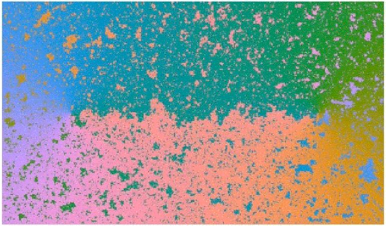
2D Ising model at criticality: correlations, interfaces, estimates

Dmitry Chelkak (ÉNS, Paris & PDMI, St.Petersburg)



[Sample of a critical 2D Ising configuration with two disorders, © Clément Hongler (EPFL)]

SPA2017, Moscow, 27.07.2017

2D Ising model at criticality: correlations, interfaces, estimates

[this is a long story, many people involved:


Stéphane Benoist


Hugo Duminil-Copin


Antti Kemppainen


Stanislav Smirnov


Clément Hongler


Konstantin Izyurov


Kalle Kytölä

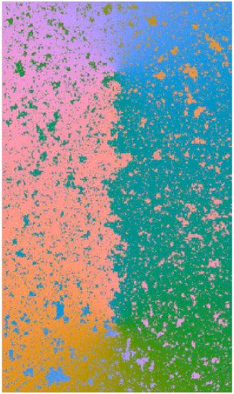

Dmitry Chelkak

David Cimasoni, Alexander Glazman, Adrien Kassel, Pierre Nolin, Frederik Viklund, ...]

SPA2017, Moscow, 27.07.2017

Nearest-neighbor critical 2D Ising model: correlations, interfaces, estimates

- Introduction: phase transition, diagonal correlations, conformal invariance
- Combinatorics: dimers, Kac-Ward, fermionic observables, double-covers
- Scaling limits at criticality via Riemann-type boundary value problems
- More fields: $\sigma, \mu, \psi, \varepsilon \rightsquigarrow$ glimpse of CFT
- Geometry: convergence of curves, convergence to CLE [Benoist–Hongler'16]
- Regularity of interfaces: a priori estimates via surgery of discrete domains
- Open questions



[Two disorders: sample of a critical 2D Ising configuration

Nearest-neighbor Ising (or Lenz-Ising) model in 2D

Definition: *Lenz-Ising model* on a planar graph G^* (dual to G) is a random assignment of $+/-$ spins to vertices of G^* (faces of G)

Q: I heard this is called a (site) percolation?
A: .. according to the following probabilities:

$$\mathbb{P}\left[\text{conf. } \sigma \in \{\pm 1\}^{V(G^*)}\right] \propto \exp\left[\beta \sum_{e=\langle uv\rangle} J_{uv}\sigma_u\sigma_v\right] \propto \prod_{e=\langle uv\rangle:\sigma_u\neq\sigma_v} x_{uv},$$

where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv\rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.

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Remark: w/o an external magnetic field
this is a **"free fermion"** model.

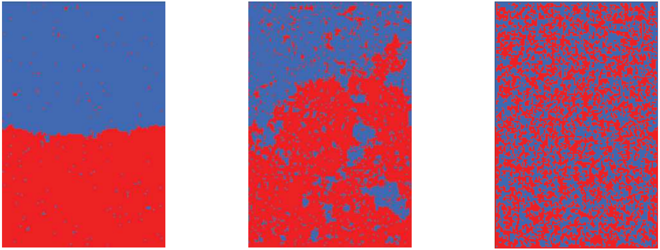
$$\mathbb{P}\left[\text{conf. } \sigma \in \{\pm 1\}^{V(G^*)}\right] \;\propto\; \exp\left[\beta \sum\nolimits_{e=\langle uv\rangle} J_{uv}\sigma_u\sigma_v\right] \;\propto\; \prod\nolimits_{e=\langle uv\rangle:\sigma_u\neq\sigma_v} x_{uv}\,,$$

where $J_{uv} > 0$ are interaction constants assigned to edges $\langle uv\rangle$, $\beta = 1/kT$ is the inverse temperature, and $x_{uv} = \exp[-2\beta J_{uv}]$.

- It is also convenient to use the parametrization $x_{uv} = \tan(\frac{1}{2}\theta_{uv})$.
- Working with subgraphs of *regular lattices*, one can consider the *homogeneous model* in which all x_{uv} are equal to each other.

