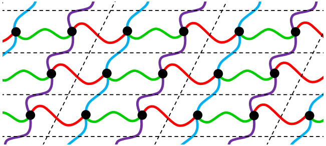
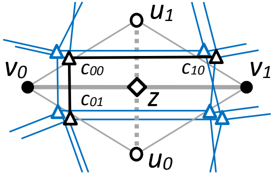


Beyond regular lattices or isoradial graphs: (periodic) s-embeddings



Assume that each $(v_0 u_0 v_1 u_1)$ is a *rhombus* with an angle $2\theta_{v_0 v_1}$ and $x_e = \tan \frac{1}{2}\theta_e$.

- **Propagation equation:**

$$X(c_{00}) = X(c_{01}) \cos\theta_e + X(c_{10}) \sin\theta_e .$$

- **S-holomorphicity:** Let $F(c) := \eta_c \delta^{-1/2} X(c)$, where $\eta_c := e^{i\frac{\pi}{4}} \exp[-\frac{i}{2}\arg(v(c)-u(c))]$.

[!] In the isoradial setup, $\mathcal{X}(c) := (v(c)-u(c))^{1/2}$ satisfies the propagation equation.

- **Criticality:** $x(\mathcal{E}_0) = x(\mathcal{E}_1)$

[Cimasoni–Duminil-Copin'12]

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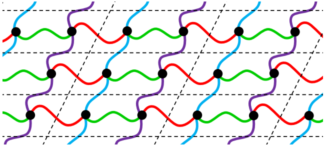
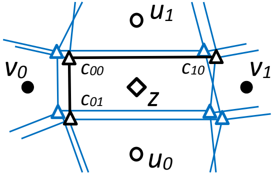


horizontal: x_1, x_2;
vertical: x_3, x_4.

How to draw graphs: (periodic) s-embeddings



At criticality, the propagation equation admits *two periodic solutions.*

- **Propagation equation:**

$$X(c_{00}) = X(c_{01})\cos\theta_e + X(c_{10})\sin\theta_e.$$

- **Definition:** Given a (periodic) complex-valued solution $\mathcal{X}$ to the PE, we define the s-embedding $\mathcal{S}_{\mathcal{X}}$ of the graph by $\mathcal{S}_{\mathcal{X}}(v)-\mathcal{S}_{\mathcal{X}}(u) := (\mathcal{X}(c))^2$.

- The function $L_{\mathcal{X}}(v) - L_{\mathcal{X}}(u) := |\mathcal{X}(c)|^2$ is also well-defined $\Rightarrow$ **tangential quads.**

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  [Cimasoni–Duminil-Copin'12]

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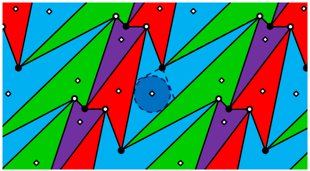
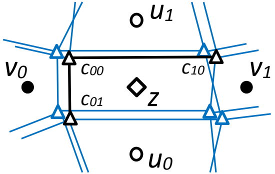


Beyond regular lattices or isoradial graphs: (periodic) s-embeddings



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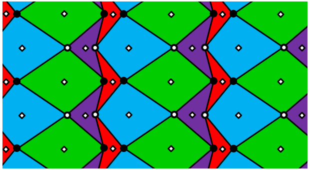
$$\mathcal{S}_{\mathcal{X}}(v) - \mathcal{S}_{\mathcal{X}}(u) := (\mathcal{X}(c))^2$$
$$L_{\mathcal{X}}(v) - L_{\mathcal{X}}(u) := |\mathcal{X}(c)|^2$$

- **Lemma:** $\exists!\mathcal{X} : L_{\mathcal{X}}$ – periodic.

- **Theorem [Ch., 2018]:** The convergence of critical FK-Ising interfaces to SLE(16/3) holds for all periodic weighted graphs.



Beyond regular lattices or isoradial graphs: (periodic) s-embeddings

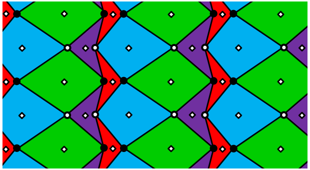
- **Key ingredients:**
  - A priori *Lipshitzness of projections* $\Pr[F(z);\alpha]$;
  - Control of discrete contour integrals of F *via* $L_{\mathcal{X}}$;
  - Positivity lemma: $\Delta_{\mathcal{S}} H_F \geqslant 0$ for some $\Delta_{\mathcal{S}} = \Delta_{\mathcal{S}}^{\top}$ ([!] $\Delta_{\mathcal{S}}$ is sign-indefinite $\rightsquigarrow$ no interpretation via RWs);
  - A priori *regularity of* H_F is nevertheless doable;
  - *Coarse-graining for* H_F*:* harmonicity in the limit;
  - Boundedness of F near *"straight" boundaries* $\Rightarrow$ convergence for (special) "straight" rectangles;
  - $\Rightarrow$ *RSW* $\Rightarrow$ convergence for arbitrary shapes Ω.

- **S-holomorphicity:** $e^{i\frac{\pi}{4}}X(c)/\mathcal{X}(c) = \Pr[F(z);\eta_c]$ for all real-valued spinors X satisfying the PE.

$$\mathcal{S}_{\mathcal{X}}(v) - \mathcal{S}_{\mathcal{X}}(u) := (\mathcal{X}(c))^2$$
$$L_{\mathcal{X}}(v) - L_{\mathcal{X}}(u) := |\mathcal{X}(c)|^2$$

- **Lemma:** $\exists!\mathcal{X} : L_{\mathcal{X}}$ – periodic.

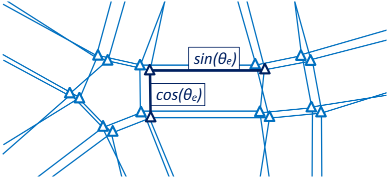
- **Theorem [Ch., 2018]:** The convergence of critical FK-Ising interfaces to SLE(16/3) holds for all periodic weighted graphs.



Some perspectives and open questions

periodic setup: other observables, 'strong' RSW,
loop ensembles, spin correlations;

your favorite object in your favorite setup:
invariance principle for the limit;



Ising model on random planar maps:
can one attack not only SLEs/CLEs but also LQG in this way?

- **Topological correlators** in the planar Ising model and CLE(3):
  is it possible to understand the convergence of 'topological correlators' for loop ensembles directly via a kind of τ-functions?
- **Supercritical regime, renormalization:** convergence to CLE(6) for $x > x_{\mathrm{crit}}$.

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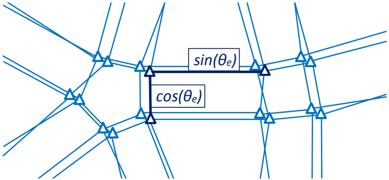

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THANK YOU FOR YOUR ATTENTION!