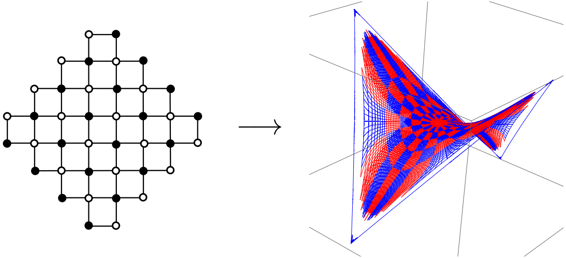
Bipartite dimer model and minimal surfaces in the Minkowski space

Dmitry Chelkak (ENS)

[joint works w/ Benoît Laslier, Sanjay Ramassamy, Marianna Russkikh]



UM – MSU Mathematics Colloquium, September 1st, 2020 @ Zoom

Outline:

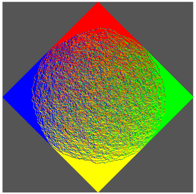
- **Basics of the bipartite dimer model:**
  - ▷ definition, Kasteleyn's theorem;
  - ▷ Thurston's height functions;
  - ▷ Temperleyan domains: $\hbar^\delta \to \mathrm{GFF}$.
- **Conjectural picture on periodic grids:**
  - ▷ Cohn–Kenyon–Propp's theorem;
  - ▷ Kenyon–Okounkov's prediction: $\hbar^\delta \to$ *GFF in a non-trivial metric.*
- **New viewpoint: t-embeddings $\mathcal{T}^\delta$**
  - ▷ basic concepts, origami maps $\mathcal{O}^\delta$;
  - ▷ **Assumptions:** perfect t-embeddings, $(\mathcal{T}^\delta, \mathcal{O}^\delta) \to$ *Lorenz-minimal surface;*
  - ▷ **Theorem** [Ch.–Laslier–Russkikh '20].
- **(Some) open questions/perspectives.**

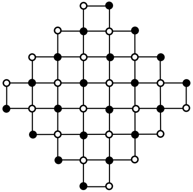
Illustration:

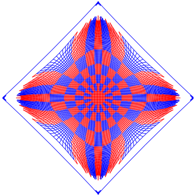
Aztec diamonds

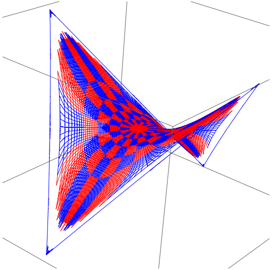
[Ch.–Ramassamy]

[arXiv:2002.07540]







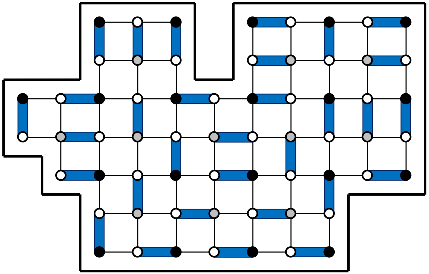


Bipartite dimer model: basics

- $(\mathcal{G}, \nu_{bw})$ – finite weighted bipartite planar graph (w/ marked outer face);
- *Dimer configuration* = perfect matching $\mathcal{D} \subset E(\mathcal{G})$: subset of edges such that each vertex is covered exactly once;
- *Probability* $\mathbb{P}(\mathcal{D}) \propto \nu(\mathcal{D}) = \prod_{e\in\mathcal{D}} \nu_e$;
- *Partition function* $\mathcal{Z}_\nu(\mathcal{G}) = \sum_{\mathcal{D}} \nu(\mathcal{D})$.

(Very) particular example:

[Temperleyan domains $\mathcal{G}_{\mathrm{T}} \subset \mathbb{Z}^2$]



Example: if all weights $\nu_{bw} = 1$, then $\mathcal{Z}$ is the number of perfect matchings in $\mathcal{G}$.

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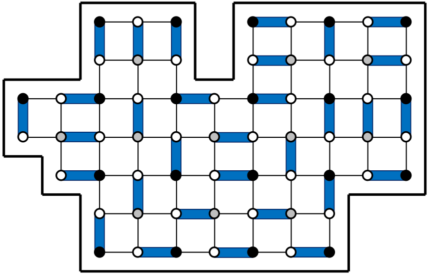

- **Theorem (Kasteleyn, 1961):** for each planar (not necessarily bipartite) graph $(\mathcal{G}, \nu)$, one can find a *signed* adjacency matrix $\mathcal{A}_\nu = -\mathcal{A}_\nu^\top$ of G:

$$\mathcal{Z}_\nu(G) = |\mathbf{Pf}\,\mathcal{A}_\nu| = |\det \mathcal{A}_\nu|^{1/2}$$

[such an orientation of edges of a planar graph $\mathcal{G}$ is called a Pfaffian orientation]

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Q: Could you remind us what $\mathrm{Pf}\,\mathcal{A}$ is?

