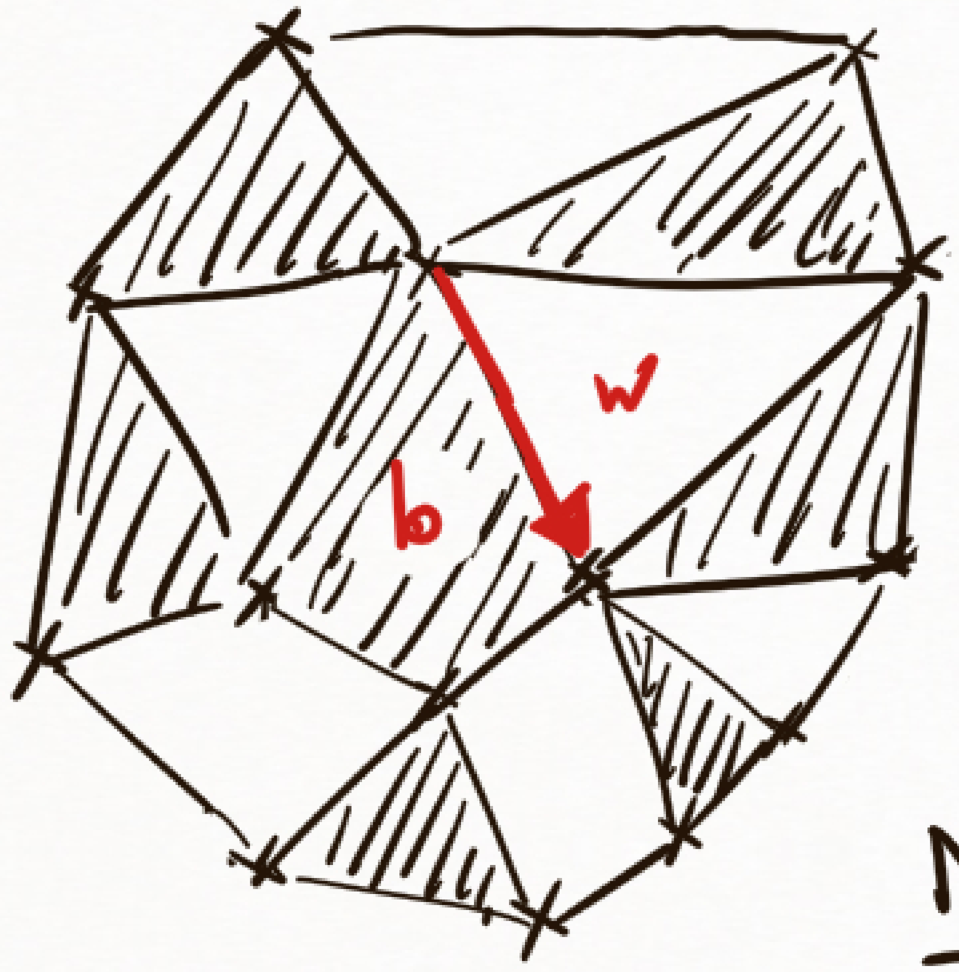
s-/t-/p-embeddings (2/3)

DIMERS Oberwolfach mini-workshop @ Zoom

November 19, 2020

RECAP: t-embeddings, origami map, T-graphs

t-embeddings = tilings w/ 'balanced' angles



[domino weights = lengths]

Motivation:

this setup is general enough to work w/ 'generic' liquid dimer model

Origami: find $|\eta_w| = |\eta_b| = 1$ s.t.

$dT((bw)^*) \in \bar{\eta}_b \bar{\eta}_w \mathbb{R}$

[existence $\Longleftrightarrow$ angle condition]

Define

$$d\mathcal{O} := \begin{cases} \eta_w^2\, dz = \eta_w^2\, dT & \text{on white faces} \\ \bar{\eta}_b^2\, d\bar{z} = \bar{\eta}_b^2\, d\bar{T} & \text{on black faces} \end{cases}$$

and integrate.

Remark: (i) this is equivalent to changing $(g^\circ, g^\bullet) \rightsquigarrow (g^\circ, \bar{g}^\bullet)$

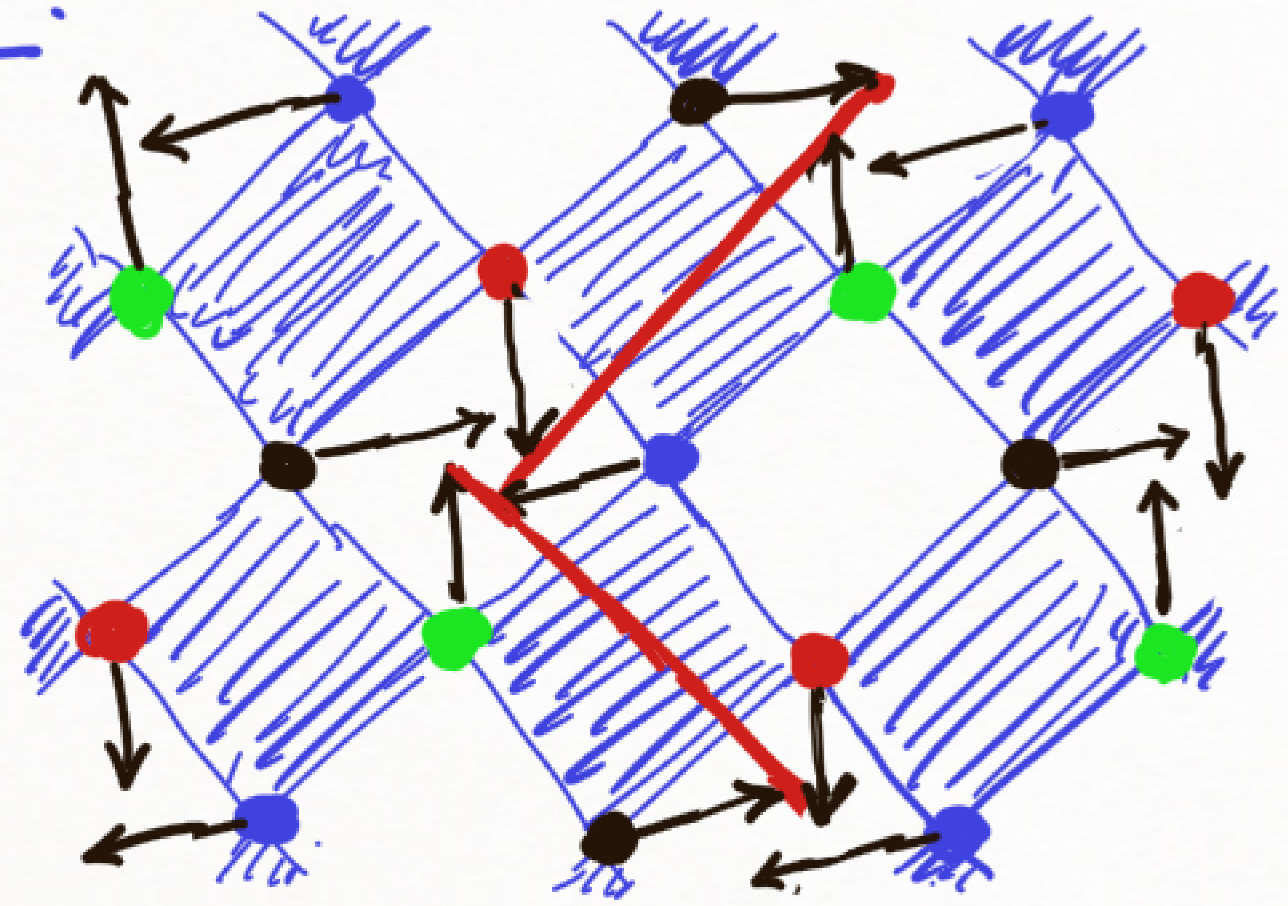
(ii) it is useful to view $\mathcal{O}$ as defined in $\mathbb{C}$ and not only on the graph

