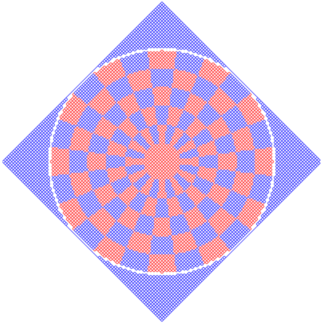
Bipartite Dimer Model:

Gaussian Free Field

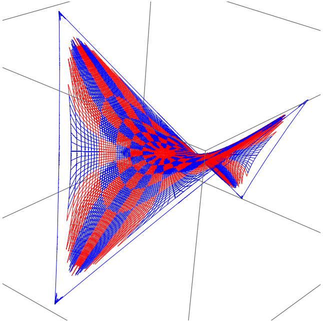
and Lorentz-minimal surfaces



Dmitry Chelkak (ENS)

[recent/in progress joint works w/ Benoît Laslier, Sanjay Ramassamy, Marianna Russkikh]

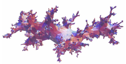
SCGP, March 9, 2020



Outline of the talk:

▷ *Running illustration:* Aztec diamonds (w/ Ramassamy, `arXiv:2002.07540`).

▷ Intro: Thurston's height functions, conv. to GFF in a *non-trivial metric.*

▷ Long[!]-term motivation: 

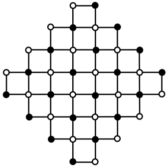
▷ *T-embeddings:* basic concepts and *a priori regularity estimates* (w/ Laslier and Russkikh, `arXiv:2001.11871`).

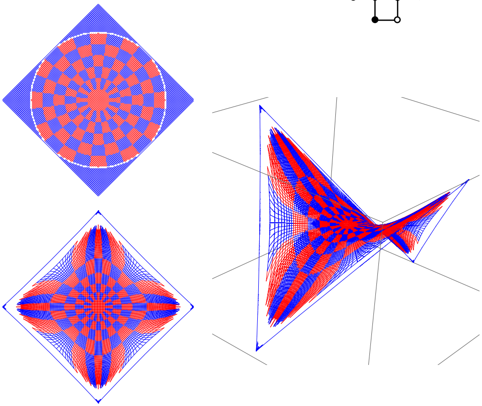
▷ *Perfect t-embeddings* and Lorentz-minimal surfaces. *Main theorem* (w/ Laslier and Russkikh, `arXiv:20**.**`).

▷ Open questions/perspectives.

Illustration:

(homogeneous) Aztec diamonds $A_n \subset n^{-1}\mathbb{Z}^2$





Theorem: [Ch. – Laslier – Russkikh]
[`arXiv:2001.11871 + 20**.**`]

Let $\mathcal{G}^\delta$, $\delta \to 0$, be finite weighted bipartite planar graphs. Assume that

- $\mathcal{T}^\delta$ are *perfect t-embeddings* of $(\mathcal{G}^\delta)^*$ [satisfying assumption $\textsc{Exp-Fat}(\delta)$];
- as $\delta \to 0$, the images of $\mathcal{T}^\delta$ converge to a domain $\mathrm{D} \subset \mathbb{C}$, tangential to $\mathbb{D}$;
- *origami maps* $(\mathcal{T}^\delta, \mathcal{O}^\delta)$ converge to a *Lorentz-minimal surface* $\mathrm{S_D} \subset \mathrm{D} \times \mathbb{R}$.

Then, *height functions fluctuations* in the dimer models on $\mathcal{T}^\delta$ converge to the *standard Gaussian Free Field* in the *intrinsic metric of* $\mathrm{S_D} \subset \mathbb{R}^{2+1} \subset \mathbb{R}^{2+2}$.

