2D Ising model: combinatorics, CFT/CLE description at criticality [and beyond...]

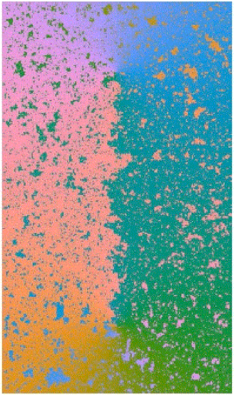
Dmitry Chelkak (ÉNS)

[Sample of a critical 2D Ising configuration (with two disorders), © Clément Hongler (EPFL)]

Les probabilités de demain. IHÉS, 11.05.2017

Nearest-neighbor 2D Ising model

- Combinatorics:
  - dimers and fermionic observables
  - double-covers and spin correlations
  - spin-disorder formalism
- Holomorphicity and phase transition: some classical computations revisited
- CFT: correlation functions at criticality
  - Riemann-type boundary value problems
  - Convergence and conformal covariance
  - Fusion rules $(\psi, \varepsilon, \mu, \sigma)$ etc
- Convergence to CLE [Benoist–Hongler'16]
  - Convergence of curves via martingales
  - Crossing estimates (precompactness)
- Open questions



[Two disorders: sample of a critical 2D Ising configuration

Nearest-neighbor Ising (or Lenz-Ising) model in 2D

Definition: *Lenz-Ising model* on a planar graph G^* (dual to G) is a random assignment of $+/-$ spins to vertices of G^* (faces of G)

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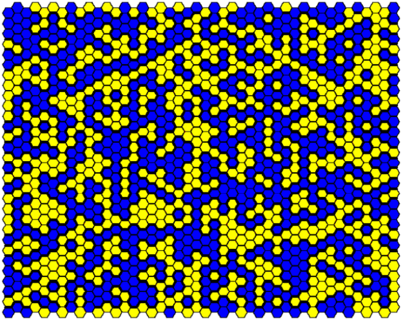

[sample of a honeycomb percolation]

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A: .. according to the following probabilities:

$$\mathbb{P}\left[\text{conf. } \sigma \in \{\pm 1\}^{V(G^*)}\right] \propto \exp\left[\beta \sum_{e=\langle uv \rangle} J_{uv}\sigma_u\sigma_v\right] \propto \prod_{e=\langle uv \rangle : \sigma_u \neq \sigma_v} x_{uv} ,$$

where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv \rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.

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Remark: w/o an external magnetic field this is a **"free fermion"** model.

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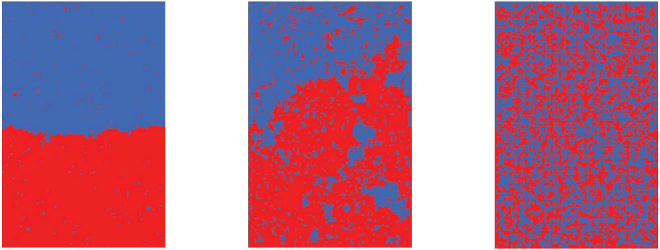
where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv\rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.

- It is also convenient to use the parametrization $x_{uv} = \tan(\frac{1}{2}\theta_{uv})$.
- Working with subgraphs of *regular lattices*, one can consider the *homogeneous model* in which all x_{uv} are equal to each other.

Lenz-Ising model: phase transition (e.g., on $\mathbb{Z}^2$)

E.g., Dobrushin boundary conditions: $+1$ on (ab) and -1 on (ba):



$x < x_{\mathrm{crit}}$ $\qquad$ $x = x_{\mathrm{crit}}$ $\qquad$ $x > x_{\mathrm{crit}}$

- Ising (1925): no phase transition in 1D $\rightsquigarrow$ doubts about 2+D;
- Peierls (1936): existence of the phase transition in 2D;
- Kramers-Wannier (1941): $x_{\text{self-dual}} = \sqrt{2} - 1 = \tan(\frac{1}{2} \cdot \frac{\pi}{4})$;
- Onsager (1944): sharp phase transition at $x_{\text{crit}} = \sqrt{2} - 1$.

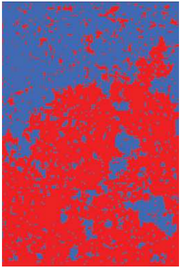
At criticality (e.g., on $\mathbb{Z}^2$):

- **scaling exponent $\frac{1}{8}$** for the magnetization [Kaufman–Onsager(1948), Yang(1952), via "diagonal" spin-spin correlations at $x \uparrow x_{\mathrm{crit}}$]

◦ [Wu (1966), correlations at $x = x_{\mathrm{crit}}$]
$\rightsquigarrow$ as $\Omega_\delta \to \Omega$, it should be $\mathbb{E}_{\Omega_\delta}[\sigma_u] \asymp \delta^{\frac{1}{8}}$.

- Existence of the **scaling limits** as $\Omega_\delta \to \Omega$:

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}[\sigma_{u_1} \dots \sigma_{u_n}] \quad \to \quad \langle \sigma_{u_1} \dots \sigma_{u_n} \rangle_\Omega$$



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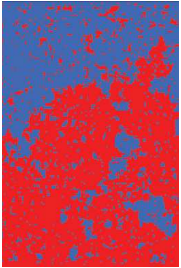

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Conformal covariance: $\quad = \quad \langle \sigma_{\varphi(u_1)} \dots \sigma_{\varphi(u_n)} \rangle_{\varphi(\Omega)} \cdot \prod_{s=1}^n |\varphi'(u_s)|^{\frac{1}{8}}$

- Basing on this, one can also deduce the convergence of the random fields $(\delta^{-\frac{1}{8}}\sigma_u)_{u\in\Omega}$ to a (non-Gaussian!) limit as $\delta \to 0$ [Camia–Garban–Newman '13, Furlan–Mourrat '16; see also Caravenna–Sun–Zygouras '15 on disorder-relevance results].

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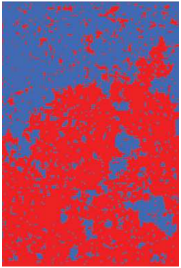

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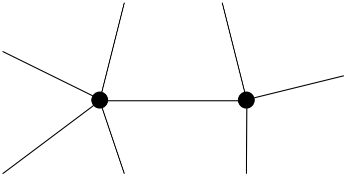
- Instead of studying correlation functions, one can describe the limit geometrically: convergence of **curves** (e.g., domain walls generated by Dobrushin boundary conditions) and **loop ensembles** (either outermost or nested) to **conformally invariant limits.**

2D Ising model as a dimer model (on a non-bipartite graph) [Fisher, Kasteleyn ('60s+),..., Kenyon, Dubédat ('00s+),...]

