s-/t-/p-embeddings (3/3)

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mini-workshop @ Zoom

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ISING ON S-EMBEDDINGS : convergence to SLE(16/3)

☹ As for now, under very restrictive assumptions:

lengths $\asymp \delta$, angles $\geq \eta_0 > 0$, $Q^\delta = O(\delta)$

Still, this covers

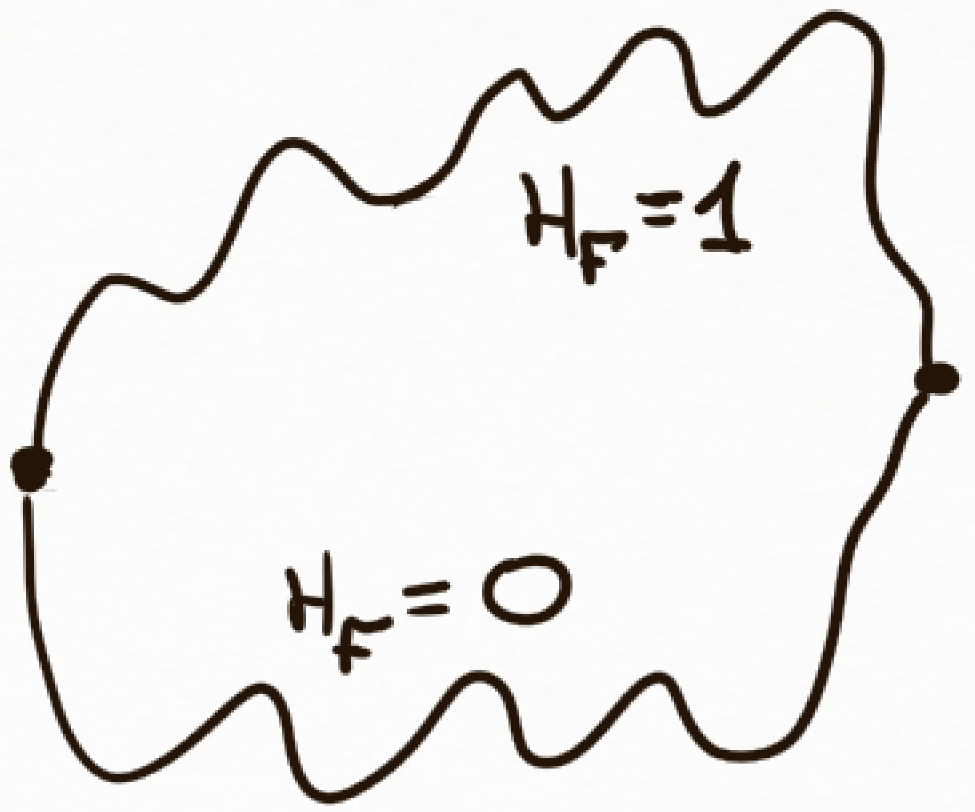
- all critical periodic models
  [$\exists$ 'canonical' periodic $\mathcal{S}^\delta$ s.t. Q^δ is periodic $\Rightarrow O(\delta)$]
- Lis' Ising on circle patterns
  [w/ bdd angles and radii $\asymp \delta$]

New scheme of the proof:

① RSW ('straight' self-dual rectangles)