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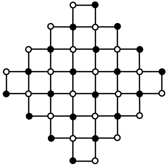
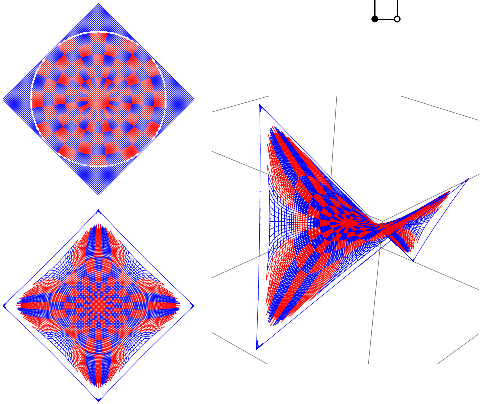


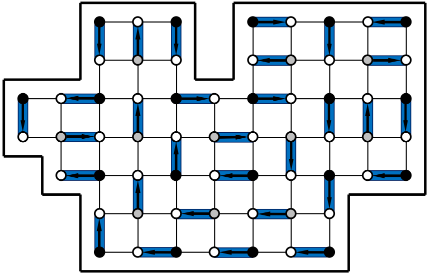


Bipartite dimer model: basics

- $(\mathcal{G}, \nu_{bw})$ – finite weighted bipartite planar graph (w/ marked outer face);
- Dimer configuration = perfect matching $\mathcal{D} \subset E(\mathcal{G})$: subset of edges such that each vertex is covered exactly once;
- *Probability* $\mathbb{P}(\mathcal{D}) \propto \prod_{e\in\mathcal{D}} \nu_e$.

(Very) particular example:

[Temperleyan domains $\mathcal{G}_{\mathrm{T}} \subset \mathbb{Z}^2$]



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- *In Temperleyan domains,* random walks and discrete harmonic functions with 'nice' boundary conditions naturally appear. This is a *very special case.*

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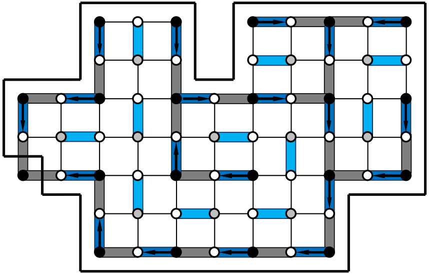
[Temperleyan domains $\mathcal{G}_{\mathrm{T}} \subset \mathbb{Z}^2$]



Temperley bijection: dimers on $\mathcal{G}_{\mathrm{T}}$ $\leftrightarrow$ *spanning trees* on another graph. This procedure is highly sensitive to the *microscopic structure* of the boundary.

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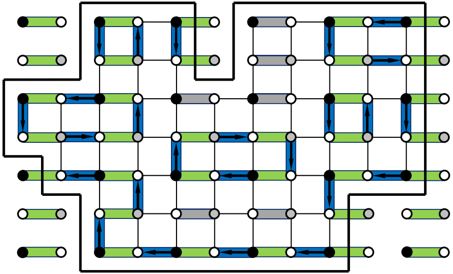

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- **Gaussian Free Field:** $\mathbb{E}[\hbar(z)] = 0$, $\mathbb{E}[\hbar(z)\hbar(w)] = G_\Omega(z,w) = -\Delta_\Omega^{-1}(z,w)$.

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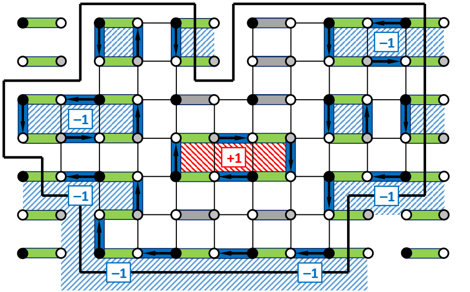
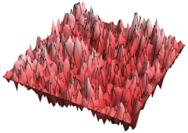


Theorem [Kenyon'00]**:**

$\delta\mathbb{Z}^2 \supset \mathcal{G}^\delta_{\mathbf{T}} \to \Omega \subset \mathbb{C}$