• **Partition function** $\mathcal{Z} = \sum_{\sigma\in\{\pm1\}^{V(G^*)}} \prod_{e=\langle uv\rangle:\sigma_u\neq\sigma_v} x_{uv}$

• There exist various representations of the 2D Ising model via dimers on an auxiliary graph



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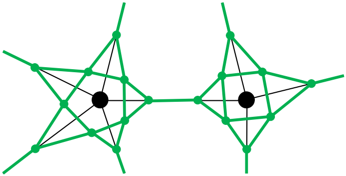

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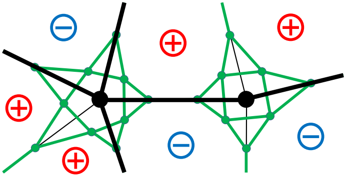

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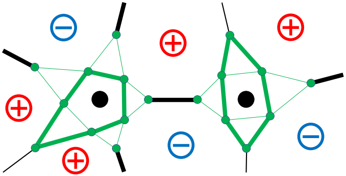

- **Kasteleyn's theory:** $\mathcal{Z} = \mathrm{Pf}[\mathrm{K}]$ [$\mathrm{K}=-\mathrm{K}^\top$ is a weighted adjacency matrix of G_F]

- **Reminder:** Let $\mathrm{K} = -\mathrm{K}^\top$ be a $2N \times 2N$ antisymmetric matrix.

$$\mathrm{Pf}[\mathrm{K}] := \frac{1}{2^N N!}\sum_\sigma (-1)^{\mathrm{sign}(\sigma)}\mathrm{K}_{\sigma(1)\sigma(2)}\ldots\mathrm{K}_{\sigma(2N-1)\sigma(2N)} = (\det[\mathrm{K}])^{\frac{1}{2}}$$

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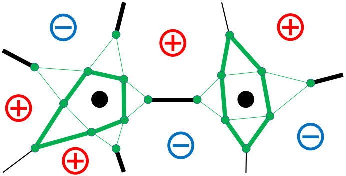

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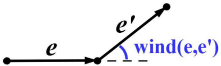
$\Leftrightarrow$ **Kac–Ward formula (1952–...,1999–...):** $\mathcal{Z}^2 = \det[\mathrm{Id} - \mathrm{T}]$,

$$\mathrm{T}_{e,e'} = \begin{cases} \exp[\tfrac{i}{2}\mathrm{wind}(e,e')]\cdot(x_e x_{e'})^{1/2} \\ 0 \end{cases}$$



"Revisiting 2D Ising combinatorics" [Ch.–Cimasoni–Kassel'15]

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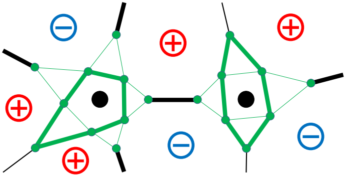

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- **Energy density field:** note that $\mathbb{P}[\sigma_{e^\sharp}\sigma_{e^\flat} = -1] = |\,\mathrm{K}^{-1}_{e,\bar{e}}\,|$.

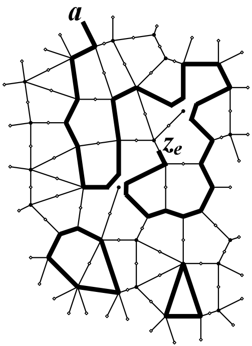
- **Local relations** for the entries $\mathrm{K}^{-1}_{a,e}$ and $\mathrm{K}^{-1}_{a,c}$ of the inverse Kasteleyn (or the inverse Kac–Ward) matrix:
(an equivalent form of) the identity $\mathrm{K}\cdot\mathrm{K}^{-1} = \mathrm{Id}$

Fermionic observables: combinatorial definition [Smirnov'00s]

For an oriented edge a and a midedge z_e (similarly, for a corner c),

$$F_G(a, z_e) := \overline{\eta}_a \sum_{\omega \in \mathrm{Conf}_G(a,z_e)} \Big[e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)} \prod_{\langle uv \rangle \in \omega} x_{uv} \Big]$$

where η_a denotes the (once and forever fixed) square root of the direction of a.



- The factor $e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z_e)}$ does not depend on the way how ω is split into non-intersecting loops and a path $a \rightsquigarrow z_e$.

- **Via dimers on G_F:** $F_G(a, c) = \overline{\eta}_c \mathrm{K}^{-1}_{c,a}$

$$F_G(a, z_e) = \overline{\eta}_e \mathrm{K}^{-1}_{e,a} + \overline{\eta}_{\overline{e}} \mathrm{K}^{-1}_{\overline{e},a}$$

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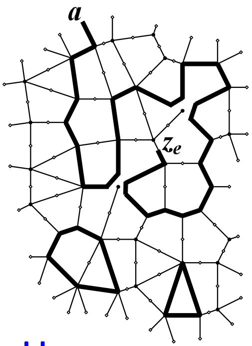

• **Local relations: at criticality,** can be thought of as a special form of **discrete Cauchy–Riemann equations.**

• **Boundary conditions $F(a, z_e) \in \overline{\eta}_{\overline{e}}\mathbb{R}$** ($\overline{e}$ is oriented outwards) uniquely determine F as a solution to an appropriate **discrete Riemann-type boundary value problem.**

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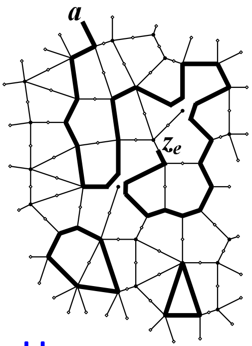

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$\rightsquigarrow$ **Scaling limit of energy densities** [Hongler–Smirnov'10]

Spin correlations and spinor observables: combinatorics

- spin configurations on G^*
  $\leftrightsquigarrow$ domain walls on G
  $\leftrightsquigarrow$ dimers on G_F
- **Kasteleyn's theory: $\mathcal{Z} = \mathrm{Pf}[\,\mathbf{K}\,]$**

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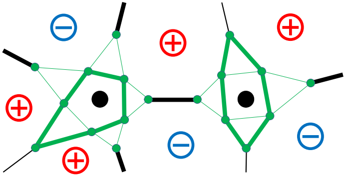

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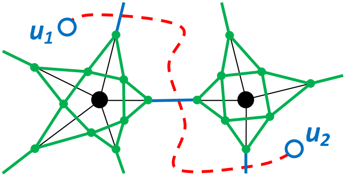

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- **Claim:**

$$\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] \;=\; \frac{\mathrm{Pf}[\,\mathrm{K}_{[u_1,\ldots,u_n]}\,]}{\mathrm{Pf}[\,\mathrm{K}\,]},$$

where $\mathrm{K}_{[u_1,\ldots,u_n]}$ is obtained from K by changing the sign of its entries on **slits linking $u_1, \ldots, u_n$** (and, possibly, u_{out}) pairwise.