$\Downarrow$

② 'Quantitative' convergence of FK-Dobrushin observables



Rem: $\Downarrow$ I. Binder, L. Richards

polynomial rate of conv. to SLE

FUNCTIONS $H_F = \int \mathrm{Re}\,[F^2 dT + |F|^2 d\theta]$

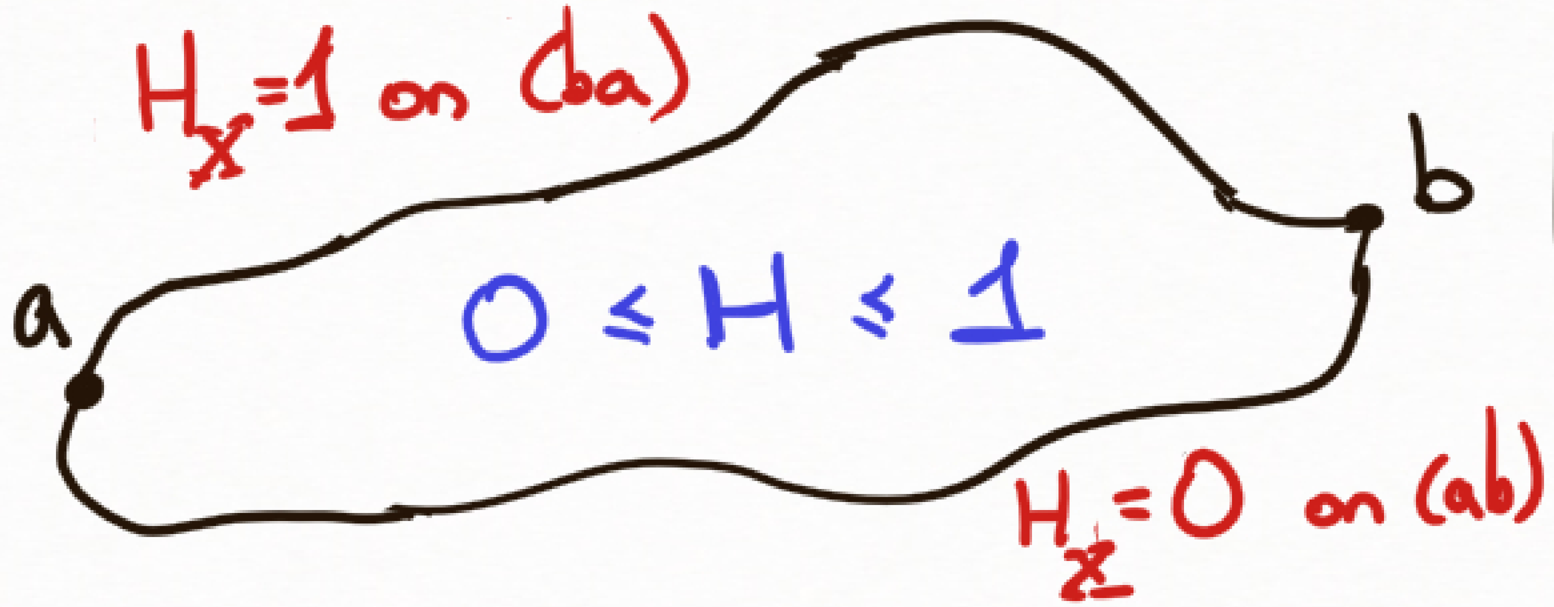
Remark: $H_F = H_X$ can be defined w/o embedding by setting $H_X(v) - H_X(u) := |X(e)|^2$

Lemma: Functions H_X satisfy the maximum principle

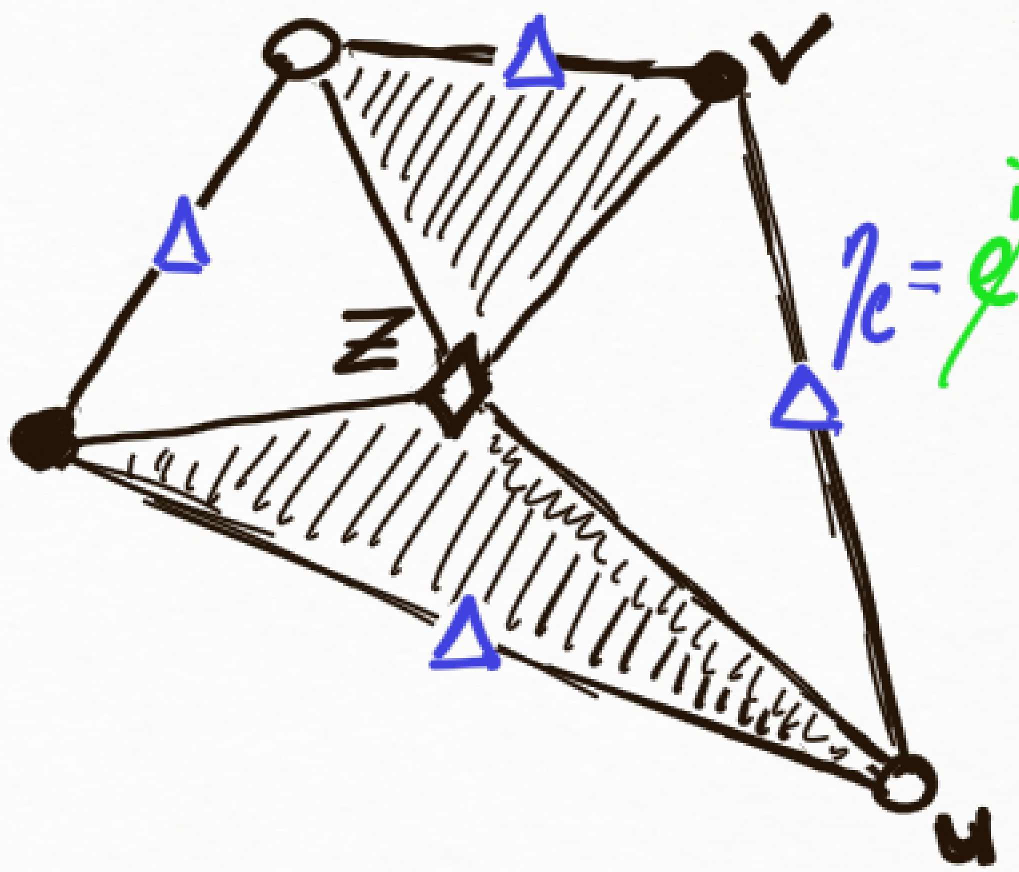
Moreover [S.C. Park], the comparison principle: $H_{X_1} \geq H_{X_2}$ along a contour $\Rightarrow$ inside