A: If $\mathcal{A} = -\mathcal{A}^\top$ is a $2n \times 2n$ matrix, then

$$\mathrm{Pf}\,\mathcal{A} := \frac{1}{2^n n!} \sum (-1)^{s(\sigma)} a_{\sigma_1\sigma_2} .. a_{\sigma_{2n-1}} a_{\sigma_{2n}}$$

Example:

$$\mathrm{Pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + cd$$

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- **$\mathcal{G}$ – bipartite** $\Rightarrow$ $\mathcal{A}_\nu = \begin{bmatrix} 0 & \mathcal{K}_\nu \\ -\mathcal{K}_\nu^\top & 0 \end{bmatrix}$

and $|\mathrm{Pf}\,\mathcal{A}_\nu| = |\det \mathcal{K}_\nu|$.

- **Corollary:** If $b \sim w$ in $\mathcal{G}$, then

$$\mathbb{P}[\,(bw) \in \mathcal{D}\,] = |\mathcal{K}_\nu^{-1}(w,b)|.$$

Moreover, the edges of a random dimer configuration $\mathcal{D}$ form a determinantal process with the kernel $\mathcal{K}_\nu^{-1} : \mathbb{C}^B \to \mathbb{C}^W$.

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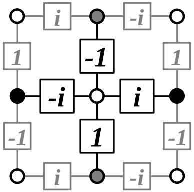
• **[Kenyon, 2000]:** it is often convenient to introduce *complex signs* in $\mathcal{K}_\nu$. E.g., *on* $\mathbb{Z}^2$, the following choice works:



$$\mathcal{K}\cdot\mathcal{K}^{-1} = \mathrm{Id}$$
$$\Downarrow$$
$\mathcal{K}^{-1}(w,\cdot) : B \to \mathbb{C}$ are *discrete holomorphic functions*.

• $\mathcal{G}$ **– bipartite** $\Rightarrow \mathcal{A}_\nu = \begin{bmatrix} 0 & \mathcal{K}_\nu \\ -\mathcal{K}_\nu^\top & 0 \end{bmatrix}$ and $|\mathrm{Pf}\,\mathcal{A}_\nu| = |\det \mathcal{K}_\nu|$.

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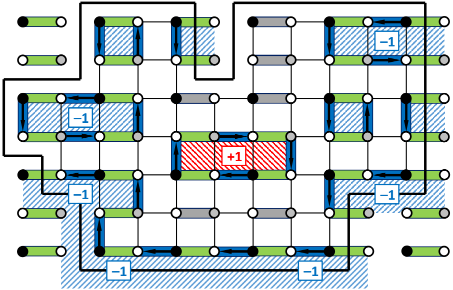
GFF and random height fluctuations

- $\mathcal{D}$ – random dimer configuration
- Random *height function h on $\mathcal{G}^*$:* fix $\mathcal{D}_0$, view $\mathcal{D} \cup \mathcal{D}_0$ as a topographic map.
- *Height fluctuations* $\hbar := h - \mathbb{E}[h]$ do *not* depend on the choice of $\mathcal{D}_0$.

(Very) particular example:

[Temperleyan domains $\mathcal{G}_{\mathrm{T}} \subset \mathbb{Z}^2$]



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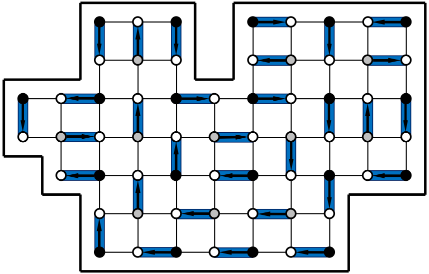

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- *Height fluctuations* $\hbar := h - \mathbb{E}[h]$ do *not* depend on the choice of $\mathcal{D}_0$.

- **Theorem (Kenyon, 2000):** Let $\mathcal{G}_{\mathrm{T}}^{\delta} \subset \delta\mathbb{Z}^2$ be *Temperleyan approximations* to a given domain $\Omega \subset \mathbb{C}$. Then,

$$\hbar^{\delta} \to \pi^{-\frac{1}{2}}\mathrm{GFF}_{\Omega} \quad \text{as} \quad \delta \to 0,$$

where GFF_{Ω} is the *Gaussian Free Field* in Ω with Dirichlet boundary conditions.