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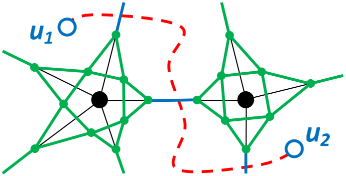

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- If one shifts u_1 to a neighboring face $\widetilde{u}_1$, the "spatial derivative"

$$\frac{\mathbb{E}[\sigma_{\widetilde{u}_1} \ldots \sigma_{u_n}]}{\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}]}$$ can be expressed via the entries of $\mathbf{K}^{-1}_{[u_1,\ldots,u_n]}$.

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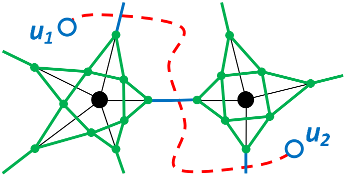

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- **More invariant way** to think about entries of $\mathrm{K}^{-1}_{[u_1,\ldots,u_n]}$:

  double-covers of G branching over $u_1, \ldots, u_n$

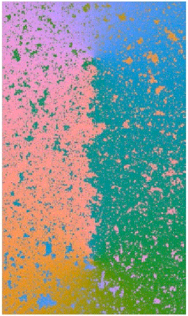
- Similarly to K^{-1}, these entries can be defined "combinatorially" [though most probably you do not like to see this definition...]
- **Alternative route: σ–μ formalism [Kadanoff–Ceva (1971)]**

σ–μ formalism [Kadanoff–Ceva]

• Recall that spins σ_u are assigned to the faces of G. Given (an even number of) *vertices* $v_1, \ldots, v_m$, link them pairwise by a collection of paths $\varkappa = \varkappa^{[v_1,\ldots,v_m]}$ and replace x_e by x_e^{-1} for all $e \in \varkappa$. Denote

$$\langle \mu_{v_1} \ldots \mu_{v_m} \rangle_G \ := \ \mathcal{Z}_G^{[v_1,\ldots,v_m]} / \mathcal{Z}_G .$$

• Equivalently, one may think of the Ising model on a double-cover $G^{[v_1,\ldots,v_m]}$ that branches over each of $v_1, \ldots, v_m$ with the *spin-flip symmetry* constrain $\sigma_{u^\sharp} = -\sigma_{u^\flat}$ if $u^\sharp$ and $u^\flat$ lie over the same face of G.



[two disorders inserted]

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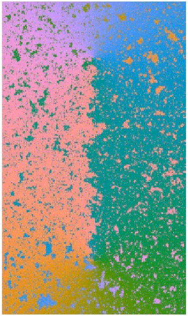

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• Equivalently, one may think of the Ising model on a double-cover $G^{[v_1,\ldots,v_m]}$ that branches over each of $v_1, \ldots, v_m$ with the *spin-flip symmetry* constrain $\sigma_{u^\sharp} = -\sigma_{u^\flat}$ if $u^\sharp$ and $u^\flat$ lie over the same face of G. Let

$$\langle \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle_G \;:=\; \mathbb{E}_{G^{[v_1,\ldots,v_m]}}[\sigma_{u_1} \ldots \sigma_{u_n}] \cdot \langle \mu_{v_1} \ldots \mu_{v_m} \rangle_G\,.$$

• By definition, $\langle \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle_G$ changes the sign when one of the faces u_k goes around of one of the vertices v_s.

σ–μ formalism [Kadanoff–Ceva]

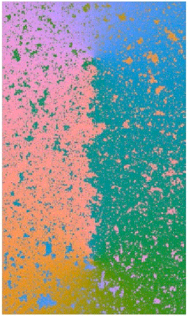
- By definition, $\langle \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle_G$ changes the sign when one of the faces u_k goes around of one of the vertices v_s.
- For a corner c lying in $u(c)$ near $v(c)$,

$$\psi_c := \delta^{\frac{1}{2}} (u(c) - v(c))^{-\frac{1}{2}} \mu_{v(c)} \sigma_{u(c)}$$

$\rightsquigarrow$ **the same fermionic observables**

$$\langle \psi_{c_1} \ldots \psi_{c_{2k}} \rangle_G = \mathrm{Pf}[\, \langle \psi_{c_p} \psi_{c_q} \rangle_G \,]_{p,q=1}^{2k}$$

as before (provided $v(c_p) \neq v(c_q)$).



[two disorders inserted]

"Revisiting 2D Ising combinatorics" [Ch.–Cimasoni–Kassel'15]

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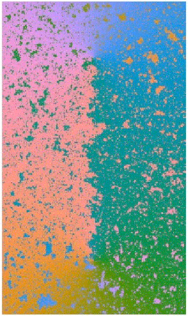
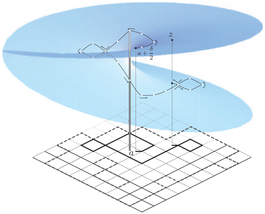


[two disorders inserted]

- **Remark:** This also works in presence of other spins and/or disorders. The antisymmetry $\langle \psi_d \psi_c \rangle_G = -\langle \psi_c \psi_d \rangle_G$ is caused by the sign change of the corresponding spin-disorder correlation.
- $x = x_{\mathrm{crit}} \Rightarrow \langle \psi_c \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle$ are **discrete holomorphic**
  [this observation goes back at least to 1980s (Perk, Dotsenko)]

σ–μ formalism [Kadanoff–Ceva]