$\Rightarrow \hbar^\delta \to \pi^{-\frac{1}{2}}\mathrm{GFF}(\Omega)$



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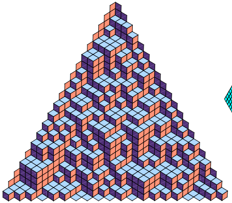
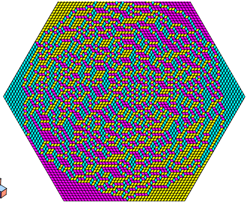
[!!!] Still, the limit of $\hbar^\delta$ as $\delta \to 0$ *heavily depends* on the limit of (deterministic) **boundary profiles of δh^δ.**

Examples (on $\mathbf{Hex}^*$) [(c) Kenyon]:





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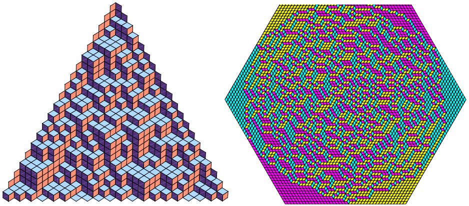

- [Cohn–Kenyon–Propp'00] the random profile δh^δ concentrates near a surface maximizing certain *entropy functional.*
- [Kenyon–Okounkov–Sheffield'06] gen. periodic lattices; **prediction on $\hbar^\delta$:** *GFF in the profile-dependent metric.*
- **Problematic beyond periodic case.**

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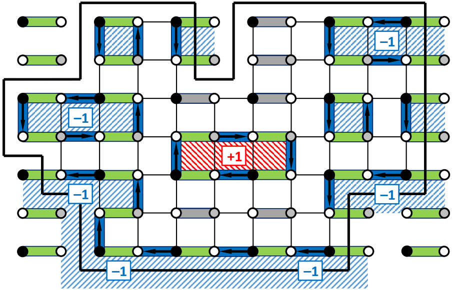
(Very) particular example:

[Temperleyan domains $\mathcal{G}_{\mathrm{T}} \subset \mathbb{Z}^2$]



Remark: If G^δ_{T} are Temperleyan, *then* the boundary profiles of δh^δ are 'flat'.

The *converse* is (by far) *false:* e.g., domains composed of 2×2 blocks are 'flat'.

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- **Double-dimer model:** two independent random configurations $\mathcal{D}$ and $\mathcal{D}'$.

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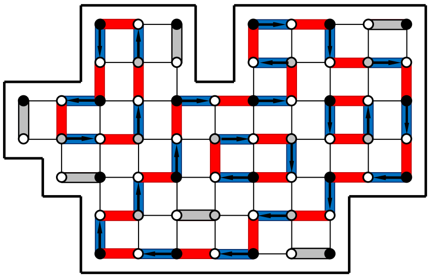
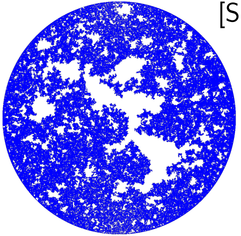


- Height functions $h_{\text{dbl-d}} = \hbar_1 - \hbar_2$.
- Loop ensembles $\mathcal{L}_{\text{dbl-d}} = \mathcal{D} \cup \mathcal{D}'$.
- $\mathcal{L}_{\text{dbl-d}} = \{\mathbb{Z}$ level lines of $h_{\text{dbl-d}}\}$.

Known results: $\delta\mathbb{Z}^2 \supset \mathcal{G}_{\mathrm{T}}^{\delta} \to \Omega \subset \mathbb{C}$

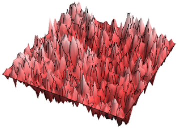
- $\hbar^\delta \to \pi^{-1/2} \cdot \mathrm{GFF}(\Omega)$ [Kenyon'00]
- $\mathcal{L}^{\delta}_{\text{dbl-d}} \xrightarrow{[?]} \mathrm{CLE}_4(\Omega)$ [Kenyon'10] $\rightsquigarrow$ [Dubédat'14] $\rightsquigarrow$ [Basok–Ch.'18] $\rightsquigarrow$ ...