FK-Dobrushin:

$$X(e) = \langle \chi_e\, \mu_{(ba)}\, \sigma_{(ab)} \rangle$$



s-holomorphicity:



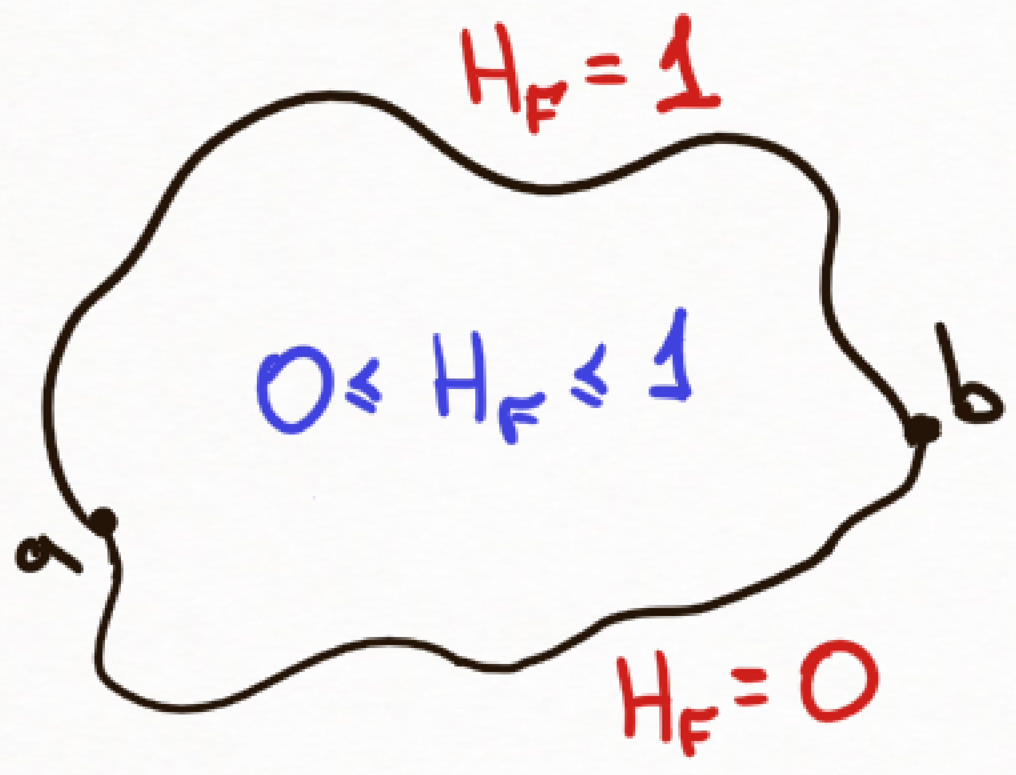
$$\eta_e = e^{i\frac{\pi}{4}} \sqrt{\frac{\overline{v-u}}{|v-u|}}$$