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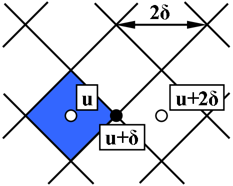
- $x = x_{\mathrm{crit}} \;\Rightarrow\; \langle \psi_c \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle$ are **discrete holomorphic**
  [with **square-root type branchings** over $v_1, \ldots, v_m, u_1, \ldots, u_n$]

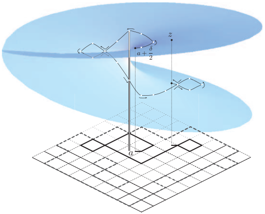
- Denote $F_{\Omega_\delta}(c) := \dfrac{\langle \psi_c \mu_{u_1+\delta} \sigma_{u_2} \ldots \sigma_{u_n} \rangle}{\langle \sigma_{u_1} \sigma_{u_2} \ldots \sigma_{u_n} \rangle}$

  [normalization: $F_{\Omega_\delta}(u_1 + \frac{1}{2}\delta) = \pm i$]

$$F_{\Omega_\delta}(u_1 + \tfrac{3\delta}{2}) = \frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1+2\delta} \sigma_{u_2} \ldots \sigma_{u_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1} \sigma_{u_2} \ldots \sigma_{u_n}]}$$



σ–μ formalism [Kadanoff–Ceva]



- By definition, $\langle \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle_G$ changes the sign when one of the faces u_k goes around of one of the vertices v_s.
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- $x = x_{\mathrm{crit}} \;\Rightarrow\; \langle \psi_c \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle$ are **discrete holomorphic** [with **square-root type branchings** over $v_1, \ldots, v_m, u_1, \ldots, u_n$]

- Denote $F_{\Omega_\delta}(c) := \dfrac{\langle \psi_c \mu_{u_1+\delta} \sigma_{u_2} \ldots \sigma_{u_n} \rangle}{\langle \sigma_{u_1} \sigma_{u_2} \ldots \sigma_{u_n} \rangle}$

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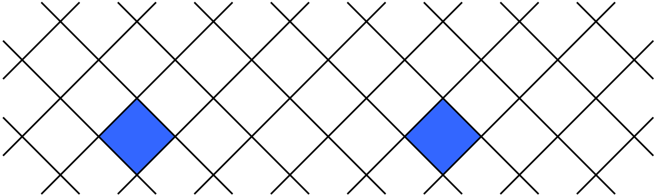
$$F_{\Omega_\delta}(u_1 + \tfrac{3\delta}{2}) = \frac{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1+2\delta} \sigma_{u_2} \ldots \sigma_{u_n}]}{\mathbb{E}^+_{\Omega_\delta}[\sigma_{u_1} \sigma_{u_2} \ldots \sigma_{u_n}]}$$

- As before, these functions can be thought of as solutions to some **Riemann-type boundary value problems in Ω_δ.**

Phase transition: classical computations revisited

Let $x = \tan \frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan \frac{\pi}{8}$, $\boldsymbol{D_n(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(0,0)}\sigma_{(2n,0)}]}$ where $\mathbb{C}^\diamond = \{(k,s) : k,s \in \mathbb{Z}, k+s \in 2\mathbb{Z}\}$ is the $\frac{\pi}{4}$-rotated $\mathbb{Z}^2$.



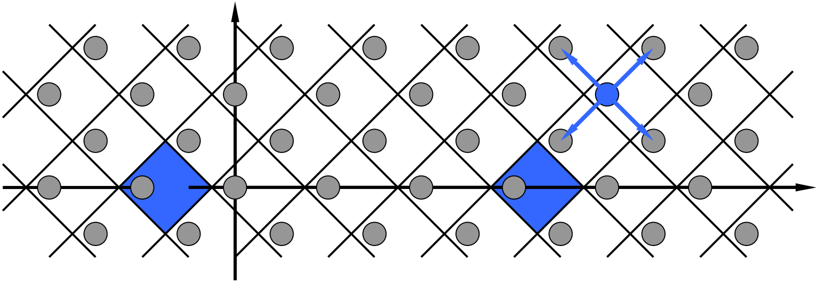
Theorem (revisited): [Kaufman – Onsager(1948), Yang(1952)]

$\boldsymbol{\lim_{n\to\infty} D_n(x)} = (1-\tan^4\theta)^{\frac{1}{4}} \sim \mathrm{cst}\cdot \boldsymbol{(x_{\mathrm{crit}} - x)^{\frac{1}{4}}}$ for $x < x_{\mathrm{crit}}$

[Wu(1966)] $\boldsymbol{D_n(x_{\mathrm{crit}})} = \left(\frac{2}{\pi}\right)^n \prod_{s=1}^{n-1}\left(1-\frac{1}{4s^2}\right)^{s-n} \sim \mathrm{cst}\cdot \boldsymbol{(2n)^{-\frac{1}{4}}}$

Phase transition: classical computations revisited

Let $x = \tan \frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan \frac{\pi}{8}$, $D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]$



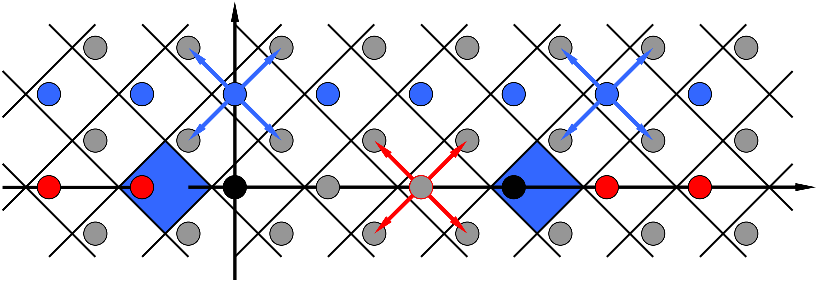
• **Local relations:** $F_{\mathbb{C}^\diamond}(d) = \frac{m}{4}\sum_{d'\sim d} F_{\mathbb{C}^\diamond}(d')$, $m := \sin(2\theta)$.

[Above, we focus on purely real values of the observable on one particular type of corners.] **Note that $m = 1$ iff $x = x_{\mathrm{crit}}$.**

• **Decay at infinity** $\rightsquigarrow$ there exists only two-parameter family of such functions. Moreover, they can be constructed "explicitly".