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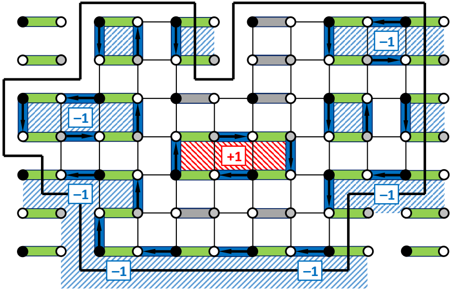
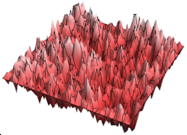


Q: What is GFF_{Ω}?

A: $\mathbb{E}[\hbar(z)] = 0$, $z \in \Omega$;

$$\mathbb{E}[\hbar(z)\hbar(w)] = -\Delta_{\Omega}^{-1}(z,w).$$



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Q: why are Temperleyan $\mathcal{G}_{\mathrm{T}}$ *so special?*

A1: 'nice' boundary conditions for discrete holomorphic functions $\mathcal{K}^{-1}(w, \cdot)$.

A2: *Wilson's algorithm for UST* $\Rightarrow$ random walks with 'nice'(=absorbed) boundary conditions naturally appear.

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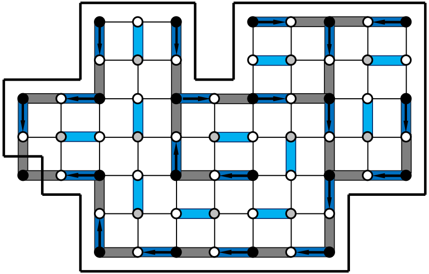


Temperley bijection: dimers on $\mathcal{G}_{\mathrm{T}}$ $\leftrightarrow$ *spanning trees* on a related graph. This procedure is highly sensitive to the *microscopic structure of the boundary.*

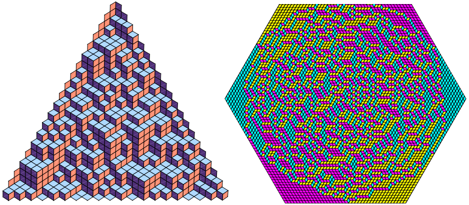
Conjectural picture on periodic grids

• [Cohn–Kenyon–Propp, 2000]:
random profiles δh^δ *concentrate* near a surface (with given boundary) that maximizes certain *entropy functional.*

▷ Example: flat height profile at $\partial\Omega$
$\rightsquigarrow$ flat surface in the bulk of Ω.

▷ Remark: the entropy functional is non-trivial and *lattice-dependent.*

Examples on $\mathbb{H}\mathrm{ex}^*$ [(c) Kenyon] :



[!!!] Though the law of $\hbar^\delta$ is independent of the choice of $\mathcal{D}_0^\delta$, the limit of $\hbar^\delta$ as $\delta \to 0$ *heavily depends on* the limit of deterministic *boundary profiles of δh^δ:*

• frozen/liquid/(gaseous) zones in Ω;

• ‘arctic curves’ $\rightsquigarrow$ algebraic geometry;

• ‘polygonal’ examples are well-studied.

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$$\hbar^\delta \to \mathrm{GFF}_{(\boldsymbol{\Omega},\boldsymbol{\mu})}\,,$$

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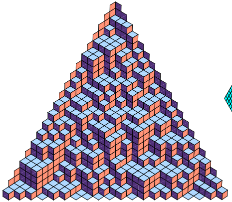

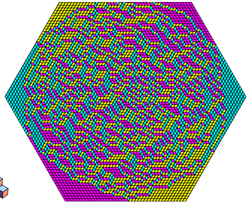

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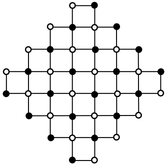
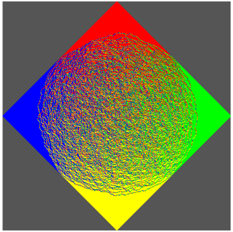
[!] This is **not** proven even for $\Omega^\delta \subset \delta\mathbb{Z}^2$ composed of 2×2 blocks [$\Rightarrow$ 'flat' μ].

- **Classical example studied in detail:**

Aztec diamonds

[Elkies–Kuperberg–Larsen–Propp'92,...]