Recap: t-embeddings, origami, T-graphs = $T + \alpha^2 \mathcal{O}$, $|\alpha| = 1$

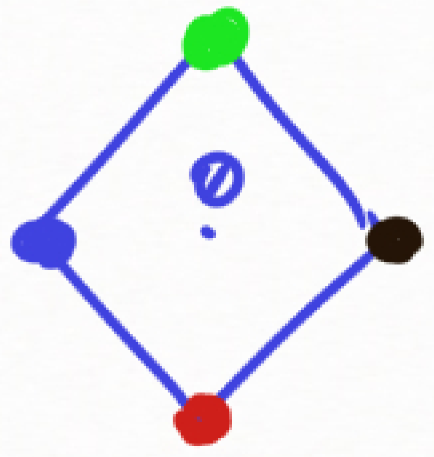
(or $T + \alpha^2 \overline{\mathcal{O}}$)

Example: $\mathbb{Z}^2$, HARM

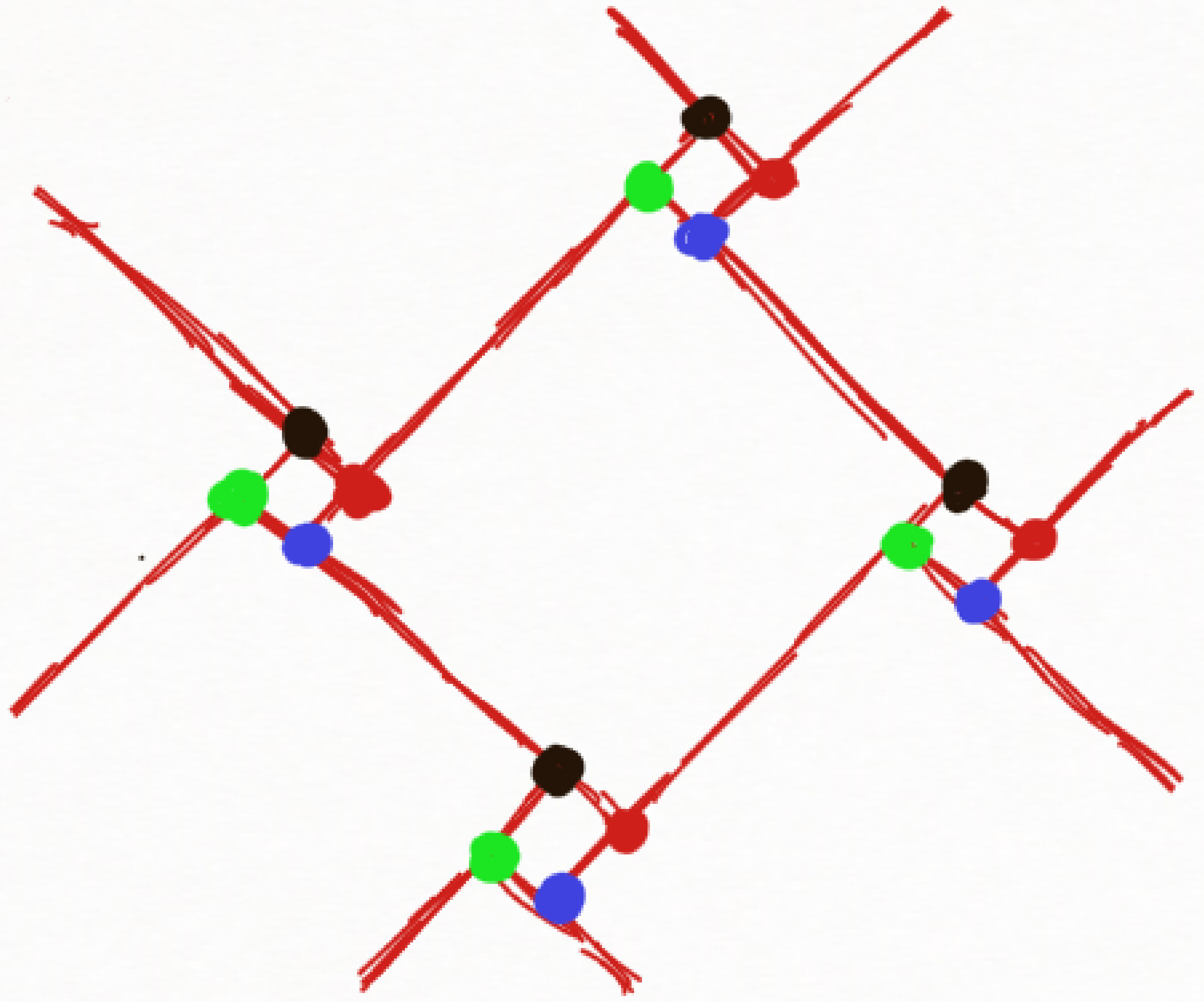
T:



$\mathcal{O}$:



$T + \alpha^2 \mathcal{O}$:



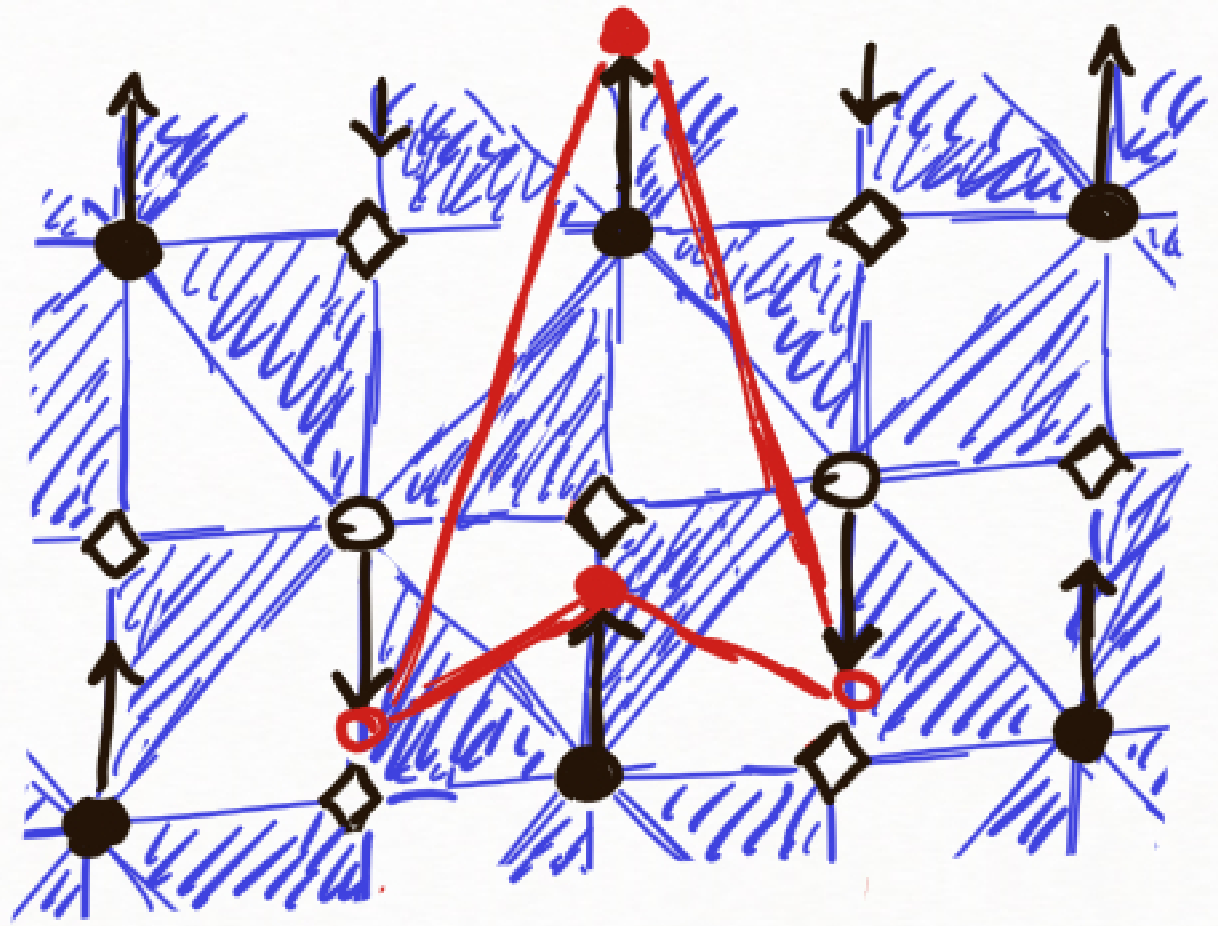
$\alpha = 1$

$\alpha = i$

usual RW's on bi-art. embeddings

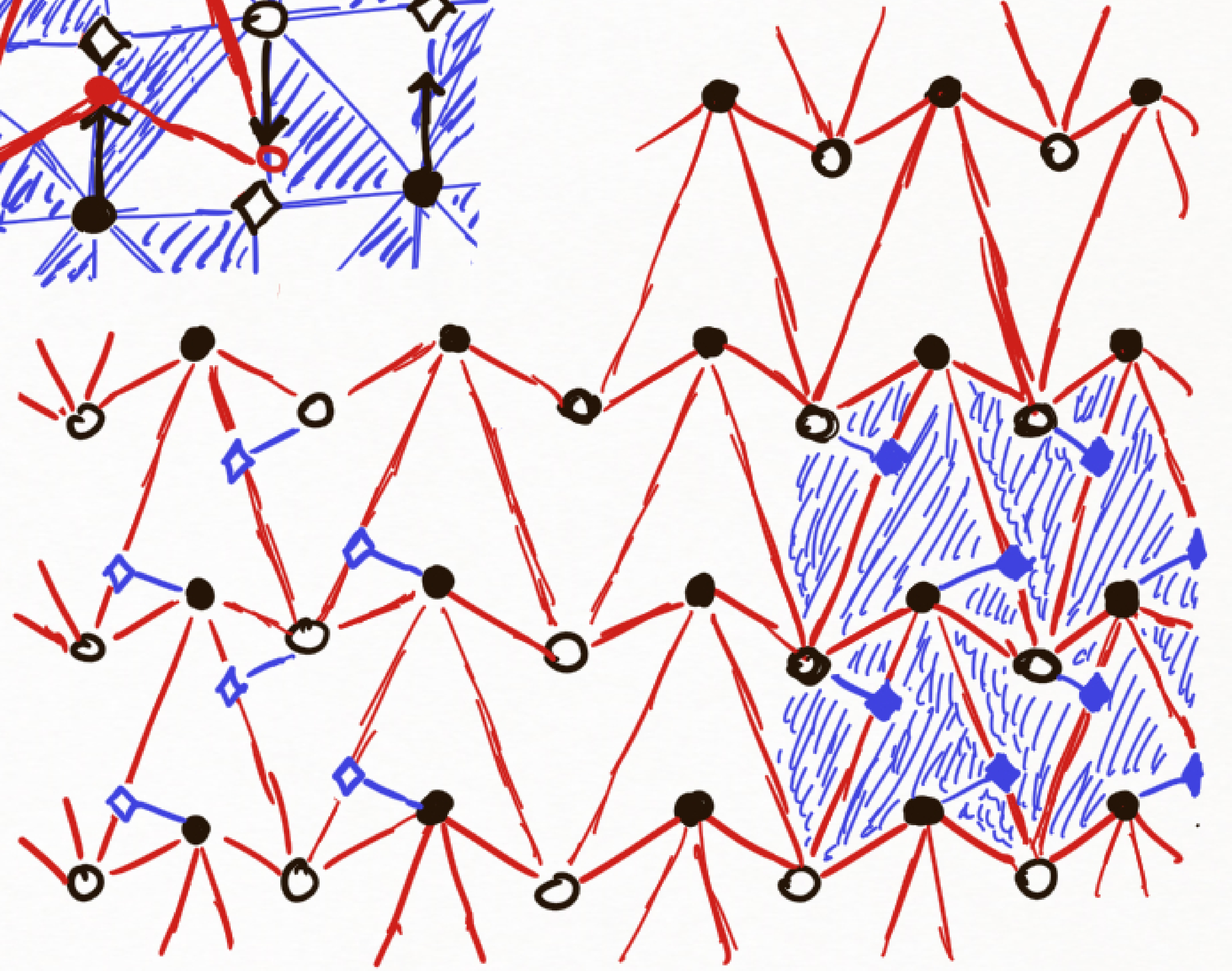
Example: $\mathbb{Z}^2$, ISING

$\mathcal{S}$:

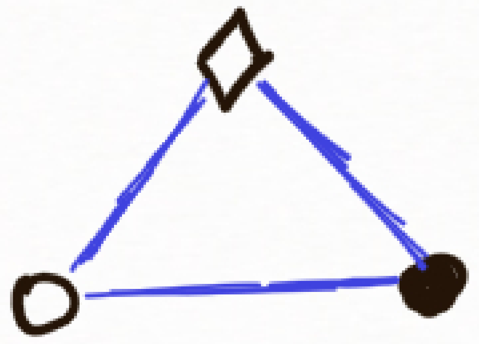


$\mathcal{S} + \alpha^2 Q$:

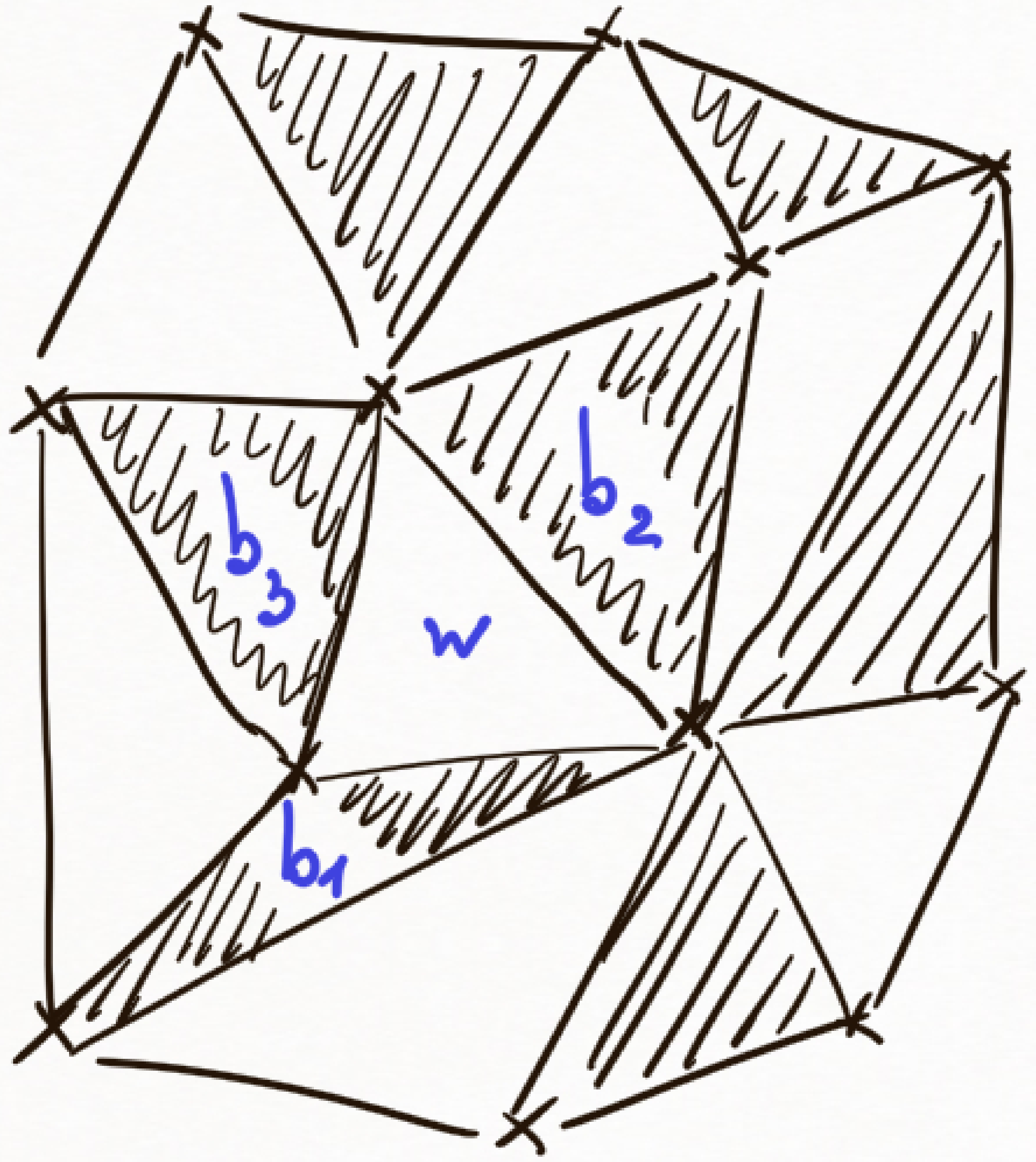
$(\alpha^2 = +i)$



Q:



RE-INTERPRETATION: $\mathbb{R}$-valued fermionic observables $\rightsquigarrow$ $\mathbb{C}$-valued



Consider functions defined, say, on B s.t.

① $F^\bullet(b) \in \eta_b\mathbb{R} = \overline{g^\bullet(b)}\,\mathbb{R}$ $\quad$ [e.g., $F^\bullet_{w_0}(b) := g^\circ(w_0)\,\mathcal{K}^{-1}(w_0,b)$]

② $F^\bullet \mathcal{K} = 0$ (locally)

Lemma: On triangulations, ① + ② $\Longleftrightarrow$

$\exists\, F^\circ(w) \in \mathbb{C}:\ F^\bullet(b_k) = \mathrm{Pr}\,[F^\circ(w);\, \eta_{b_k}\mathbb{R}],\ k = 1,2,3$

For simplicity, let T be a triangulation [otherwise, split white faces into triangles]