Phase transition: classical computations revisited

Let $x = \tan \frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan \frac{\pi}{8}$, $\boldsymbol{D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]}$



- Fourier transform: $\boldsymbol{Q_{n,s}(e^{it}) := \sum_{k\in\mathbb{Z}:k+s\in 2\mathbb{Z}} e^{\frac{1}{2}ikt} F_{\mathbb{C}^\diamond}(k,s).}$

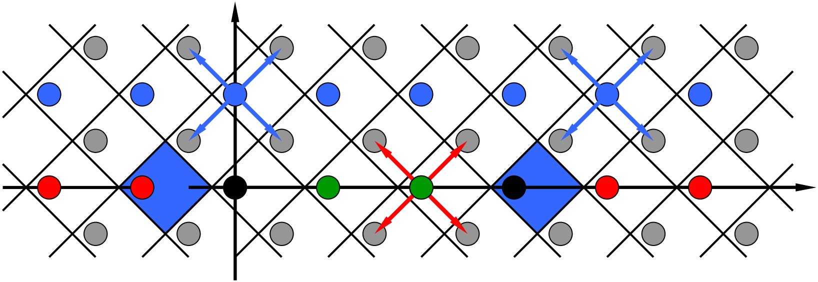
Combinatorics of observables $\Rightarrow$ the following values on $\mathbb{R}$:

$$D_{n+1}Q_{n,0}(e^{it}) = \mathbf{0} + D_n + \ldots + D_n^* e^{int} + \mathbf{0}$$
$$w(e^{it}) \cdot D_{n+1}Q_{n,0}(e^{it}) = \ldots + D_{n+1} + \mathbf{0} + q^2 D_{n+1}^* e^{int} + \ldots$$

where $\boldsymbol{w(e^{it}) = |1 - q^2 e^{it}|}$, $q := \tan\theta \leqslant 1$ and $D_n^* := D_n(\tan(\frac{\pi}{4} - \theta))$.

Phase transition: classical computations revisited

Let $x = \tan\frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan\frac{\pi}{8}$, $\boldsymbol{D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]}$



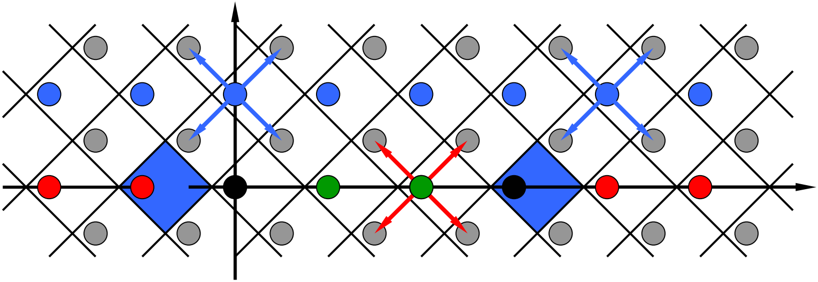
• $\rightsquigarrow$ **the values of these full-plane observables** on the real line are coefficients of certain **orthogonal polynomials Q_n wrt $w(e^{it})$** [which are simply Legendre polynomials if $x = x_{\mathrm{crit}}$].

$\Longrightarrow$ one can express D_{n+1}, D^*_{n+1} via D_n, D^*_n and norms of Q_n,

where $\boldsymbol{w(e^{it}) = |1 - q^2 e^{it}|}$, $q := \tan\theta \leqslant 1$ and $D^*_n := D_n(\tan(\frac{\pi}{4} - \theta))$.

Phase transition: classical computations revisited

Let $x = \tan \frac{1}{2}\theta \leqslant x_{\mathrm{crit}} = \tan \frac{\pi}{8}$, $\boldsymbol{D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]}$



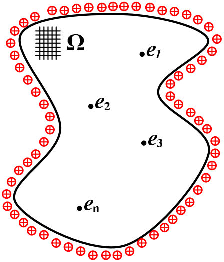
Theorem (revisited): [Kaufman – Onsager(1948), Yang(1952)]

$\lim_{n\to\infty} D_n(x) = (1-\tan^4\theta)^{\frac{1}{4}} \sim \mathrm{cst} \cdot (x_{\mathrm{crit}} - x)^{\frac{1}{4}}$ for $x < x_{\mathrm{crit}}$

[Wu(1966)] $D_n(x_{\mathrm{crit}}) = \left(\frac{2}{\pi}\right)^n \prod_{s=1}^{n-1}\left(1-\frac{1}{4s^2}\right)^{s-n} \sim \mathrm{cst} \cdot (2n)^{-\frac{1}{4}}$

• **Remark:** similar computations for the **magnetization** (single spin expectation) in the half-plane and for the "layered" model.

Scaling limits via Riemann-type b.v.p.'s: ε (energy density)



- Three local primary fields:
  1, **σ (spin)**, **ε (energy density)**;
  Scaling exponents: **0**, $\mathbf{\frac{1}{8}}$, **1**.

- **Theorem:** [Hongler–Smirnov, Hongler'10]
  If $\Omega_\delta \to \Omega$ and $e_k \to z_k$ as $\delta \to 0$, then

$$\delta^{-n} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\varepsilon_{e_1} \ldots \varepsilon_{e_n}] \underset{\delta \to 0}{\to} \mathcal{C}_\varepsilon^n \cdot \langle \varepsilon_{z_1} \ldots \varepsilon_{z_n} \rangle^{+}_{\Omega}$$

where $\mathcal{C}_\varepsilon$ is a lattice-dependent constant,

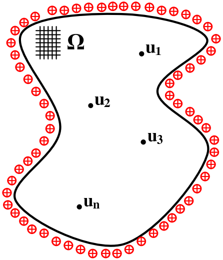
$$\langle \varepsilon_{z_1} \ldots \varepsilon_{z_n} \rangle^{+}_{\Omega} = \langle \varepsilon_{\varphi(z_1)} \ldots \varepsilon_{\varphi(z_n)} \rangle^{+}_{\Omega'} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|$$

for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\langle \varepsilon_{z_1} \ldots \varepsilon_{z_n} \rangle^{+}_{\mathbb{H}} = i^n \cdot \mathrm{Pf}\left[(z_s - z_m)^{-1}\right]_{s,m=1}^{2n}, \quad z_s = \overline{z}_{2n+1-s}\,.$$

- **Ingredients:** convergence of **basic fermionic observables** (via Riemann-type b.v.p.) and (built-in) **Pfaffian formalism**

Scaling limits via Riemann-type b.v.p.'s: σ (spin)