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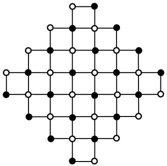

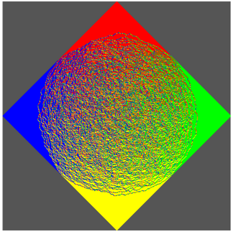

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Q: How can holomorphic/harmonic functions on $\delta\mathbb{Z}^2$ lead to a non-trivial complex structure in the limit $\delta \to 0$?

"A": Think about functions $h(n,m) = \sin(\alpha n)\sinh(bm)$ with $\cos\alpha + \cosh b = 2$.

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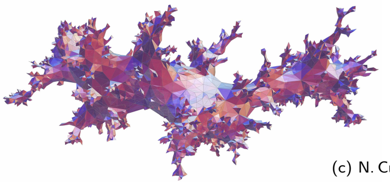
- **Known tools: problematic to apply**

$\updownarrow$ [???] to irregular graphs $(\mathcal{G},\nu)$

- **Long [!!!] -term motivation:**

random maps carrying bipartite dimers [or the Ising model, via bosonization] and their scaling limits *(Liouville CFT).*



(c) N. Curien

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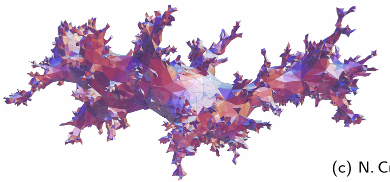

(c) N. Curien

- **Wanted:** *special embeddings* of abstract weighted bipartite planar graphs
$+$ *'discrete complex analysis' techniques*
$\rightsquigarrow$ *complex structure in the limit.*

Theorem: [Ch. – Laslier – Russkikh] [`arXiv:2001.11871 + 20**.**`]

Assume that, for finite weighted bipartite planar graphs $\mathcal{G}^\delta = (\mathcal{G}^\delta, \nu^\delta)$,

- $\mathcal{T}^\delta$ are *perfect t-embeddings* of $(\mathcal{G}^\delta)^*$ [satisfying assumption $\textsc{Exp-Fat}(\delta)$];
- as $\delta \to 0$, the images of $\mathcal{T}^\delta$ converge to a *domain* D_ξ [$\xi \in \mathrm{Lip}_1(\mathbb{T})$, $|\xi| < \frac{\pi}{2}$];
- *origami maps* $(\mathcal{T}^\delta, \mathcal{O}^\delta)$ converge to a *Lorentz-minimal surface* $\mathrm{S}_\xi \subset \mathrm{D}_\xi \times \mathbb{R}$.

Then, the height fluctuations $\hbar^\delta$ in the dimer models on $\mathcal{T}^\delta$ converge to the standard Gaussian Free Field in the *intrinsic metric of* $\mathrm{S}_\xi \subset \mathbb{R}^{2+1} \subset \mathbb{R}^{2+2}$.

Illustration:

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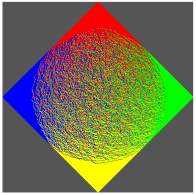
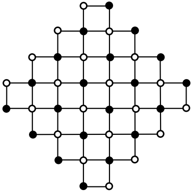
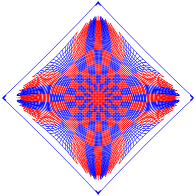
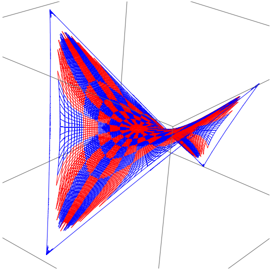
[Ch. – Ramassamy]

[`arXiv:2002.07540`]









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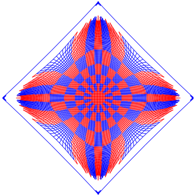
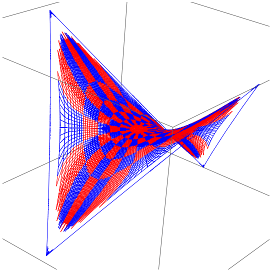
- **Domains D_ξ, surfaces S_ξ:**
- $\xi : \mathbb{T} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ – 1-Lipschitz function;
- D_ξ : bounded by $z(\phi) = e^{i\phi} \cdot (\cos \xi(\phi))^{-1}$;
- S_ξ spans $\mathrm{L}_\xi := (z(\phi), \tan(\xi(\phi)))_{\phi \in \mathbb{T}}$

$\mathrm{L}_\xi \subset \{x \in \mathbb{R}^{2+1} : \|x\|^2 = x_1^2 + x_2^2 - x_3^2 = 1\}$.