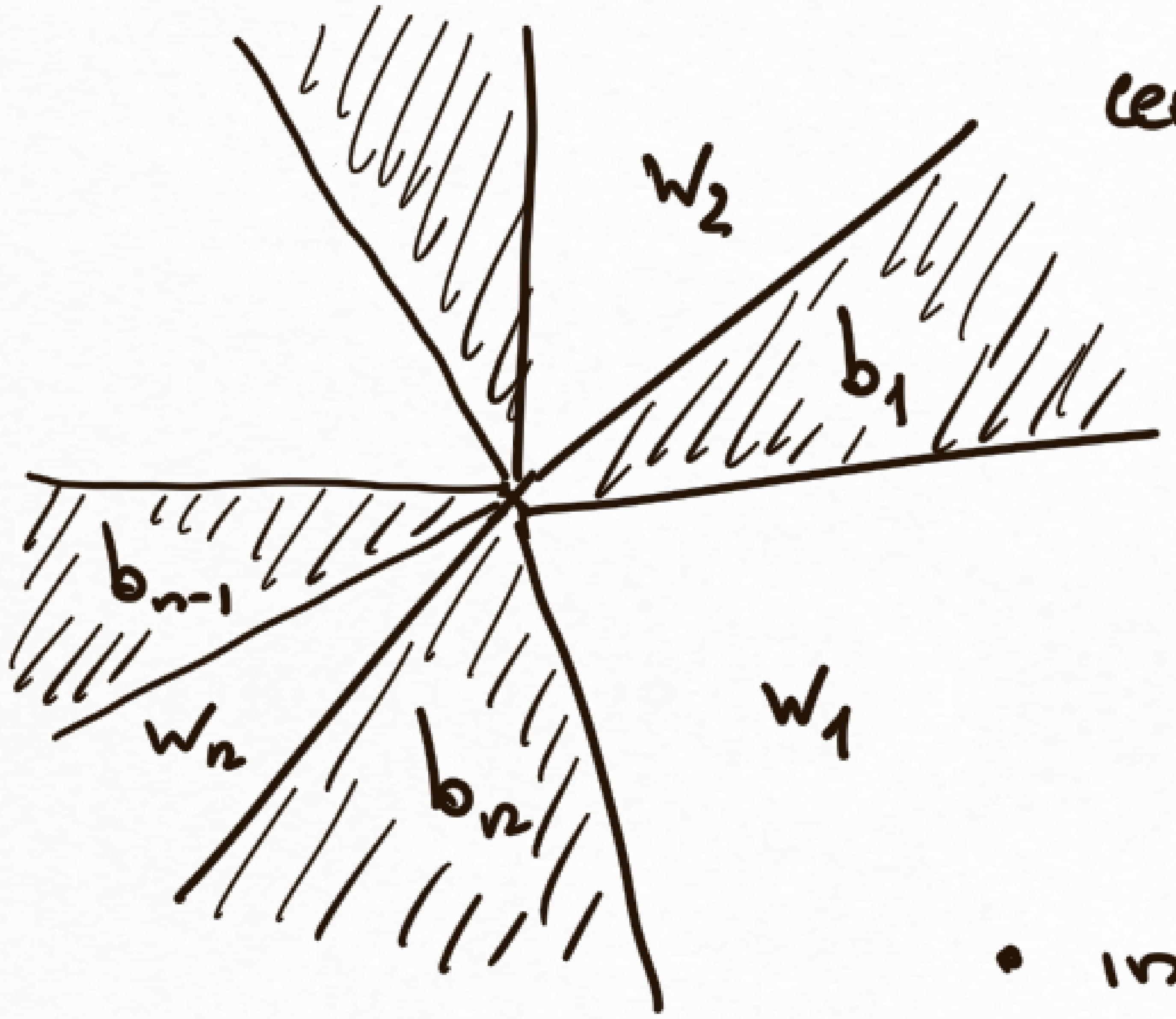
We call such functions t-white holomorphic and use the notation F_w

Similarly: t-black hol. F_b

Rem: in the Ising context, this generalizes Smirnov's s-holomorphic functions from $\mathbb{Z}^2$ to s-embeddings

A PRIORI REGULARITY of t-holomorphic functions [CLR]

Observation: $\forall \alpha$ $\Pr[F^\circ(\cdot); \alpha\mathbb{R}]$ are martingales wrt to a certain directed RW:



Proposition: this RW can be identified w/ the backward RW on $T - \alpha^2\bar{O}$

- invariant measure = Area(w)
- identification $W \leftrightarrow G^*$ from combinatorics of $T - \alpha^2\bar{O}$

$$\Pr[F^\circ(w_{k+1}) - F^\circ(w_k); \eta_{b_k}\mathbb{R}] = 0$$

$$\Updownarrow$$

$$\frac{\operatorname{Im}(\eta_{b_k})}{\operatorname{Re}(\eta_{b_k})}(\operatorname{Im} F^\circ(w_k) - \operatorname{Im} F^\circ(w_{k+1})) = \operatorname{Re} F^\circ(w_k) - \operatorname{Re} F^\circ(w_{k+1})$$

$$\Downarrow$$

$$\sum \underbrace{\left(\frac{\operatorname{Im}\eta_{b_k}}{\operatorname{Re}\eta_{b_k}} - \frac{\operatorname{Im}\eta_{b_{k-1}}}{\operatorname{Re}\eta_{b_{k-1}}}\right)}_{\text{all coeff. except one are} > 0} \operatorname{Im} F^\circ(w_k) = 0$$

Thus, we need some ellipticity ($\Leftrightarrow$ uniform crossings) of these RW's

A PRIORI REGULARITY of t-hol. jets (cont.)

Proposition: ([CLR])

$\mathrm{Lip}(k,\delta)$, $k<1$

Let $\boxed{|\mathcal{O}(x)-\mathcal{O}(y)| \leq k\,|\mathcal{T}(x)-\mathcal{T}(y)| \text{ for all } |\mathcal{T}(x)-\mathcal{T}(y)|\geq\delta}$

Then, continuous-time RW's on T-graphs $\mathcal{T}+\alpha^2\mathcal{O}$ are uniformly elliptic on time scales $\geq\delta^2$:
$\exists\, t_0=t_0(k)$ and $\sigma_0=\sigma_0(k)$ s.t.

$$\mathrm{Var}\left(\mathrm{Re}\left(\bar\beta\left(X_{t_0\delta^2}-X_0\right)\right)\right)\geq\sigma_0\delta^2 \quad \forall\beta:|\beta|=1$$

Remark: We do not use any other (than $\mathrm{Lip}(k,\delta)$ assumption) on $\mathcal{T}$ here!

It is a (relatively) standard corollary of that t-holomorphic functions are Hölder above scale δ:

$\exists\,\beta=\beta(k),\ C=C(k),\ cst=cst(k)$ s.t. for all $r\geq cst\cdot\delta$

$$\operatorname*{osc}_{B(z,r)}(F^\circ)\leq C\left(\frac{r}{R}\right)^{\beta}\cdot\operatorname*{osc}_{B(z,R)}(F^\circ)$$

Rem: note that distances in $\mathcal{T}$ and $\mathcal{T}+\alpha^2\mathcal{O}$ are unif. comparable above scale δ

A PRIORI REGULARITY of t-holomorphic functions (cont.)

- Given a t-hol. fct F_w one can define its primitive
  $I_{\mathbb{C}}[F_w] := \int (F_w d\mathcal{T} + \overline{F_w} d\mathcal{O})$
  ↑ this lives on a t-embedding

Rem: $I_{\alpha\mathbb{R}}[F_w] := \mathrm{Pr}(I_{\mathbb{C}}[F_w], \alpha\mathbb{R})$ is harmonic on $\mathcal{T} + \alpha^2\mathcal{O}$

Exp-Fat(δ): (for triangulations)

$\forall \beta > 0$, if one removes all '$\exp(-\beta\delta^{-1})$-fat' triangles from $\mathcal{T}^\delta$, then the size of remaining vertex-connected components $\xrightarrow[\delta\to 0]{} 0$

A priori Höl of F_w

EXP-FAT(δ) $\Longrightarrow$ control of F_w via $I_{\alpha\mathbb{R}}[F_w]$ (or via H_F for Ising)

"blow-up of oscillations":

F_w - big & $I[F_w]$ bounded

$\Rightarrow$ big oscillations

(Höl) $\Rightarrow$ even bigger oscillations and bigger values nearby

RE-INTERPRETATION OF FERMIONIC OBSERVABLES (continued)