- Three local primary fields:
  1, **σ (spin)**, **ε (energy density)**;
  Scaling exponents: $\mathbf{0}$, $\mathbf{\frac{1}{8}}$, $\mathbf{1}$.
- **Theorem:** [Ch.–Hongler–Izyurov'12]
  If $\Omega_\delta \to \Omega$ as $\delta \to 0$, then

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}] \underset{\delta\to 0}{\rightarrow} \mathcal{C}_\sigma^n \cdot \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^{+}_{\Omega}$$

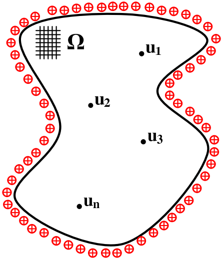
where $\mathcal{C}_\sigma$ is a lattice-dependent constant,

$$\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^{+}_{\Omega} \;=\; \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle^{+}_{\Omega'} \cdot \textstyle\prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}}$$

for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\Big[\langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle^{+}_{\mathbb{H}} \Big]^2 = \prod_{1\leqslant s\leqslant n} (2\,\mathrm{Im}\; u_s)^{-\frac{1}{4}} \times \sum_{\beta\in\{\pm 1\}^n} \prod_{s<m} \left| \frac{u_s - u_m}{u_s - \overline{u}_m} \right|^{\frac{\beta_s\beta_m}{2}}$$

Scaling limits via Riemann-type b.v.p.'s: σ (spin)



- Three local primary fields:
  1, $\boldsymbol{\sigma}$ **(spin)**, ε **(energy density)**;
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where $\mathcal{C}_\sigma$ is a lattice-dependent constant,

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for any conformal mapping $\varphi : \Omega \to \Omega'$.

- Another approach: "exact bosonization" [J. Dubédat'11],
  see also the works of C. Boutillier & B. de Tilière('08 – ...)

Scaling limits via Riemann-type b.v.p.'s: σ (spin)

- Three local primary fields:
  1, $\boldsymbol{\sigma}$ **(spin)**, $\boldsymbol{\varepsilon}$ **(energy density)**;
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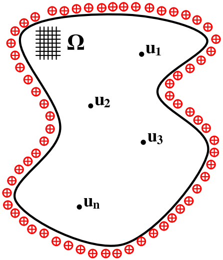


- **Theorem:** [Ch.–Hongler–Izyurov'12]
  If $\Omega_\delta \to \Omega$ as $\delta \to 0$, then

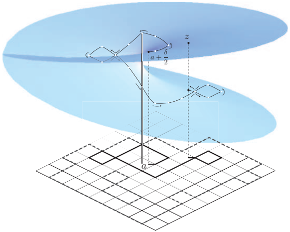
$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}] \underset{\delta \to 0}{\to} \mathcal{C}^n_\sigma \cdot \langle \boldsymbol{\sigma}_{u_1} \ldots \boldsymbol{\sigma}_{u_n} \rangle^{+}_{\boldsymbol{\Omega}}$$

- **General strategy:** ◦ spatial derivatives in discrete: encode them via holomorphic spinors F^δ solving discrete Riemann-type b.v.p.'s
  ◦ discrete ⇝ continuum: prove convergence of F^δ to solutions of similar continuous b.v.p.'s [non-trivial technicalities];
  ◦ continuum ⇝ discrete: find the limit of spatial derivatives using the convergence $F^\delta \to f$ [via coefficients at singularities];
  ◦ spatial derivatives ⇝ correlations: recover the multiplicative normalization [technicalities: "decoupling" estimates in discrete].

Scaling limits via Riemann-type b.v.p.'s: σ (spin)

Example: to handle $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_u]$, one should consider the following b.v.p.:

- $\circ$ $g(z^\sharp) \equiv -g(z^\flat)$, branches over u;
- $\circ$ $\mathrm{Im}\big[\, g(\zeta)\sqrt{\tau(\zeta)}\,\big] = 0$ for $\zeta \in \partial\Omega$;
- $\circ$ $g(z) = \frac{(2i)^{-1/2}}{\sqrt{z-u}} + \ldots$



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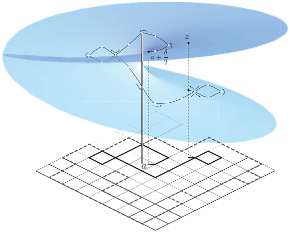

- $g(z^\sharp) \equiv -g(z^\flat)$, branches over u;
- $\mathrm{Im}\big[\, g(\zeta)\sqrt{\tau(\zeta)}\,\big] = 0$ for $\zeta \in \partial\Omega$;
- $g(z) = \frac{(2i)^{-1/2}}{\sqrt{z-u}}[1+2\mathcal{A}_\Omega(u)(z-u)+\ldots]$

Claim: If Ω_δ converges to Ω as $\delta \to 0$, then

- $(2\delta)^{-1}\log\left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2\delta}] \,/\, \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}]\right] \to \mathrm{Re}[\,\mathcal{A}_\Omega(u)\,]\,;$
- $(2\delta)^{-1}\log\left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2i\delta}] \,/\, \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}]\right] \to -\,\mathrm{Im}\,[\,\mathcal{A}_\Omega(u)\,]\,.$

Scaling limits via Riemann-type b.v.p.'s: σ (spin)

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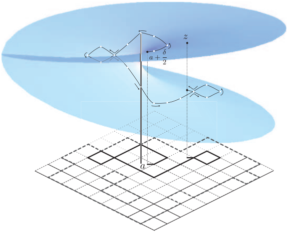

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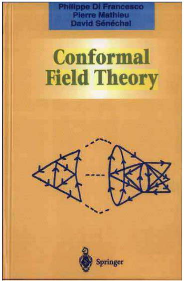
- $(2\delta)^{-1}\log\left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta+2i\delta}]\,/\,\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_\delta}]\right] \to -\,\mathrm{Im}\,[\,\mathcal{A}_\Omega(u)\,]\,.$

Conformal covariance $\frac{1}{8}$: for any conformal map $\phi : \Omega \to \Omega'$,

- $f_{[\Omega,a]}(w) = f_{[\Omega',\phi(a)]}(\phi(w)) \cdot (\phi'(w))^{1/2}\,;$
- $\mathcal{A}_\Omega(z) = \mathcal{A}_{\Omega'}(\phi(z)) \cdot \phi'(z) + \frac{1}{8} \cdot \phi''(z)/\phi'(z)\,.$

Scaling limits via Riemann-type b.v.p.'s: more fields

[Ch.–Hongler–Izyurov '17 (in progress...)]