Lenz-Ising model: phase transition (e.g., on $\mathbb{Z}^2$)

E.g., Dobrushin boundary conditions: $+1$ on (ab) and -1 on (ba):



$x < x_{\mathrm{crit}}$ $\qquad$ $x = x_{\mathrm{crit}}$ $\qquad$ $x > x_{\mathrm{crit}}$

- Ising (1925): no phase transition in 1D $\rightsquigarrow$ doubts about 2+D;
- Peierls (1936): existence of the phase transition in 2D;
- Kramers-Wannier (1941): $x_{\text{self-dual}} = \sqrt{2} - 1 = \tan(\frac{1}{2} \cdot \frac{\pi}{4})$;
- Onsager (1944): sharp phase transition at $x_{\text{crit}} = \sqrt{2} - 1$.

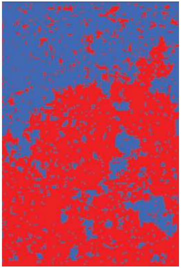
At criticality (e.g., on $\mathbb{Z}^2$):

- **scaling exponent $\frac{1}{8}$** for the magnetization [Kaufman–Onsager(1948), Yang(1952)]

$\lim_{n\to\infty} \mathbb{E}[\sigma_0\sigma_{2n}] \sim \mathrm{cst}\cdot|x-x_{\mathrm{crit}}|^{\frac{1}{4}},\ x\uparrow x_{\mathrm{crit}}$

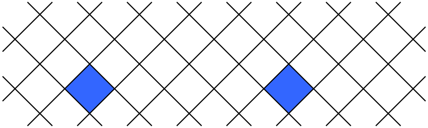
[Wu (1966), correlations at $x = x_{\mathrm{crit}}$]

$$\begin{aligned}\mathbb{E}[\sigma_0\sigma_{2n}] &= \left(\tfrac{2}{\pi}\right)^n \prod_{s=1}^{n-1}\left(1-\tfrac{1}{4s^2}\right)^{s-n} \\ &\sim \mathrm{cst}\cdot(2n)^{-\frac{1}{4}},\ n\to\infty\end{aligned}$$



$x = x_{\mathrm{crit}}$

Remark: "modern" proofs (Fourier transform applied to full-plane observables) take several pages only.



[see arXiv:1605.09035]. Similarly, "explicit" computations can be done in the "layered" case [Ch.–Hongler, still in preparation], i.e. when all interactions are the same in each of the zig-zag columns.

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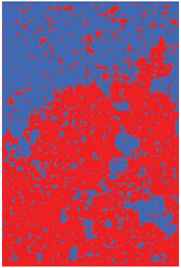

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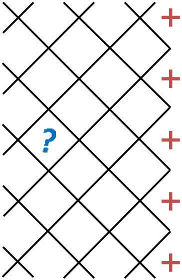

Theorem (layered half-plane): [Ch.–Hongler]

$$\mathbb{E}^{+}_{\mathbf{i}\mathbb{H}_\diamond}[\sigma_{-2n}] = \frac{\det \mathrm{H}_n[t^{1/2}\mu]}{(\det \mathrm{H}_n[\mu]\det \mathrm{H}_n[t\mu])^{1/2}},$$

where $\det \mathrm{H}_n[\mu] := \det\left[\int_0^1 t^{i+j}\mu(dt)\right]_{i,j=0}^{n-1}$ and μ is the spectral measure of the Jacobi matrix $\langle y, Wy\rangle = \sum_{n\geqslant 0}(a_{2n}a_{2n+1}y_n - b_{2n+1}b_{2n+2}y_{n+1})^2$.