$\mathbb{R}$-valued sets on an "abstract" graph [K^{-1} or spin-disorders]

$\xrightarrow[\mathcal{T} \text{ or } \mathcal{S}]{\text{choice of}}$

$\mathbb{C}$-valued (piece-wise constant) functions defined in $\mathbb{C}$ s.t.

$$\boxed{\begin{array}{l} F_w\, d\mathcal{T} + \overline{F_w}\, d\mathcal{O} \\ F_b\, d\mathcal{T} + \overline{F_b}\, d\overline{\mathcal{O}} \end{array} \text{ are closed forms}}$$

Dimers (cf. Kenyon): height correlations are linear combinations of

$$\int \mathrm{Re}\left[F_w F_b\, d\mathcal{T} + \overline{F_w} F_b\, d\mathcal{O} \right]$$

[$\int$ in each of the variables]

Ising (cf. Smirnov):

$H_F := \int \mathrm{Re}\left[F^2 d\mathcal{T} + |F|^2 d\mathcal{O} \right]$ is a crucial tool to work with b.v.p.'s

RE-WRITING "$f\,dz + \bar{f}\,d\theta$ - closed" : massive holomorphicity

① Ising $(\theta \in \mathbb{R})$

$f\,dz + \bar{f}\,d\theta$ - closed $\Longleftrightarrow$ $\boxed{\partial_{\bar{\varrho}} \psi = m\,\bar{\psi}}$

ϱ - conformal parametrization of the surface $\{(z, \theta(z))\} \subset \mathbb{R}^{2+1}$

m = mean curvature ($\times$ metric element)

$\psi := (z_\varrho)^{1/2} f + (\bar{z}_\varrho)^{1/2} \bar{f}$

② Dimers $(\theta \in \mathbb{C})$:

similar w/ $\partial_{\bar{\varrho}} \psi_w = m\,\bar{\psi}_w$
$\partial_{\bar{\varrho}} \psi_b = \bar{m}\,\bar{\psi}_b$ $\rightsquigarrow$ height correlations ("bosonization") of the form $\int \mathrm{Re}\,[\psi_w \psi_b\, d\varrho]$

$$\boxed{\text{Minimal surfaces } (m=0) \Longleftrightarrow \text{holomorphic jets in } \varrho}$$

THEOREM [CLR2]:

existence is an open question!

Let T^δ be p-embeddings satisfying $\mathrm{Lip}(k,\delta)$ and EXP-FAT(δ) on compacts.