- **Aztec case surface S_ξ:**





Theorem: [Ch. – Laslier – Russkikh]
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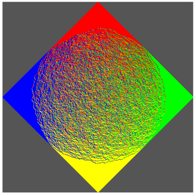
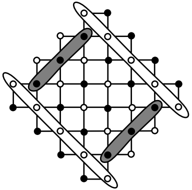
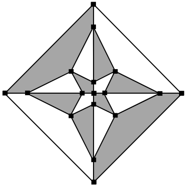
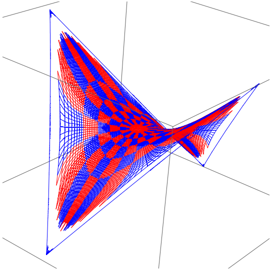
Illustration:

Aztec diamonds
[Ch. – Ramassamy]
[`arXiv:2002.07540`]









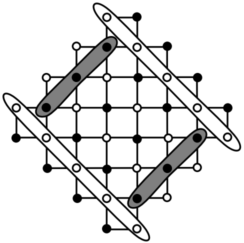
Embeddings of weighted bipartite planar graphs carrying the dimer model

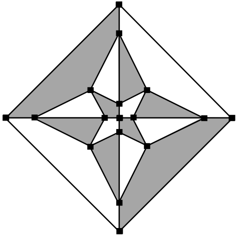
[and admitting reasonable notions of discrete complex analysis]

- *t-embeddings = Coulomb gauges:* given $(\mathcal{G},\nu)$, find $\mathcal{T}:\mathcal{G}^*\to\mathbb{C}$ $[\mathcal{G}^*$ – augmented dual$]$ s.t.

▷ weights ν_e are *gauge equivalent* to $\chi_{(vv')^*}:=|\mathcal{T}(v')-\mathcal{T}(v)|$ (i.e., $\nu_{bw}=g_b\chi_{bw}g_w$ for some $g:B\cup W\to\mathbb{R}_+$) and

▷ at each inner vertex $\mathcal{T}(v)$, the sum of black angles $=\pi$.





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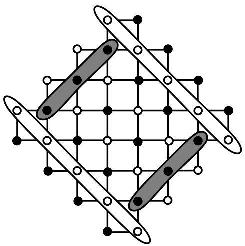

- *p-embeddings* = *perfect t-embeddings:*
  - ▷ outer face is a tangential (possibly, non-convex) polygon,
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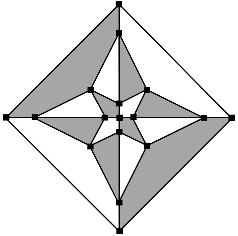

- **Warning:** for general $(\mathcal{G}, \nu)$, the *existence* of perfect t-embeddings is *not known* though they do exist in particular cases + the count of #(degrees of freedom) matches.

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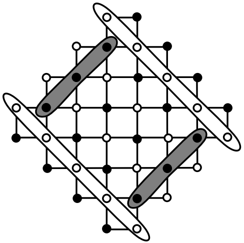
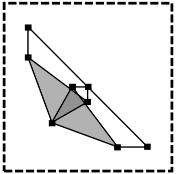
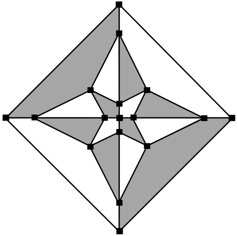


- *origami maps $\mathcal{O}$: $\mathcal{G}^*\to\mathbb{C}$*

  "fold $\mathbb{C}$ along segments of $\mathcal{T}$"

- the mapping $(\mathcal{T},\mathcal{O})$ can be viewed as a 'piece-wise linear embedding' of $\mathcal{G}^*$ into $\mathbb{R}^{2+2}$.