[Notation: $a_k = \cos\theta_k$, $b_k = \sin\theta_k$, where $x_k = \tan\frac{1}{2}\theta_k$ is the interaction constant in the k-th zig-zag column]

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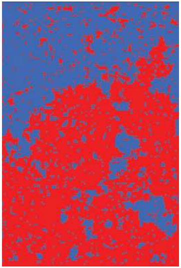

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- Existence of **scaling limits** as $\Omega_\delta \to \Omega$:
  [Ch.–Hongler–Izyurov, arXiv:1202.2838]

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}_{\Omega_\delta}[\sigma_{u_1} \ldots \sigma_{u_n}] \quad \to \quad \langle \sigma_{u_1} \ldots \sigma_{u_n} \rangle_\Omega$$

Conformal covariance: $$= \quad \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle_{\varphi(\Omega)} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}}$$

Remark. Basing on this, one can study the convergence of random fields $(\delta^{-\frac{1}{8}}\sigma_u)_{u\in\Omega}$ to a (non-Gaussian!) limit as $\delta \to 0$ [Camia–Garban–Newman '13, Furlan–Mourrat '16]

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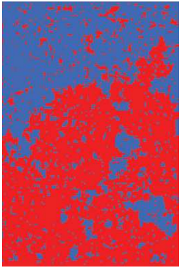

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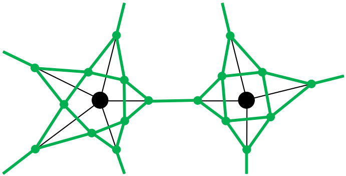
$$= \quad \langle \sigma_{\varphi(u_1)} \ldots \sigma_{\varphi(u_n)} \rangle_{\varphi(\Omega)} \, .$$

- Instead of correlation functions, one can study convergence of **curves** (e.g., domain walls generated by Dobrushin boundary conditions) and **loop ensembles** (either outermost or nested) to **conformally invariant limits: SLE(3)'s and CLE(3).**

2D Ising model as a dimer model [Fisher, Kasteleyn ('60s), ...]

- **Partition function** $\mathcal{Z} = \sum_{\sigma \in \{\pm 1\}^{V(G^*)}} \prod_{e=\langle uv \rangle : \sigma_u \neq \sigma_v} x_{uv}$

- There exist various representations of the 2D Ising model via dimers on an auxiliary graph: e.g. 1-to-$2^{|V(G)|}$ correspondence of $\{\pm 1\}^{V(G^*)}$ with dimers on **this G_F**



Reference: "Revisiting 2D Ising combinatorics" arXiv:1507.08242

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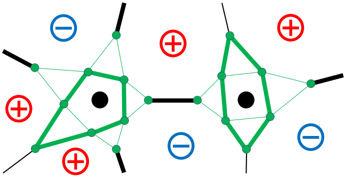

- **Kasteleyn's theory:** $\mathcal{Z} = \mathbf{Pf}[\,\mathbf{K}\,]$ [$\mathbf{K}=-\mathbf{K}^\top$ is a weighted adjacency matrix of G_F]

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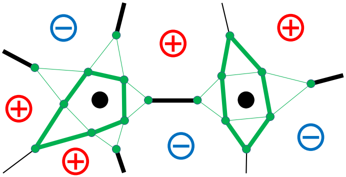

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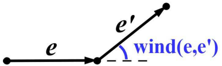
- **Kac–Ward formula (1952–...,1999–...):** $\mathcal{Z}^2 = \det[\mathrm{Id} - \mathrm{T}]$,

$$\mathrm{T}_{e,e'} = \begin{cases} \exp[\tfrac{i}{2}\mathrm{wind}(e,e')]\cdot(x_e x_{e'})^{1/2} \\ 0 \end{cases}$$



[**is equivalent** to the **Kasteleyn theorem for dimers on G_F**]

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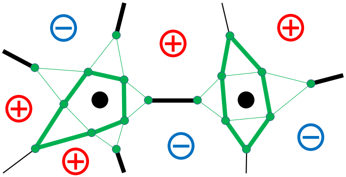

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- **Energy density field:** note that $\mathbb{P}[\sigma_{e^\sharp}\sigma_{e^\flat} = -1] = |\,\mathrm{K}^{-1}_{e,\bar{e}}\,|$.