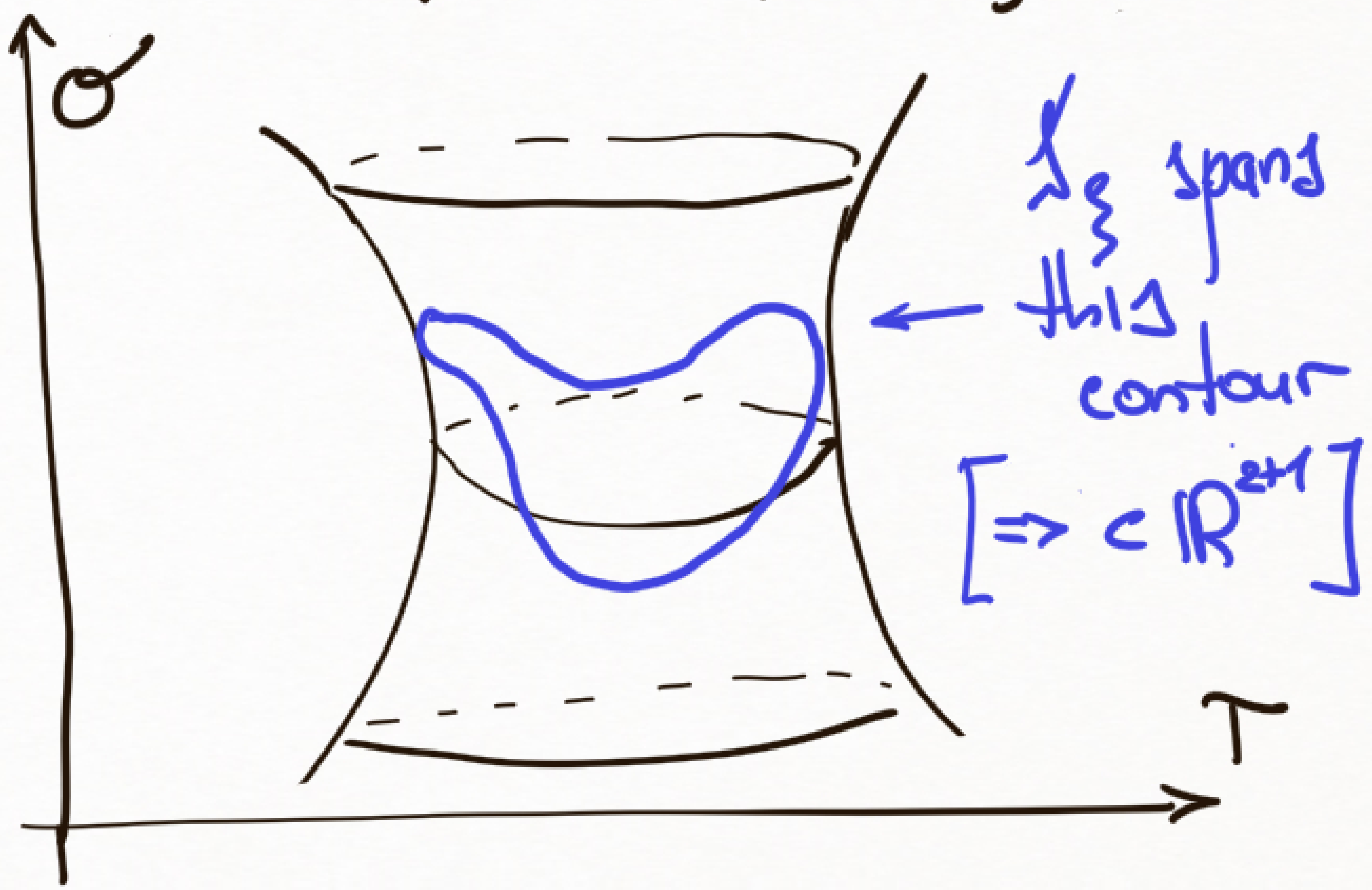
Assume that the discrete surfaces

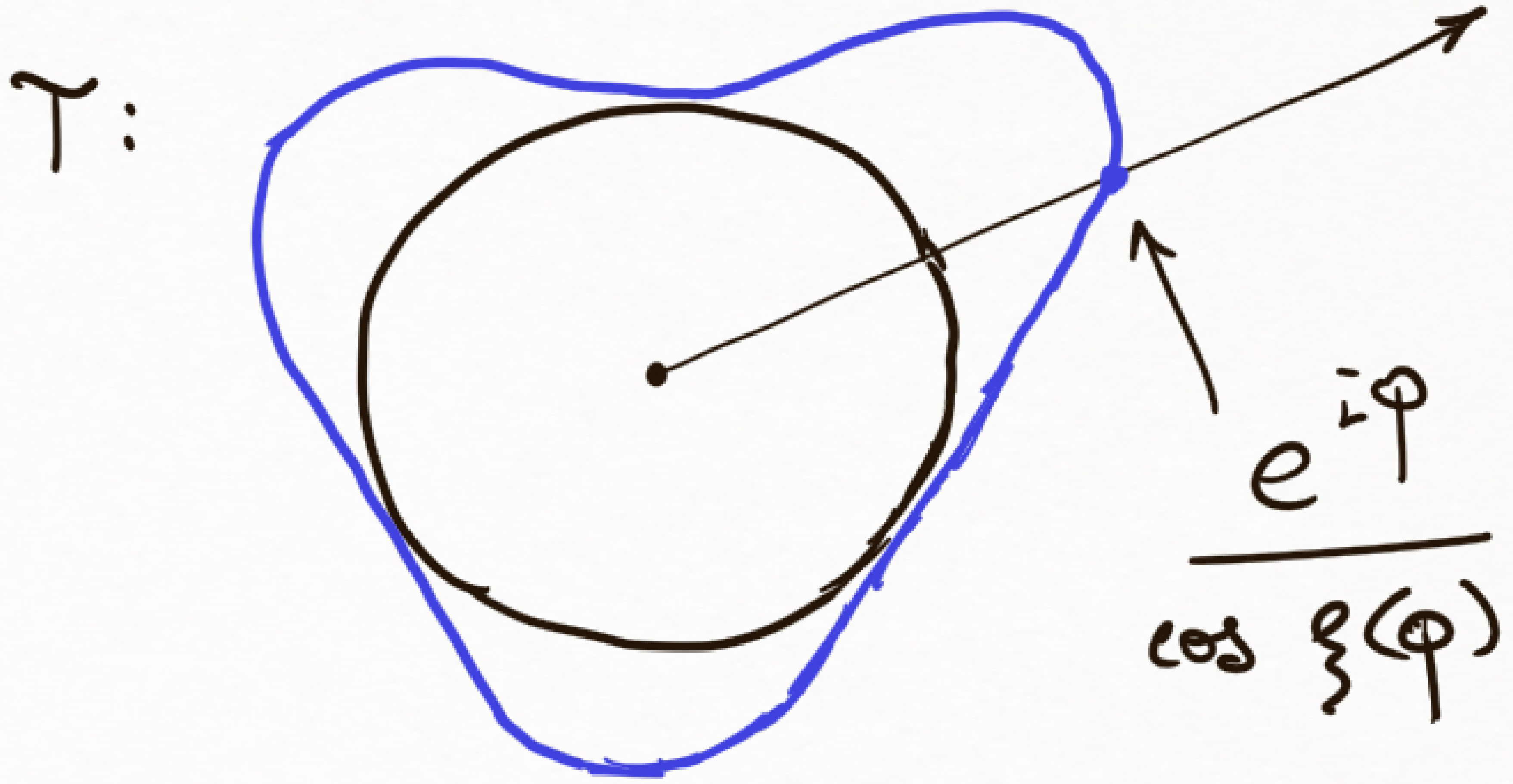
$$\{(T^\delta, O^\delta)\} \xrightarrow[\delta\to 0]{} S_\xi \subset \mathbb{R}^{2+1} \subset \mathbb{R}^{2+2},$$

where S_ξ is a minimal surface bdd by a contour $\left(\frac{e^{i\varphi}}{\cos\xi(\varphi)}, \tan\xi(\varphi)\right)$, where $\xi : \mathbb{T} \to (-\pi/2, \pi/2)$ is 1-Lip.

Then, height fluctuations $\to \pi^{-\frac{1}{2}}$ GFF [in the conformal parametrization of S_ξ]

$$|T|^2 - |O|^2 = 1, \quad O \in \mathbb{R}$$





Comments on Thm / Prf:

(1) In this 'perfect Coulomb gauge' the following holds:

$$\boxed{(\mathcal{K}^\delta)^{-1}(w,b) = O(1)}$$

for w,b away from each other and **at least one** in the bulk

Prf: two inputs

(a) b_0 - bdry face $\Rightarrow$ $0 \le I_{\bar\eta_{b_0}\mathbb{R}}[\bar\eta_{b_0}\mathcal{K}^{-1}(\cdot,b_0)] \le 1$ along the boundary

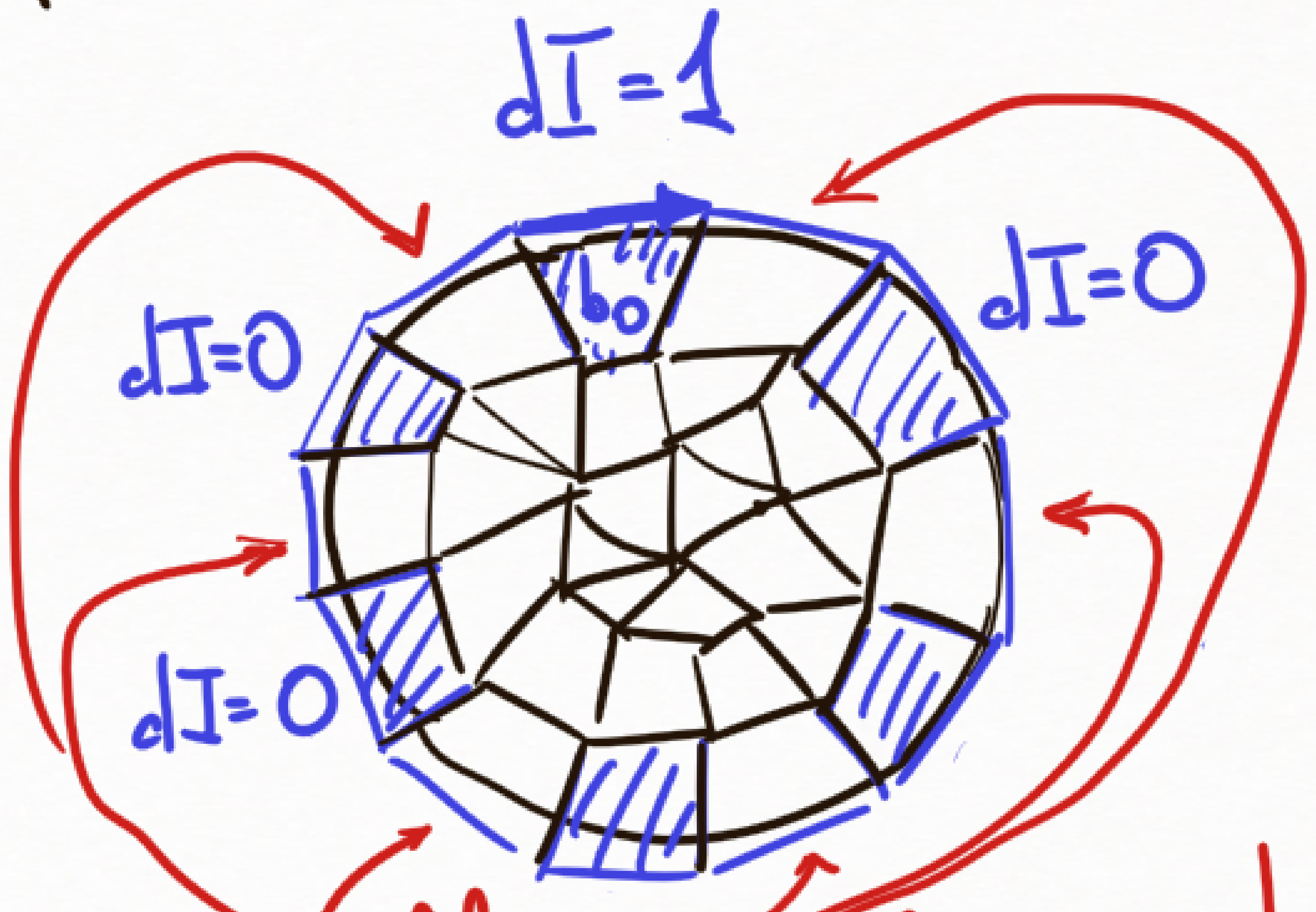
$\Downarrow$ (max principle)