Theorem: [Ch. – Laslier – Russkikh] [`arXiv:2001.11871 + 20**.**`]

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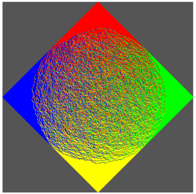

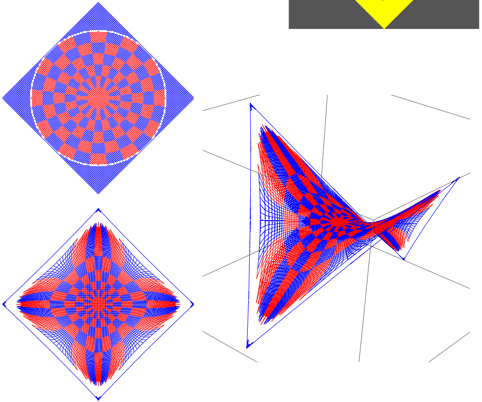

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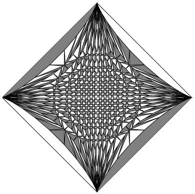
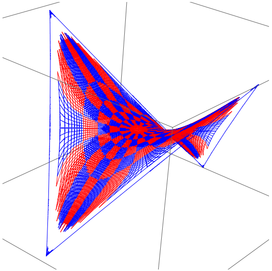
- $\mathrm{Exp\text{-}Fat}(\delta)$ for triangulations $\mathcal{T}^\delta$:

for each $\beta > 0$, if one removes all '$\exp(-\beta\delta^{-1})$-fat' triangles from $\mathcal{T}^\delta$, then the size of remaining (in the bulk of D_ξ) vertex-connected components $\to_{\delta \to 0} 0$.

[non-triangulations: split either black or white faces into triangles]

- **Aztec case p-embeddings:**





Embeddings of weighted bipartite planar graphs carrying the dimer model

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Coulomb gauges [Kenyon – Lam – Ramassamy – Russkikh, `arXiv:1810.05616`]

$\Updownarrow$ *(circle patterns, cluster algebras)* [+ Affolter `arXiv:1808.04227`]

t-embeddings [Ch. – Laslier – Russkikh, `arXiv:2001.11871`, `arXiv:20**.**`]

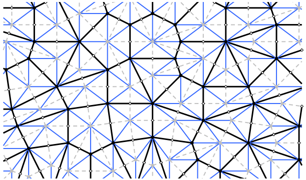
(discrete complex analysis framework & a priori regularity estimates)

Particular cases: harmonic/ *Tutte's embeddings* [via the Temperley bijection]

Ising model s-embeddings [Ch., `arXiv:1712.04192`, `2006.14559`]

Very particular case: Baxter's Z-invariant Ising model: *rhombic lattices/isoradial graphs*

[Ch.–Smirnov, `arXiv:0808.2547`, `0910.2045` "*Universality in the 2D Ising model and conformal invariance of fermionic observables*"]



Open questions, perspectives *[general $(\mathcal{G},\nu)$]*

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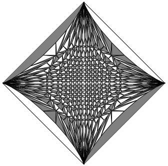

- $\deg f_{\mathrm{out}} = 4$: OK [KLRR]
- #(degrees of freedom): OK

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Another simple example: annulus-type graphs

$\rightsquigarrow$ *Lorentz-minimal cusp* $(z, \operatorname{arcsinh}|z|)$.

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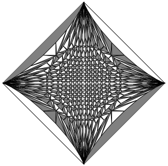
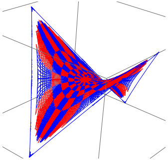
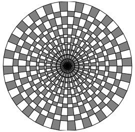
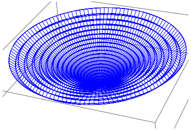
$\leftrightsquigarrow$ embeddings to the Klein/**Plücker quadric** [?]









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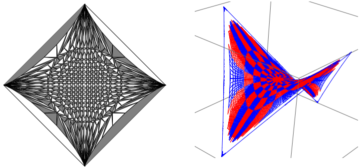

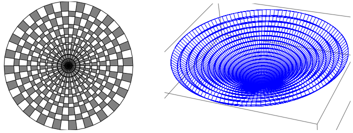

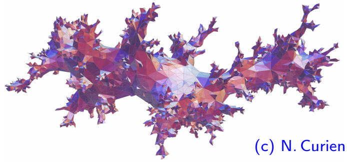

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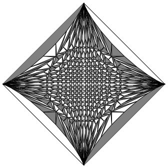

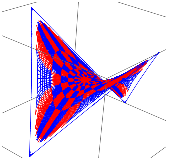

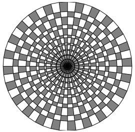

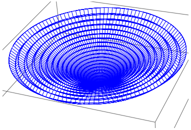

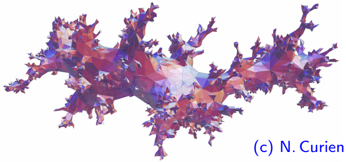

(c) N. Curien

Thank you!