• Convergence of **mixed correlations: spins (σ), disorders (μ), fermions (ψ), energy densities (ε)** (in multiply connected domains Ω, with mixed fixed/free boundary conditions $\mathfrak{b}$) to conformally covariant limits, which can be defined via solutions to appropriate Riemann-type boundary value problems in Ω.

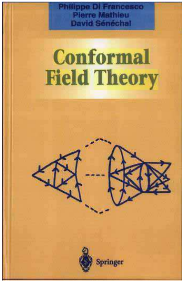
• Standard **CFT fusion rules**

$$\sigma\mu \rightsquigarrow \overline{\eta}\psi + \eta\overline{\psi}, \qquad \psi\sigma \rightsquigarrow \mu, \qquad \psi\mu \rightsquigarrow \sigma,$$
$$i\psi\overline{\psi} \rightsquigarrow \varepsilon, \qquad \sigma\sigma \rightsquigarrow 1 + \varepsilon, \qquad \mu\mu \rightsquigarrow 1 - \varepsilon$$

can be deduced directly from the analysis of these b.v.p.'s

Scaling limits via Riemann-type b.v.p.'s: more fields

[Ch.–Hongler–Izyurov '17 (in progress...)]



• Convergence of **mixed correlations: spins (σ), disorders (μ), fermions (ψ), energy densities (ε)** (in multiply connected domains Ω, with mixed fixed/free boundary conditions $\mathfrak{b}$) to conformally covariant limits, which can be defined via solutions to appropriate Riemann-type boundary value problems in Ω.

• Standard **CFT fusion rules**, e.g. $\sigma\sigma \rightsquigarrow 1 + \varepsilon$:

$$\langle \sigma_{u'} \sigma_u \ldots \rangle_\Omega^{\mathfrak{b}} = |u'-u|^{-\frac{1}{4}} \left[\langle \ldots \rangle_\Omega^{\mathfrak{b}} + \tfrac{1}{2} |u'-u| \langle \varepsilon_u \ldots \rangle_\Omega^{\mathfrak{b}} + \ldots \right],$$

can be deduced directly from the analysis of these b.v.p.'s

• **More CFT:** stress-energy tensor [Ch. – Glazman – Smirnov'16]; Virasoro algebra on local fields [Honlger–Kytölä–Viklund('13–17)]

Geometric viewpoint: conformal loop ensembles (CLEs)

Question: What could be a good candidate for the *scaling limit of loops* surrounding clusters (e.g., with **"+" b.c.**)?

Intuition: Distribution of loops should
(a) be conformally invariant
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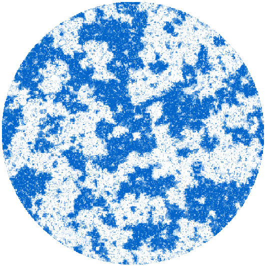
given the loops intersecting $D_2 \setminus D_1$, the remaining ones form an independent CLE in each component of the complement.



critical Ising sample with **free b.c.**, © C. Hongler

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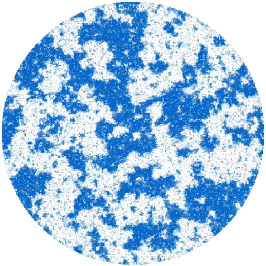

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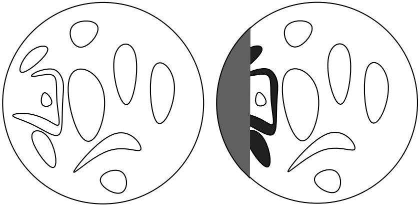

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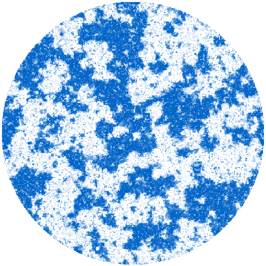

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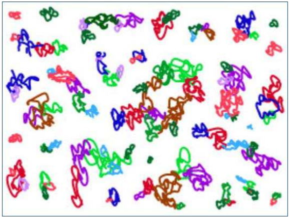

Loop-soup construction:

• sample a (countable) set of Brownian loops using some natural conformally-friendly Poisson process of intensity c.

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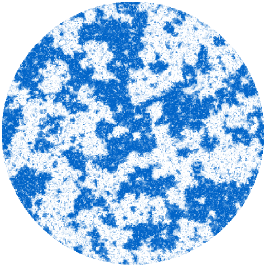

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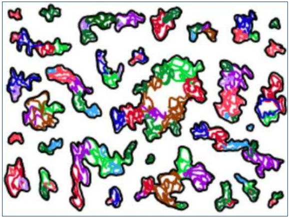

Loop-soup construction:

- sample a (countable) set of Brownian loops using some natural conformally-friendly Poisson process of intensity c.
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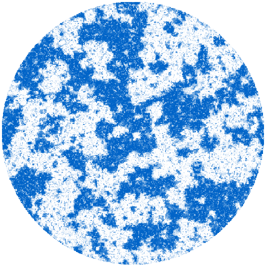

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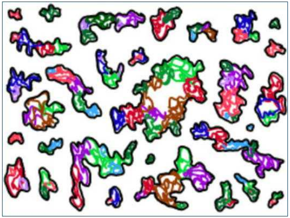

Thm [Sheffield–Werner'10]: provided that loops do not touch each other, (a) and (b) imply that CLE has the law of loop-soup boundaries for some intensity $c \in (0, 1]$.

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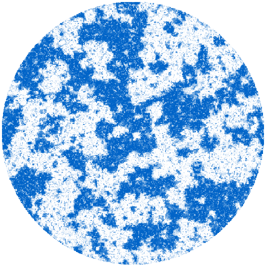

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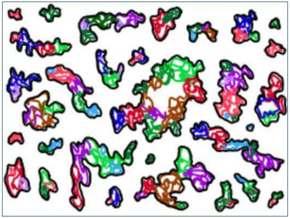
Theorem [Benoist – Hongler'16]:
The limit of critical spin-Ising clusters is a (nested) CLE corresponding to $c = \frac{1}{2}$.