$$\mathrm{Pr}\,[F(z), \eta_e \mathbb{R}] = \frac{\eta_e X(e)}{\sqrt{|v-u|}}$$

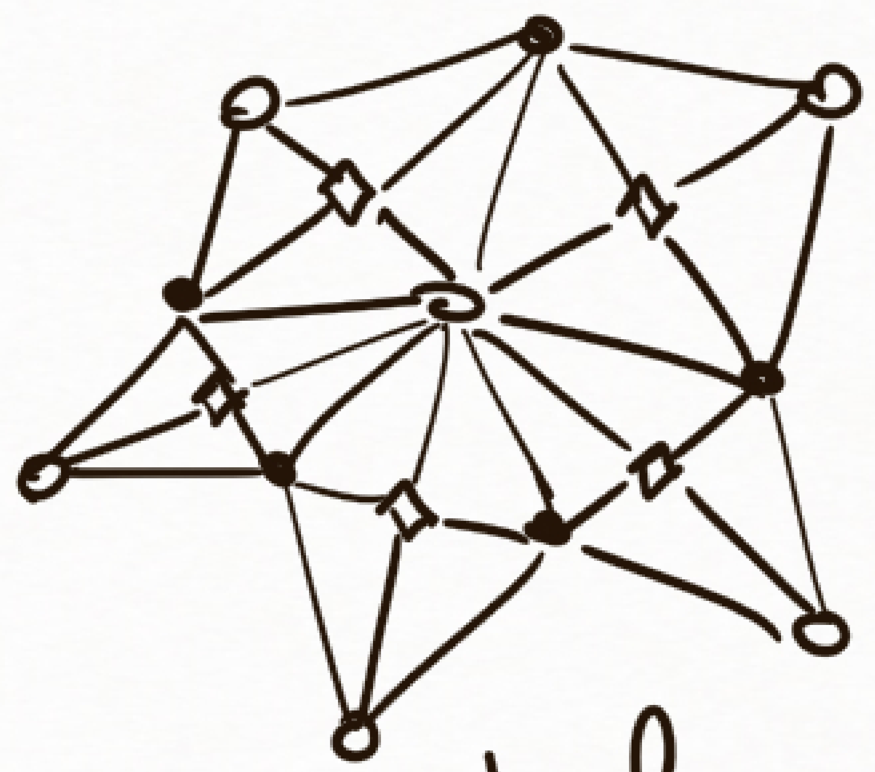
$$X(e) = \langle \underbrace{\mu_{v(e)}\, \sigma_{u(e)}}_{\chi_e :=} \cdots\cdots \rangle$$

Rem: $F \sim 1$, independently of the local scale

FUNCTIONS $H_F = \int \mathrm{Re}[F^2 dT + |F|^2 d\theta]$ AND BOUNDARY CONDITIONS



Unlike the isoradial case there is <u>no</u> 'magic' link of H_F and discrete harmonic fcts



ISORAD:

$\Delta H_F^\bullet \ge 0$

$\Delta H_F^\circ \le 0$

$H_F^\circ \le H_F^\bullet$

$\rightsquigarrow$ control of b.c. by comparison w/ discrete harm. measures

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A priori regularity (see 2/3) $\Rightarrow$ $F^\delta(z) = O(1/d_z^{1/2})$, $d_z := \mathrm{dist}(z, \partial\Omega)$

This implies the existence of subseq. (holomorphic !) limits $F^\delta \underset{\delta\to 0}{\rightrightarrows} f$ and hence $H^\delta \underset{\delta \to 0}{\to} h = \int \mathrm{Re}[f^2 dz]$ but does <u>not</u> say anything on $h|_{\partial\Omega}$.