- **Local relations** for the entries $\mathbf{K}^{-1}_{a,e}$ and $\mathbf{K}^{-1}_{a,c}$ of the inverse Kasteleyn (or the inverse Kac–Ward) matrix:

  (an equivalent form of) the identity $\mathbf{K}\cdot\mathbf{K}^{-1} = \mathbf{Id}$

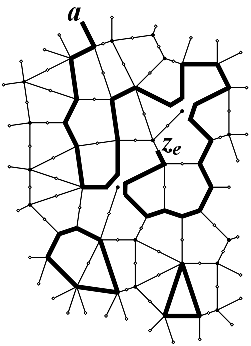
Reference: "Revisiting 2D Ising combinatorics" arXiv:1507.08242

Fermionic observables: combinatorial definition [Smirnov'00s]

For an oriented edge a and a midedge z_e (similarly, for a corner c),

$$F_G(a,z_e) \ := \ \overline{\eta}_a \sum_{\omega\in\mathrm{Conf}_G(a,z_e)} \Big[e^{-\frac{i}{2}\mathrm{wind}(a\rightsquigarrow z_e)} \prod_{\langle uv\rangle\in\omega} x_{uv} \Big]$$

where η_a denotes the (once and forever fixed) square root of the direction of a.



- The factor $e^{-\frac{i}{2}\mathrm{wind}(a\rightsquigarrow z_e)}$ does not depend on the way how ω is split into non-intersecting loops and a path $a \rightsquigarrow z_e$.

- **Via dimers on G_F:** $F_G(a,c) = \overline{\eta}_c \mathrm{K}^{-1}_{c,a}$

$$F_G(a,z_e) = \overline{\eta}_e \mathrm{K}^{-1}_{e,a} + \overline{\eta}_{\overline{e}} \mathrm{K}^{-1}_{\overline{e},a}$$

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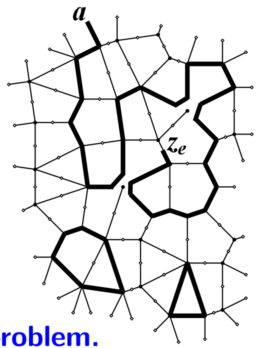


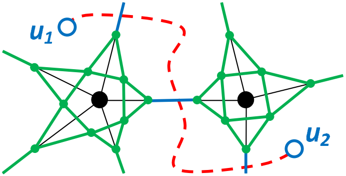
• **Local relations: at criticality,** can be thought of as a special form of **discrete Cauchy–Riemann equations.**

• **Boundary conditions $F(a, z_e) \in \overline{\eta}_{\bar{e}}\mathbb{R}$** ($\bar{e}$ is oriented outwards) uniquely determine F as a solution to an appropriate **discrete Riemann-type boundary value problem.**

⇝ **Scaling limit** of **fermions** [Smirnov'06, Ch.–Smirnov'09] and of **energy densities** [Hongler–Smirnov, Hongler'10]

Derivatives of spin correlations $\leftrightarrow$ fermions on double-covers

- spin configurations on G^*
  $\leftrightsquigarrow$ domain walls on G
  $\leftrightsquigarrow$ dimers on G_F



- **Kasteleyn's theory:** $\mathcal{Z} = \mathrm{Pf}[\mathbf{K}]$

[$\mathbf{K} = -\mathbf{K}^\top$ is a weighted adjacency matrix of G_F]

- **Claim:**

$$\mathbb{E}[\sigma_{u_1} \ldots \sigma_{u_n}] \;=\; \frac{\mathrm{Pf}[\,\mathbf{K}_{[u_1,\ldots,u_n]}\,]}{\mathrm{Pf}[\,\mathbf{K}\,]},$$

where $\mathbf{K}_{[u_1,\ldots,u_n]}$ is obtained from $\mathbf{K}$ by changing the sign of its entries on **slits linking** $u_1, \ldots, u_n$ (and, possibly, u_{out}) pairwise.

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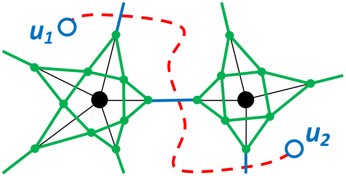

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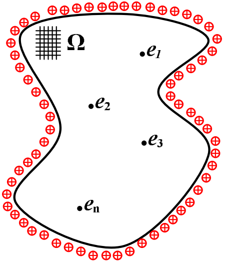
More invariant way: **double-covers branching over $u_1, \dots, u_n$.**

- If one shifts u_1 to a neighboring face $\widetilde{u}_1$, the "spatial derivative"

$$\frac{\mathbb{E}[\sigma_{\widetilde{u}_1}\sigma_{u_2}\dots\sigma_{u_n}]}{\mathbb{E}[\sigma_{u_1}\sigma_{u_2}\dots\sigma_{u_n}]}$$ can be expressed via the entries of $\mathrm{K}^{-1}_{[u_1,\dots,u_n]}$.