• *The intensity in the loop-soup construction coincide with the central charge in the CFT formalism for correlations.*



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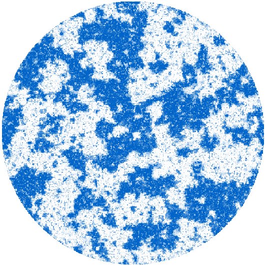

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[among others]

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Stanislav Smirnov

Stéphane Benoist

Antti Kemppainen

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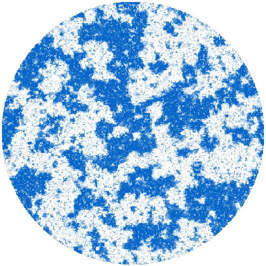
- This is the tip of the iceberg, which is built upon a work of many people. Preliminary results ['06 – '16] include:



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◦ **Convergence of individual curves** (via *martingale observables*) for both spin- and FK-representations of the model [Smirnov'06, Ch. – Smirnov, Hongler – Kytölä / Izyurov, Kemppainen – Smirnov]

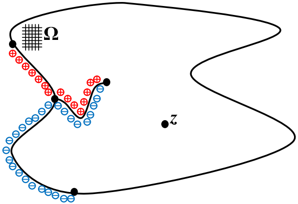
◦ **Uniform RSW-type bounds** [Ch. – Duminil-Copin – Hongler] based on *discrete complex analysis estimates* in rough domains.

Convergence of correlations $\rightsquigarrow$ convergence of interfaces

[see Ch. – Duminil-Copin – Hongler – Kemppainen – Smirnov '13]

- "Martingale observables": choose a function $M_{\Omega_\delta}(z)$, $z \in \Omega_\delta$, such that $M_{\Omega_\delta \setminus \gamma_\delta[0,n]}(z)$ is a martingale wrt the filtration $\mathcal{F}_n := \sigma(\gamma_\delta[0,n])$.

Example: $\mathbb{E}_{\Omega_\delta}[\sigma_z]$ for $+/-/\text{free}$ b. c.



- Convergence of observables: prove uniform (wrt Ω_δ) convergence of the (re-scaled) martingales $M_{\Omega_\delta}(z)$to $M_\Omega(z)$ as $\delta \to 0$.

Remark: technically, the martingale above is (by far) not an optimal choice: fermionic correlations are much easier to handle [Smirnov '06; Ch. – Smirnov '09; Hongler – Kytölä '11; Izyurov '14]

Convergence of correlations ⇝ convergence of interfaces

[see Ch. – Duminil-Copin – Hongler – Kemppainen – Smirnov '13]

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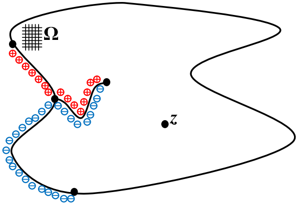


- RSW-type crossing estimates ⇒ tightness of the family $(\gamma_\delta)_{\delta\to 0}$:

  [Aizenmann – Burchard (1999), Kemppainen – Smirnov '12];

◦ Crossings in rectangles: [Duminil-Copin – Hongler –Nolin '09];
◦ Rough domains: [Ch. '12 ⇝ Ch. – Duminil-Copin – Hongler '13]

- Identification of subsequential limits: for each $\gamma = \lim_{\delta_k \to 0} \gamma_{\delta_k}$, the quantities $M_{\Omega\setminus\gamma[0,t]}(z)$ are martingales wrt $\mathcal{F}_t := \sigma(\gamma[0,t])$.

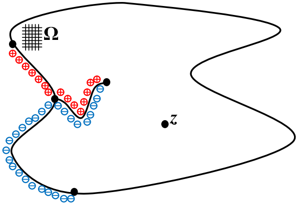
- **conformal covariance of M_Ω ⇒ conformal invariance of γ**

Convergence of correlations ⇝ convergence of interfaces

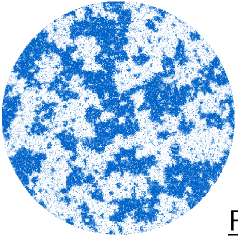
[see Ch. – Duminil-Copin – Hongler – Kemppainen – Smirnov '13]

- "Martingale observables"
- Convergence of observables
- **Uniform RSW-type estimates**

⇝ control of "pinning points" arising along the exploration



Convergence and conformal invariance of the loop ensemble



- Iterative "exploration algorithm"

[Benoist – Hongler '16], switching between spin- and FK(random-cluster)-representations of the model, see also [Benoist – Duminil-Copin – Hongler '14].

Related work: [Kempainnen – Smirnov '15–'16]

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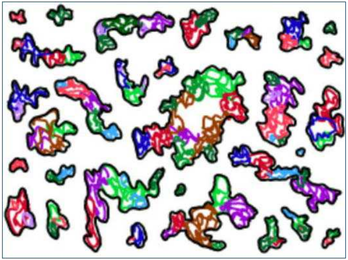
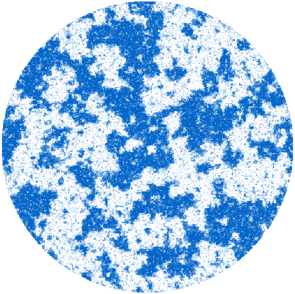
• Can one find **(semi-)discrete precursors of loop-soups** just in some combinatorial representation of the critical Ising model?





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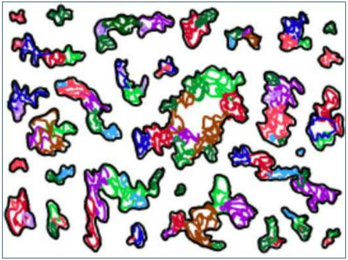

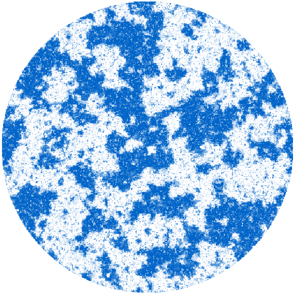

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• **Irregular graphs/interactions,**
Ising model on planar maps etc: (infinitely) many questions...

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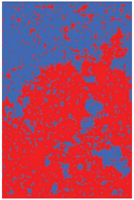

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• **Renormalization**

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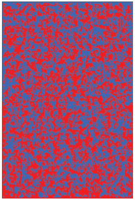

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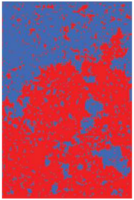

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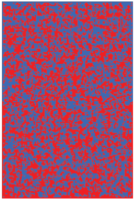

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THANK YOU!