Conformal Loop Ensemble $\mathrm{CLE}_4(\Omega)$:



(c) David Wilson

[Schramm–Sheffield'10]
[Sheffield–Werner'10]
$\sqrt{\pi/2}\cdot\mathbb{Z}$ level lines of $\mathrm{GFF}(\Omega)$



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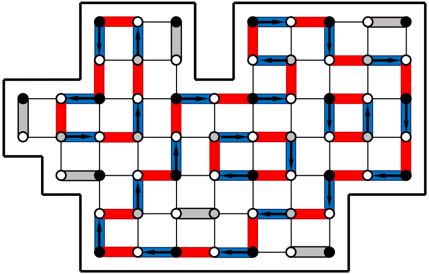

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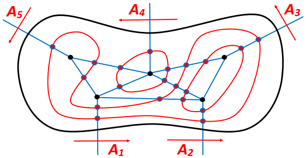
- $\mathrm{GFF}(\Omega) \not\Rightarrow \mathrm{CLE}_4(\Omega)$ ["$\Leftarrow$" is OK]

additional tool: topological observables

$$\tau^\delta(\rho) := \mathbb{E}\big[\prod_{\gamma\in\mathcal{L}^\delta_{\text{dbl-d}}} \tfrac{1}{2}\mathrm{Tr}\rho(\gamma)\big]$$



ρ: locally unipotent connections $\pi_1(\Omega\setminus\{\lambda_k\}) \to \mathrm{SL}_2(\mathbb{C})$

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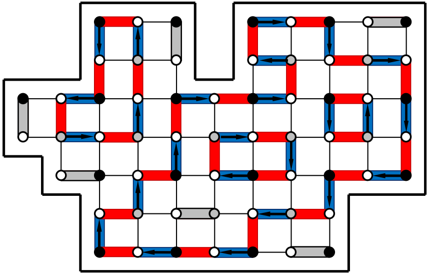

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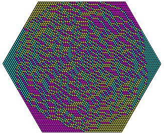
- **Non-flat case:** $\mathbf{GFF}_{\boldsymbol{\mu}}(\boldsymbol{\Omega})$

▷ *Temperleyan-type domains* $\subset \mathrm{Hex}^*$ coming from T-graphs [Kenyon'04]

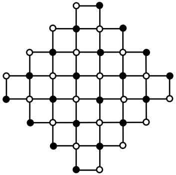
▷ *'polygons' via 'integrable probability'* and (rather hard) asymptotic analysis [Petrov, Bufetov–Gorin, ... '12+]

▷ thorough analysis of *concrete setups (e.g., Aztec diamonds)* w/ interesting behavior [Chhita–Johansson–Young, ... '12+]

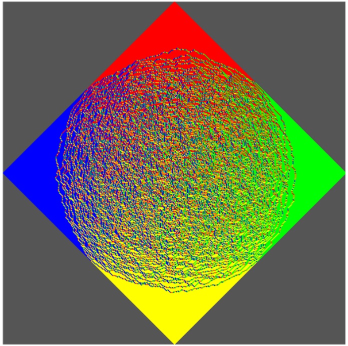


Aztec diamonds $\boldsymbol{A_n \subset n^{-1}\mathbb{Z}^2}$**:**

[Elkies – Kuperberg – Larsen – Propp '92, ...]



[(c) A. & M. Borodin, S. Chhita]



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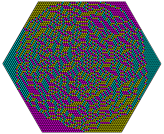

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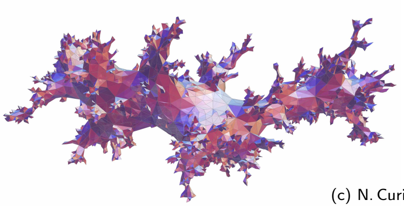
• Long[!]-term goal:

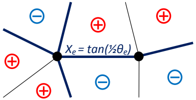
attack random maps carrying the bipartite dimer [or the *critical Ising*] model.