[Note that $F^\delta \neq O(1)$ near $\partial\Omega$)

KEY IDEA: mollifier on scale ρ_z

$$\widetilde{H}_F^\delta(z) := \int \underbrace{\frac{1}{\rho_z^2}\varphi_0\left(\frac{w-z}{\rho_z}\right)}_{\text{mollifier on scale } \rho_z} H_F^\delta(w)\, dA(w)$$

where $\rho_z := \delta^{\varepsilon} \cdot \mathrm{crad}(z, \Omega^\delta)$

ANALYSIS OF BOUNDARY CONDITIONS:

$$\tilde{H}_F^\delta(z) := \int \frac{1}{\rho_z^2}\,\varphi_0\!\left(\frac{w-z}{\rho_z}\right) dA(w) \quad \text{for } \underline{\text{all }} z:\ \operatorname{dist}(z,\partial\Omega^\delta) \geq \delta^{1-2\varepsilon}$$

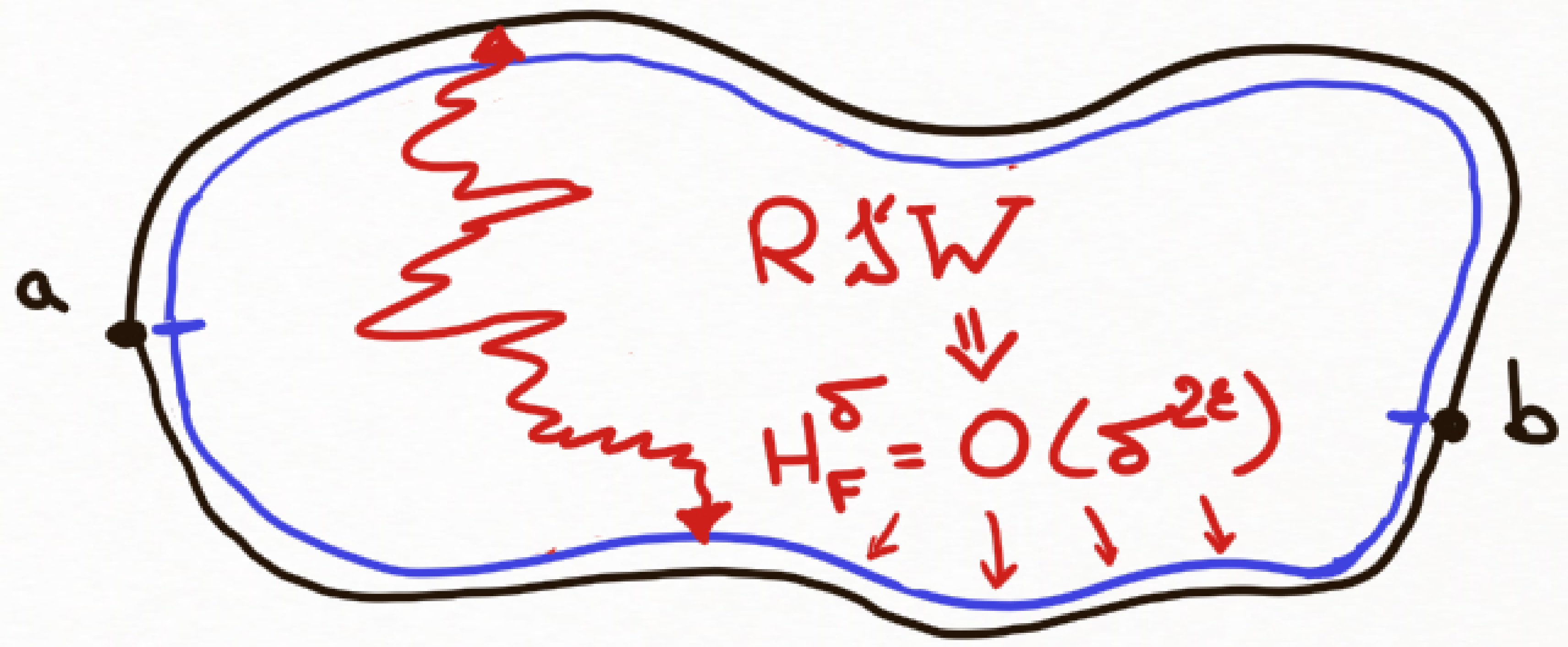
$$\rho_z = \delta^\varepsilon \operatorname{crad}(z,\Omega^\delta) \asymp \delta^\varepsilon \operatorname{dist}(z,\partial\Omega^\delta)$$