Scaling limits via Riemann-type b.v.p.'s [arXiv:1605.09035]



- Three local primary fields:
  1, $\boldsymbol{\sigma}$ **(spin)**, ε **(energy density)**;
  Scaling exponents: **0**, $\mathbf{\frac{1}{8}}$, **1**.

- **Theorem:** [Hongler–Smirnov, Hongler'10]
  If $\Omega_\delta \to \Omega$ and $e_k \to z_k$ as $\delta \to 0$, then

$$\boldsymbol{\delta^{-n}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\varepsilon_{e_1} \ldots \varepsilon_{e_n}] \underset{\delta \to 0}{\to} \mathcal{C}^n_\varepsilon \cdot \langle \boldsymbol{\varepsilon_{z_1} \ldots \varepsilon_{z_n}} \rangle^{+}_{\boldsymbol{\Omega}}$$

where $\mathcal{C}_\varepsilon$ is a lattice-dependent constant,

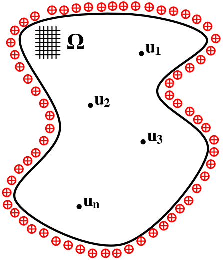
$$\langle \varepsilon_{z_1} \ldots \varepsilon_{z_n} \rangle^{+}_{\Omega} = \langle \varepsilon_{\varphi(z_1)} \ldots \varepsilon_{\varphi(z_n)} \rangle^{+}_{\Omega'} \cdot \textstyle\prod_{s=1}^{n} |\varphi'(u_s)|$$

for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\langle \boldsymbol{\varepsilon_{z_1} \ldots \varepsilon_{z_n}} \rangle^{+}_{\mathbb{H}} = i^n \cdot \mathrm{Pf}\left[(z_s - z_m)^{-1}\right]^{2n}_{s,m=1}, \quad z_s = \overline{z}_{2n+1-s} \,.$$

- **Ingredients:** convergence of **basic fermionic observables** (via Riemann-type b.v.p.) and (built-in) **Pfaffian formalism**

Scaling limits via Riemann-type b.v.p.'s [arXiv:1605.09035]



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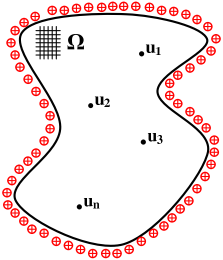
for any conformal mapping $\varphi : \Omega \to \Omega'$, and

$$\Big[\langle \sigma_{u_1} \dots \sigma_{u_n} \rangle^+_{\mathbb{H}} \Big]^2 = \prod_{1\leqslant s \leqslant n} (2\,\mathrm{Im}\ u_s)^{-\frac{1}{4}} \times \sum_{\beta\in\{\pm 1\}^n} \prod_{s<m} \left| \frac{u_s - u_m}{u_s - \overline{u}_m} \right|^{\frac{\beta_s \beta_m}{2}}$$

- Another approach (full plane): "exact bosonization" [J. Dubédat'11]

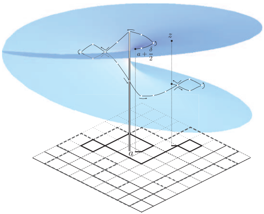
Scaling limits via Riemann-type b.v.p.'s [arXiv:1605.09035]

- Three local primary fields:
  1, $\boldsymbol{\sigma}$ **(spin)**, ε **(energy density)**;
  Scaling exponents: $\mathbf{0}$, $\mathbf{\frac{1}{8}}$, $\mathbf{1}$.