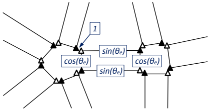


(c) N. Curien

"Bosonization": [Dubédat'11, ...]:

2D n.n. Ising $\hookrightarrow$ bipartite dimers





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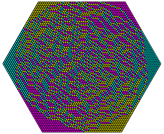

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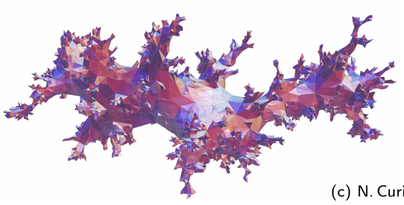

(c) N. Curien

- **Wanted:** *special embeddings* of abstract weighted bipartite planar graphs $+$ *'discrete complex analysis' techniques* on such embeddings

$\rightsquigarrow$ *complex structure in the limit.*

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[arXiv:2001.11871 + 20**.**]

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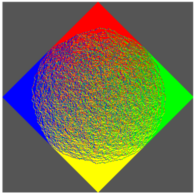
[Ch. – Ramassamy]
[arXiv:2002.07540]



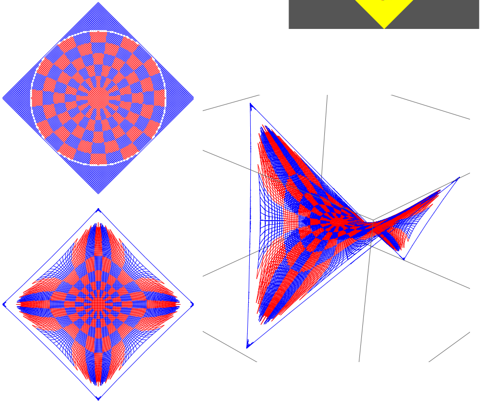


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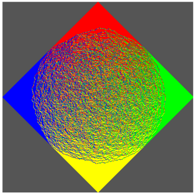
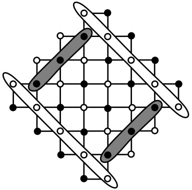
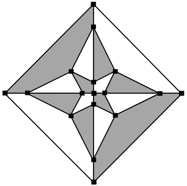
Illustration:

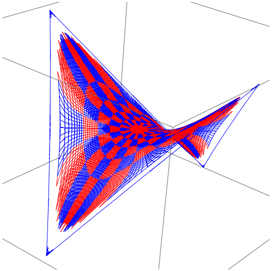
Aztec diamonds

[Ch. – Ramassamy] [`arXiv:2002.07540`]









Embeddings of weighted bipartite planar graphs carrying the dimer model

[and admitting reasonable notions of discrete complex analysis]

Coulomb gauges [Kenyon – Lam – Ramassamy – Russkikh, `arXiv:1810.05616`]

$\Updownarrow$

t-embeddings [Ch. – Laslier – Russkikh, `arXiv:2001.11871`, `arXiv:20**.**`]

Particular cases: harmonic/ *Tutte's embeddings* [via the Temperley bijection]

Ising model *s-embeddings* [`arXiv:1712.04192`, via the bosonization]

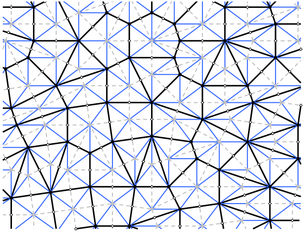
$\cup$

Extremely particular case:

Baxter's critical Z-invariant Ising model on *rhombic lattices/isoradial graphs*

[Ch. – Smirnov, `arXiv:0910.2045`

"*Universality in the 2D Ising model and conformal invariance of fermionic observables*"]