Proposition:
$$\Delta \tilde{H}_F^\delta(z) = O\!\left(\frac{\delta^\varepsilon}{d_z^{2-\varepsilon}} + \frac{\delta^{1-\varepsilon}}{d_z^3}\right)$$
$$= O\!\left(\frac{1}{d_z^{2-\varepsilon}}\right) \quad \text{if} \quad d_z \geq \delta^{1-2\varepsilon}$$

Lemma: (no regularity assumption on $\partial\Omega$)

$$h|_{\partial\Omega} = 0 \qquad \Delta h = O(1/d^{2-\varepsilon}) \implies h = O(d^{\varepsilon/5})$$

Remark: this approach can also be applied to $m \neq 0$



RSW $\implies$ $F^\delta(z) = O(\delta^{-1/2}\,\delta^{1/2})$

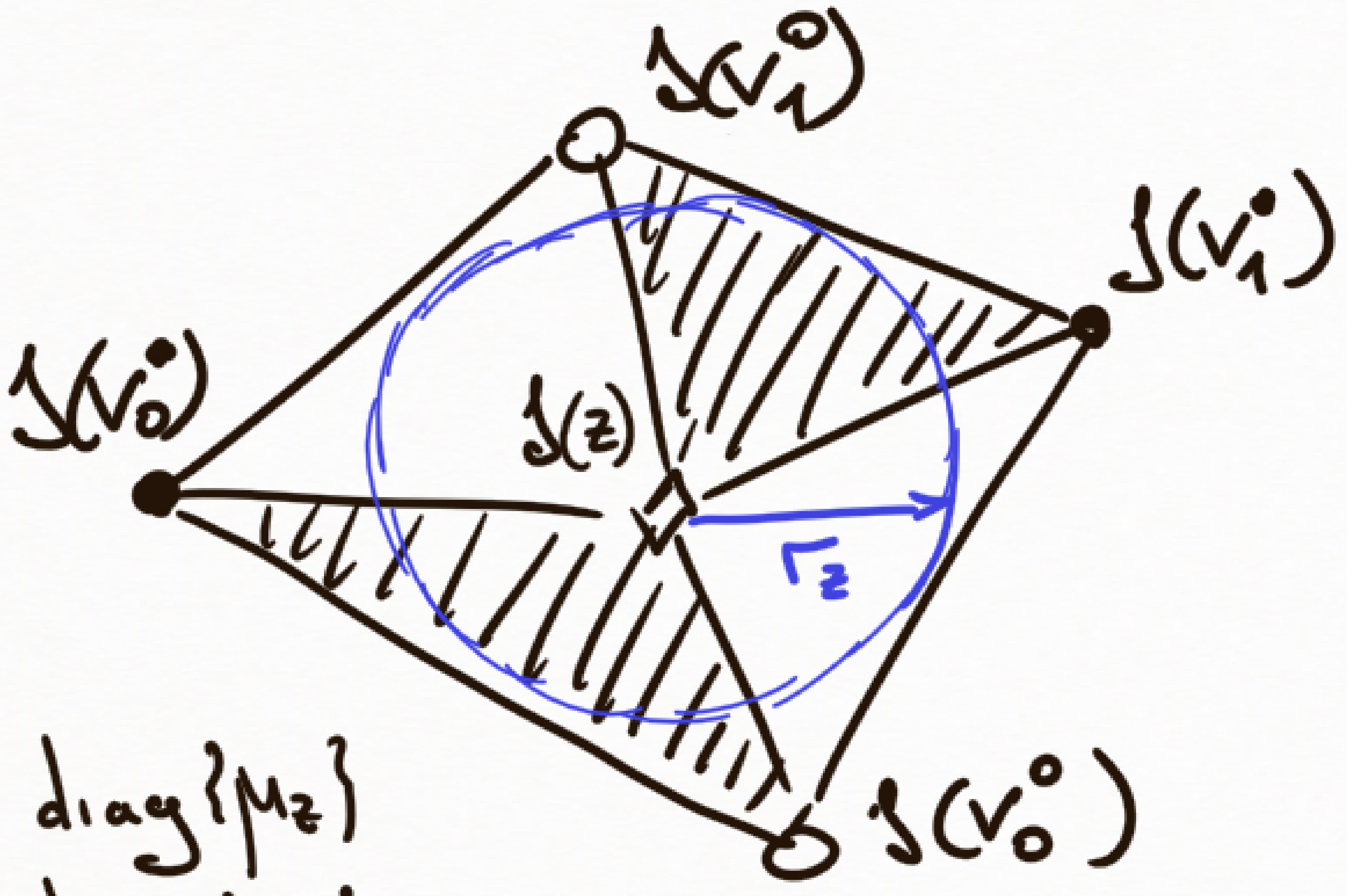
$$= O(\delta^{-1/2+2\varepsilon})$$

and hence

$$H_F^\delta(z) - H_F^\delta|_{\partial\Omega} = O(\delta^{2\varepsilon}) \quad \text{if} \quad d_z \leq \delta^{1-2\varepsilon}$$

$$\uparrow = \int \operatorname{Re}[F^2 d\tau + |F|^2 d\sigma]$$

DIFFERENTIAL OPERATORS ON $\mathcal{S}$-EMBEDDINGS:



$U = \mathrm{diag}\{\mu_z\}$

$R = \mathrm{diag}\{r_z\}$