everywhere

(b) w_0 - inner face

$\Rightarrow$ $I_{i\bar\eta_{w_0}\mathbb{R}}[\bar\eta_{w_0}\mathcal{K}^{-1}(w_0,\cdot)]$

is semi-bounded near w_0



all dI along white bdry edges are of the same sign

Comments on Thm / Prf :

② The identification of singularities $\frac{1}{2\pi}\cdot\frac{1}{z-z_k} + O(1),\ z\to z_k$ comes from the fact that $I_{i\eta_{w_0}\mathbb{R}}$ is semi-bounded near z_k & monodromy of $I_{\eta_{w_0}\mathbb{R}} = 1$

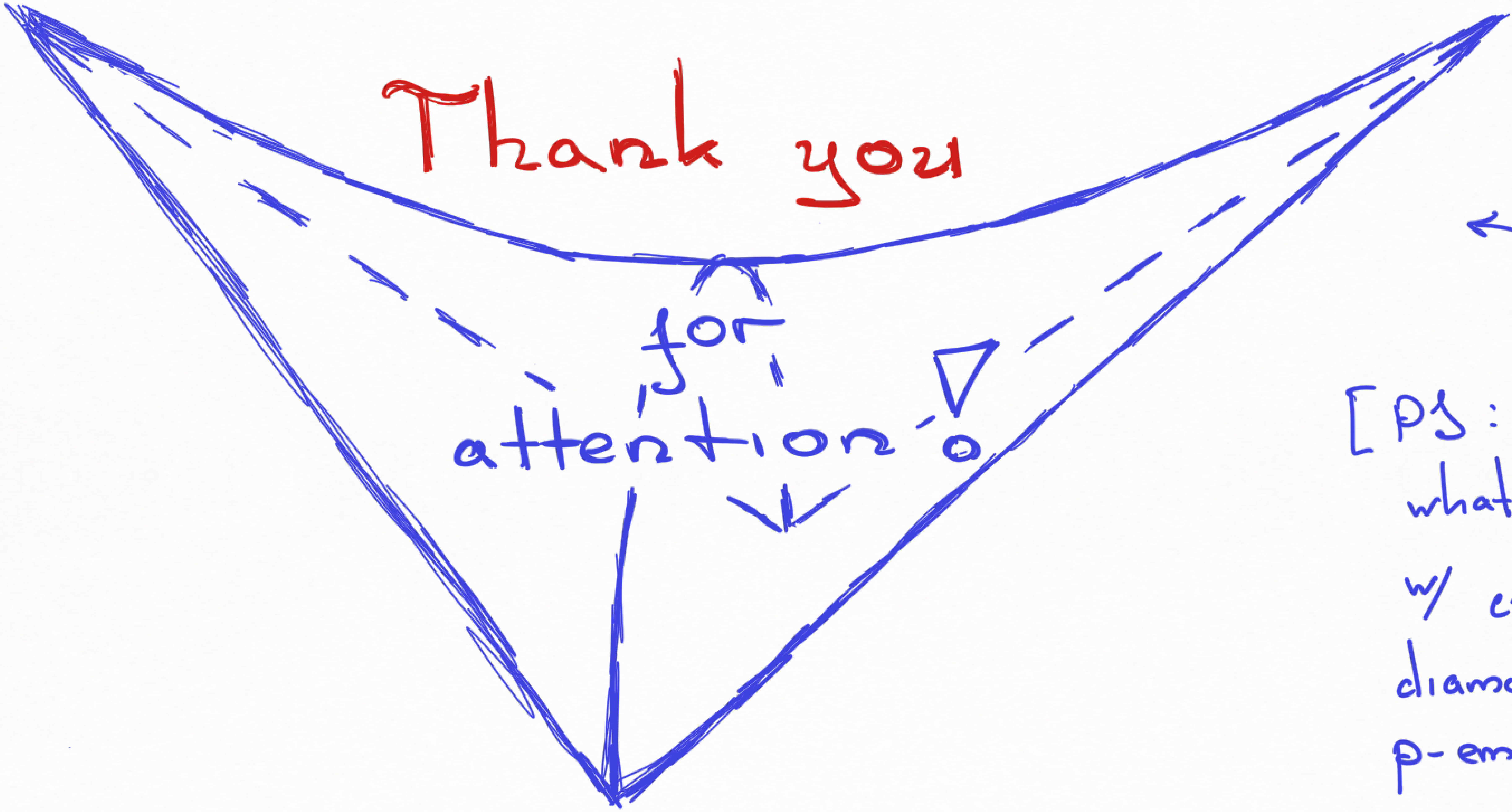
("it is useful to work with **all** $\mathcal{T}+\alpha^2\mathcal{O}$ **simultaneously**")

③ In this setup [when $\mathcal{K}^{-1}(w_0,\cdot) = O(1)$ up to the boundary]

THERE IS NO NEED TO IDENTIFY THE LIMIT OF $(\mathcal{K}^\delta)^{-1}$

height correlations $\xrightarrow[\delta\to 0]{}$ harmonic (in ρ) functions with standard singularities and Dirichlet boundary conditions

Rem: it is highly sensitive to b.c. in known cases, only (a lot of) subseq. limits exist



["illustration" joint w/ Ramassamy]

[PS: this is what happens w/ classical Aztec diamonds under p-embeddings]