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▷ weights ν_e are *gauge equivalent* to $\chi_{(vv')^*} := |\mathcal{T}(v') - \mathcal{T}(v)|$

(i.e., $\nu_{bw} = g_b \chi_{bw} g_w$ for some $g : B \cup W \to \mathbb{R}_+$) and

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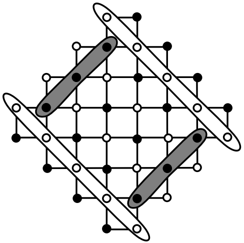

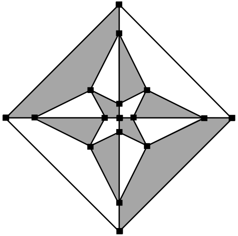

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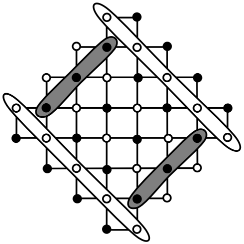
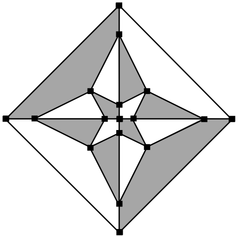
  - ▷ edges adjacent to outer vertices are bisectors.

- **Warning:** for general $(\mathcal{G},\nu)$, the *existence* of perfect t-embeddings is not known though they do exist in particular cases + the count of #(degrees of freedom) matches.





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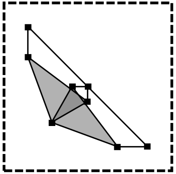
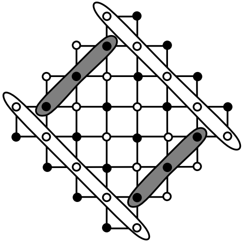
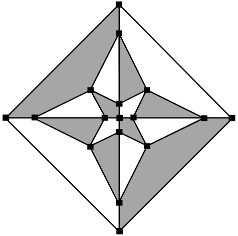
- *origami maps $\mathcal{O}$:* $\mathcal{G}^*\to\mathbb{C}$ ["fold $\mathbb{C}$ along segments of $\mathcal{T}$"]



- *T-graphs* $\mathcal{T}+\alpha^2\mathcal{O}$, $|\alpha|=1$: **[GeoGebra]**

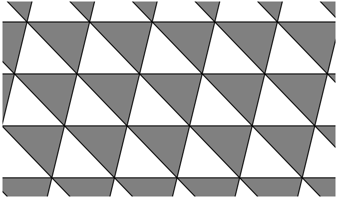


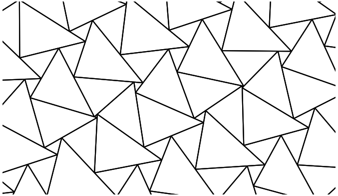


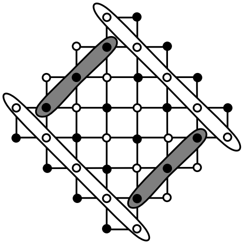
Embeddings of weighted bipartite planar graphs carrying the dimer model

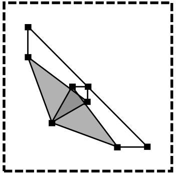
[and admitting reasonable notions of discrete complex analysis]

- **"Regular" case:** triangular grids [Kenyon'04 + Laslier'13]







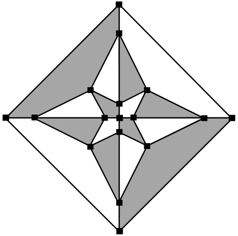


- *T-graphs* $\mathcal{T}+\alpha^2\mathcal{O}$, $|\alpha|=1$: **[GeoGebra]**

- *t-holomorphic functions* $F^\circ : W \to \mathbb{C}$

  $\overline{\alpha}\cdot\{$ gradients of harmonic on $\mathcal{T}+\alpha^2\mathcal{O}\ \}$