$$[\bar\partial_{\mathcal{S}} H](z) := \frac{\mu_z}{4}\left[\frac{H(v_0^\bullet)}{\mathcal{S}(v_0^\bullet)-\mathcal{S}(z)} + \frac{H(v_1^\bullet)}{\mathcal{S}(v_1^\bullet)-\mathcal{S}(z)} - \frac{H(v_0^\circ)}{\mathcal{S}(v_0^\circ)-\mathcal{S}(z)} - \frac{H(v_1^\circ)}{\mathcal{S}(v_1^\circ)-\mathcal{S}(z)}\right]$$

Remark:

$$\bar\partial_{\mathcal{S}} 1 = \bar\partial_{\mathcal{S}} \mathcal{S} = \bar\partial_{\mathcal{S}} Q = 0, \quad \bar\partial_{\mathcal{S}} \bar{\mathcal{S}} = 1$$

$$\Delta_{\mathcal{S}} := 16\,\partial_{\mathcal{S}}^* U^{-1} R\, \bar\partial_{\mathcal{S}} = 16\,\bar\partial_{\mathcal{S}}^* \bar{U}^{-1} R\, \partial_{\mathcal{S}}$$ – real, symmetric, not sign-def

Remark: In the isoradial case, this reads as

$$\left[\bar\partial_\Lambda = U\partial_\Lambda(-I^\bullet + I^\circ)\right]$$

$$\Delta_{\mathcal{S}} = \begin{pmatrix} \Delta^\bullet & 0 \\ 0 & -\Delta^\circ \end{pmatrix}$$

$\mathcal{S}$-POSITIVITY PHENOMENON:

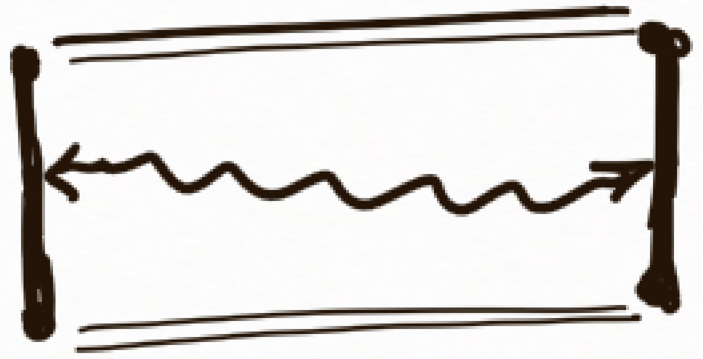
$\forall F \ [\Delta_{\mathcal{S}} H_F] \geq 0$ pointwise