- **Theorem:** [Ch.–Hongler–Izyurov'12]
  If $\Omega_\delta \to \Omega$ as $\delta \to 0$, then

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1} \dots \sigma_{u_n}] \underset{\delta\to 0}{\to} \mathcal{C}_\sigma^n \cdot \langle \boldsymbol{\sigma}_{u_1} \dots \boldsymbol{\sigma}_{u_n} \rangle^{+}_{\Omega}$$

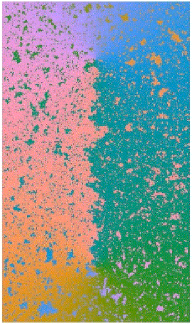


E.g., to handle $\mathbb{E}^{+}_{\Omega_\delta}[\boldsymbol{\sigma}_{\tilde{u}}]/\mathbb{E}^{+}_{\Omega_\delta}[\boldsymbol{\sigma}_{u}]$, one should consider the following b.v.p.:

- ∘ $g(z^\sharp) \equiv -g(z^\flat)$, branches over u;
- ∘ $\operatorname{Im}\big[\, g(\zeta)\sqrt{\tau(\zeta)}\,\big] = 0$ for $\zeta \in \partial\Omega$;
- ∘ $g(z) = \frac{(2i)^{-1/2}}{\sqrt{z-u}}[1+2\mathcal{A}_\Omega(u)(z-u)+\dots]$

- **Conformal covariance:** $\mathcal{A}_\Omega(z) = \mathcal{A}_{\Omega'}(\phi(z)) \cdot \phi'(z) + \frac{1}{8} \cdot \frac{\phi''(z)}{\phi'(z)}$.

σ–μ formalism [Kadanoff–Ceva'71]



[two disorders inserted]

- Given (an even number of) *vertices* $v_1, \ldots, v_m$, consider the Ising model on a double-cover $G^{[v_1,\ldots,v_m]}$ ramified at each of $v_1, \ldots, v_m$ with the *spin-flip symmetry* constrain $\sigma_{u^\sharp} = -\sigma_{u^\flat}$ if $u^\sharp$ and $u^\flat$ lie over the same face of G. Let

$$\langle \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle_G$$
$$:= \mathbb{E}_{G^{[v_1,\ldots,v_m]}}[\sigma_{u_1} \ldots \sigma_{u_n}] \cdot \mathcal{Z}_G^{[v_1,\ldots,v_m]} / \mathcal{Z}_G \, .$$

[by definition, the (formal) correlator $\langle \mu_{v_1} \ldots \mu_{v_m} \sigma_{u_1} \ldots \sigma_{u_n} \rangle_G$ changes the sign when one of u_k goes around of one of v_s]

Reference: "Revisiting 2D Ising combinatorics" arXiv:1507.08242

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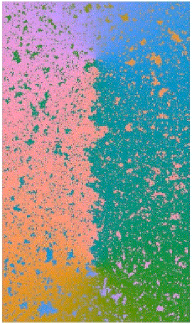


[two disorders inserted]

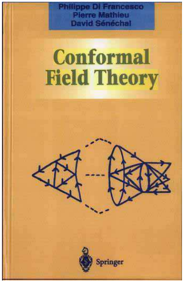
• For a corner c lying in the face $u(c)$ near the vertex $v(c)$, set $\psi_c := \delta^{\frac{1}{2}}(u(c) - v(c))^{-\frac{1}{2}} \mu_{v(c)} \sigma_{u(c)}$. Provided $v(c_p) \neq v(c_q)$,

$\rightsquigarrow$ **the same fermions** $\langle \psi_{c_1} \dots \psi_{c_{2k}} \rangle_G = \mathrm{Pf}[\langle \psi_{c_p} \psi_{c_q} \rangle_G]_{p,q=1}^{2k}$, this also works in presence of other spins and/or disorders.

Reference: "Revisiting 2D Ising combinatorics" arXiv:1507.08242

Scaling limits via Riemann-type b.v.p.'s: more fields

[Ch.–Hongler–Izyurov '17 (to appear soon...)]



• Convergence of **mixed correlations: spins (σ), disorders (μ), fermions (ψ), energy densities (ε)** (in multiply connected domains Ω, with mixed fixed/free boundary conditions $\mathfrak{b}$) to conformally covariant limits, which can be defined via solutions to appropriate Riemann-type boundary value problems in Ω.