  [this notion does not depend on α *]*



Embeddings of weighted bipartite planar graphs carrying the dimer model

[and admitting reasonable notions of discrete complex analysis]

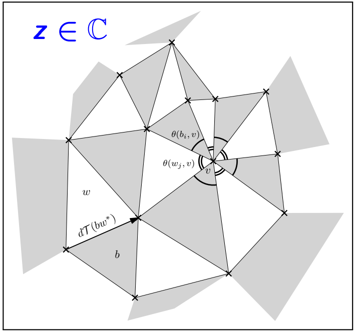
A priori regularity theory [arXiv:2001.11871]

- $\mathcal{T}^\delta$ satisfies $\text{Lip}(\kappa,\delta)$ for $\kappa < 1$ and $\delta > 0$ if

$$|z' - z| \geq \delta \quad \Rightarrow \quad |\mathcal{O}^\delta(z') - \mathcal{O}^\delta(z)| \leq \kappa \cdot |z' - z|.$$

- (triangulations) $\mathcal{T}^\delta$ satisfy $\text{Exp-Fat}(\delta)$ as $\delta \to 0$ if

for each $\beta > 0$, if one removes all '$\exp(-\beta\delta^{-1})$-fat' triangles from $\mathcal{T}^\delta$, then the size of remaining vertex-connected components tends to zero as $\delta \to 0$.



Results: • *Hölder* regularity of *t-holomorphic* functions,

• *Lipschitz* regularity of *harmonic* functions on $\mathcal{T}^\delta + \alpha^2\mathcal{O}^\delta$.

• *What can be said on subsequential limits?*

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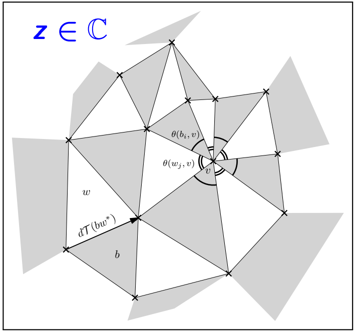
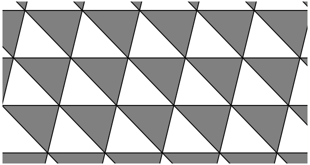
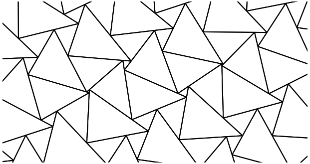


• Assume that $\mathcal{O}^\delta(z) \to \vartheta(z)$, $\delta \to 0$. Then, limits of harmonic functions on $\mathcal{T}^\delta + \alpha^2\mathcal{O}^\delta$ are martingales wrt to a *certain diffusion* whose coefficients *depend on* ϑ, α.





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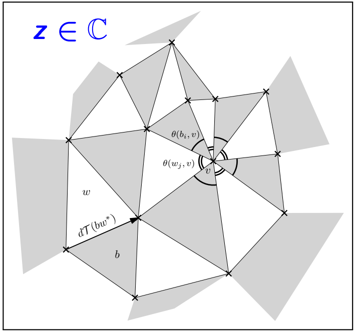

- Assume that $\mathcal{O}^\delta(z) \to \vartheta(z),\ z \in \mathrm{D},\ \delta \to 0$ and that
- $\{(z, \vartheta(z))\}_{z \in \mathrm{D}} \subset \mathbb{R}^{2+2}$ is a *Lorentz-minimal* surface.
- Let a *parametrization* ζ be *conformal* $z_\zeta \bar{z}_\zeta = \vartheta_\zeta \overline{\vartheta}_\zeta$ and *harmonic* $z_{\zeta\bar{\zeta}} = \vartheta_{\zeta\bar{\zeta}} = 0$.
- Then, subsequential limits of harmonic functions on all T-graphs $\mathcal{T}^\delta + \alpha^2\mathcal{O}^\delta$, $|\alpha| = 1$, and, moreover, all limits of dimer height functions *correlations are harmonic in* ζ.