RSW (for FK-representation): it is enough to prove

Proof: a 'brute force' adaptation of Smirnov's argument for 4-pt observables

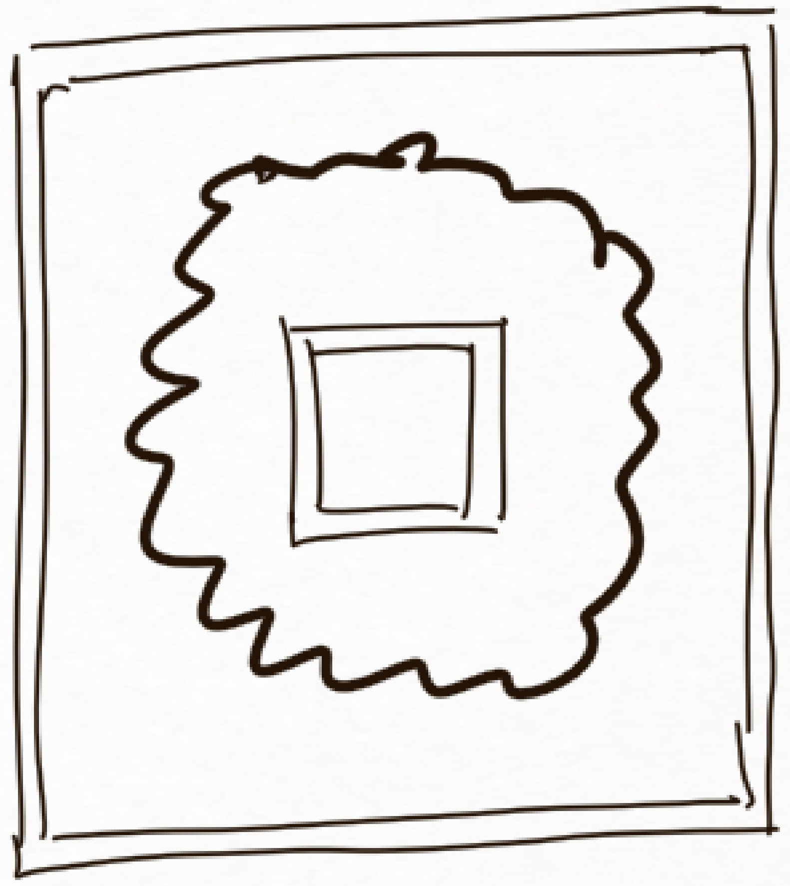
Requires a lot of technicalities:

- construction of special cuts (= straight lines on $\delta + \mathbb{Z}^2 Q$) such that $\Re[F; \alpha\mathbb{R}] \geq 0$
- harmonic measure (on δ-graphs) estimates to control F up to such 'straight' boundaries



for 'straight' rectangles w/ self-dual boundary cond.

$\Rightarrow$



$\Rightarrow \eta > 0$

RSW [$H_F^\delta = o(1)$ IF $d_z < \delta^{1-\varepsilon}$] SEEMS TO BE THE WEAKEST LINK IN THE CHAIN OF ARGUMENTS!

[though $\Delta H_F^\delta = O(1/d_z^{2-\varepsilon})$ also requires smth]

SOME PERSPECTIVES / RESEARCH ROUTES

UNDER CONSTRUCTION:

- updating techniques used in the proofs of the convergence of correlations [C. Hongler, K. Izyurov, S.C. Park]
  (⊕ spin correlations etc in the massive isoradial case w/ R. Mahfouf)

- generalizing RSW [R. Mahfouf, M. Oulamara] (ideally to $\mathcal{L}ip(k,\delta)$ but many "intermediate" setups exist)

- massive SLE(3) curves [S.C. Park, Y. Wan] (non-trivial issues 'in continuum', much less clear than mSLE(2)!)

... BUT ALSO DREAMS

DREAMS on critical planar maps ⊕ Ising (dimers)

It seems that critical planar maps weighted by Ising could/should (via s-embeddings) lead to "canonical" fluctuating space-like surfaces in $\mathbb{R}^{2+1(2)}$

- Can we "guess" the law ?
- Is there a way to speak about Dirac $\begin{pmatrix} im & \partial \\ \bar{\partial} & -im \end{pmatrix}$ on such (rough!) surfaces, where $m =$ "mean curvature"?
- Could $(\det D)^{1/4(?)}$ (presumably, additionally multiplied by certain "geometric" factors) be a relevant quantity?

COULD S-HOLOMORPHICITY / S-EMBEDDINGS BE A BEGINNING OF (ANOTHER) BEAUTIFUL STORY 12 YEARS AFTER MFO-08 ?!