• Standard **CFT fusion rules**

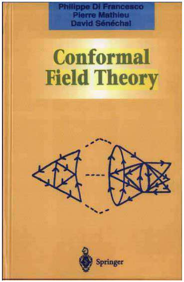
$$\sigma\mu \rightsquigarrow \tfrac{1}{2}(\overline{\eta}\psi + \eta\psi^\star), \qquad \psi\sigma \rightsquigarrow \mu, \qquad \psi\mu \rightsquigarrow \sigma,$$
$$\tfrac{i}{2}\psi\psi^\star \rightsquigarrow \varepsilon, \qquad \sigma\sigma \rightsquigarrow 1 + \tfrac{1}{2}\varepsilon, \qquad \mu\mu \rightsquigarrow 1 - \tfrac{1}{2}\varepsilon$$

can be deduced directly from the analysis of these b.v.p.'s

[cf. the invited session talk by Izyurov (on Monday...)]

Scaling limits via Riemann-type b.v.p.'s: more fields

[Ch.–Hongler–Izyurov '17 (to appear soon...)]



• Convergence of **mixed correlations: spins (σ), disorders (μ), fermions (ψ), energy densities (ε)** (in multiply connected domains Ω, with mixed fixed/free boundary conditions $\mathfrak{b}$) to conformally covariant limits, which can be defined via solutions to appropriate Riemann-type boundary value problems in Ω.

• Standard **CFT fusion rules**, e.g. $\sigma\sigma \rightsquigarrow 1+\varepsilon$:

$$\langle \sigma_{u'}\sigma_u ... \rangle_\Omega^{\mathfrak{b}} = |u'-u|^{-\frac{1}{4}} \left[\langle ... \rangle_\Omega^{\mathfrak{b}} + \tfrac{1}{2}|u'-u| \langle \varepsilon_u ... \rangle_\Omega^{\mathfrak{b}} + \ldots \right],$$

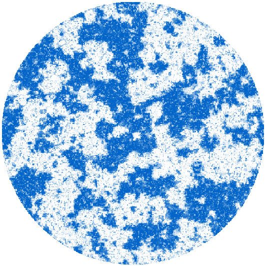
can be deduced directly from the analysis of these b.v.p.'s

• **More CFT:** stress-energy tensor [Ch. – Glazman – Smirnov'16]; Virasoro algebra on local fields [Hongler–Kytölä–Viklund('13–17)]

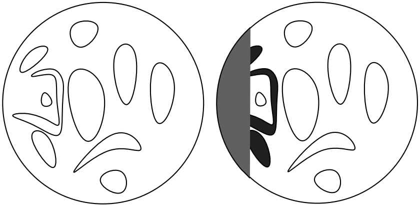
Geometric viewpoint: conformal loop ensembles (CLEs)

Question: What could be a good candidate for the *scaling limit of loops* surrounding clusters (e.g., with **"+" b.c.**)?

Intuition: Distribution of loops should
(a) be conformally invariant
(b) satisfy the domain Markov property:
given the loops intersecting $D_2 \setminus D_1$, the remaining ones form an independent CLE in each component of the complement.



critical Ising sample with **free b.c.**, © C. Hongler



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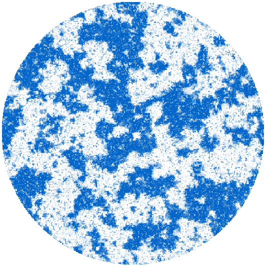

critical Ising sample with
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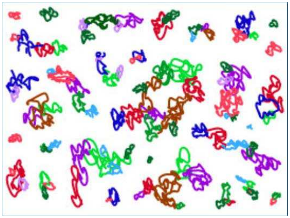

Loop-soup construction:

• sample a (countable) set of Brownian loops using some natural conformally-friendly Poisson process of intensity c.