Theorem: [Ch. – Laslier – Russkikh] [arXiv:2001.11871 + 20**.**]

Let $\mathcal{G}^\delta$, $\delta \to 0$, be finite weighted bipartite planar graphs. Assume that

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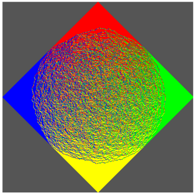
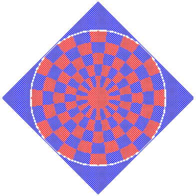
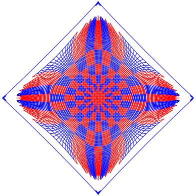
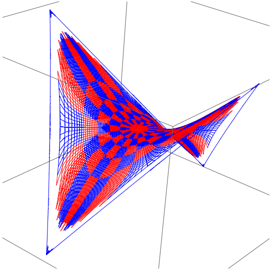
Illustration: Aztec diamonds

[Ch.– Ramassamy] [arXiv:2002.07540]









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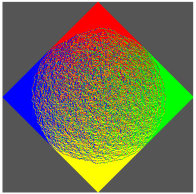

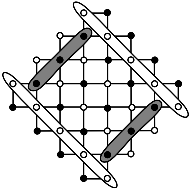

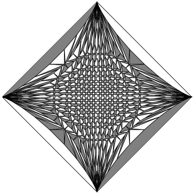

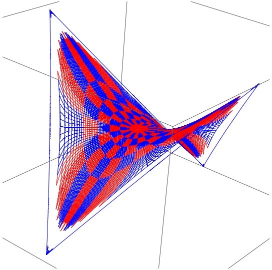

Open questions, perspectives *[<u>general</u> $(\mathcal{G}, \nu)$]*

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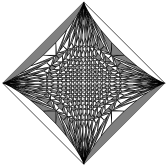

▷ $\deg f_{\text{out}} = 4$: OK [KLRR]

▷ #(degrees of freedom): OK

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  Another example: annulus-type graphs

  $\rightsquigarrow$ *Lorentz-minimal cusp* $(z, \operatorname{arcsinh}|z|)$.

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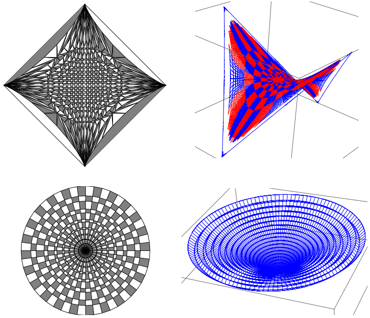

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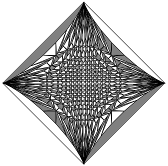

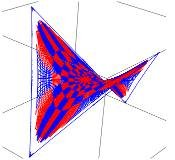

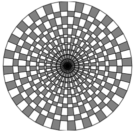

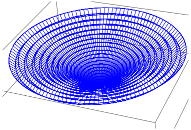

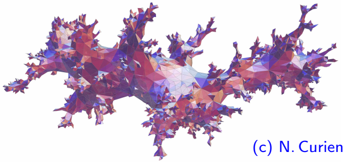

(c) N. Curien

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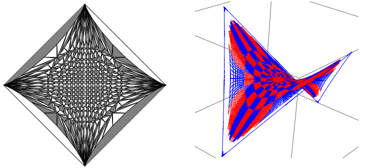

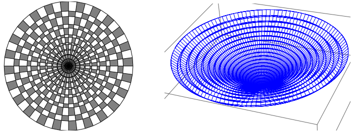

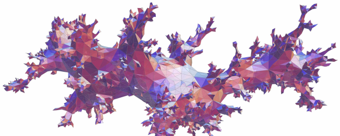

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Thank you!