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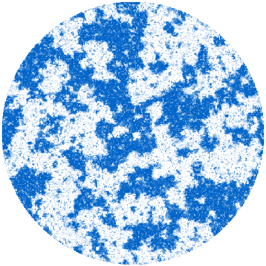

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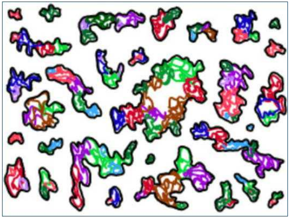

Loop-soup construction:

- sample a (countable) set of Brownian loops using some natural conformally-friendly Poisson process of intensity c.
- fill the outermost clusters

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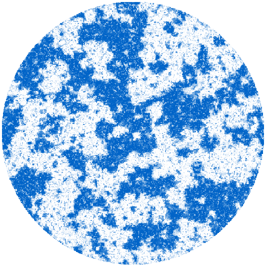

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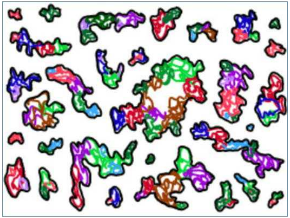

Thm [Sheffield–Werner'10]: provided that loops do not touch each other, (a) and (b) imply that CLE has the law of loop-soup boundaries for some intensity $c \in (0, 1]$.

Geometric viewpoint: conformal loop ensembles (CLEs)

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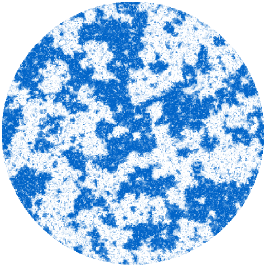
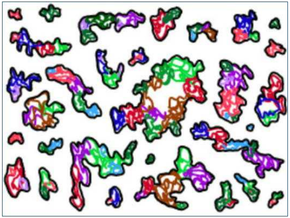


critical Ising sample with
free b.c., © C. Hongler

Theorem [Benoist – Hongler'16]:

The limit of critical spin-Ising clusters is a (nested) CLE corresponding to $c = \frac{1}{2}$.

• *The intensity in the loop-soup construction coincide with the central charge in the CFT formalism for correlations.*



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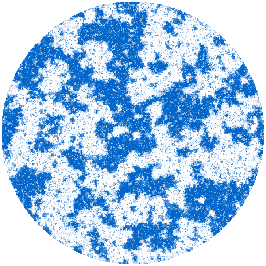
The limit of critical spin-Ising clusters is a (nested) CLE corresponding to $c = \frac{1}{2}$.

- This is the tip of the iceberg, which is built upon a work of many people. Preliminary results ['06 – '16] include:



critical Ising sample with **free b.c.**, © C. Hongler

◦ **Convergence of individual curves** (via *martingale observables*) for both spin- and FK-representations of the model [Smirnov'06, Ch. – Smirnov, Hongler – Kytölä / Izyurov, Kemppainen – Smirnov]

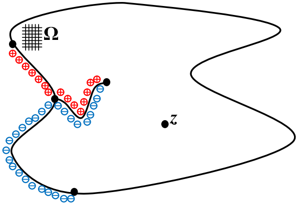
◦ **Uniform RSW-type bounds** [Ch. – Duminil-Copin – Hongler] based on *discrete complex analysis estimates* in rough domains.

Convergence of correlations ⇝ convergence of interfaces

[see Ch. – Duminil-Copin – Hongler – Kemppainen – Smirnov '13]

• "Martingale observables": choose a function $M_{\Omega_\delta}(z)$, $z \in \Omega_\delta$, such that $M_{\Omega_\delta \setminus \gamma_\delta[0,n]}(z)$ is a martingale wrt the filtration $\mathcal{F}_n := \sigma(\gamma_\delta[0,n])$.

Example: $\mathbb{E}_{\Omega_\delta}[\sigma_z]$.



• Convergence of observables: prove uniform (wrt Ω_δ) convergence of the (re-scaled) martingales $M_{\Omega_\delta}(z)$ to $M_\Omega(z)$ as $\delta \to 0$.

Remark: technically, $\mathbb{E}_{\Omega_\delta}[\sigma_z]$ is (by far) not an optimal choice of a martingale: e.g., fermionic observables are much easier to handle [Smirnov '06; Ch. – Smirnov '09; Izyurov '14]

Convergence of correlations ⇝ convergence of interfaces

[see Ch. – Duminil-Copin – Hongler – Kemppainen – Smirnov '13]

- "Martingale observables": choose a function $M_{\Omega_\delta}(z)$, $z \in \Omega_\delta$, such that $M_{\Omega_\delta \setminus \gamma_\delta[0,n]}(z)$ is a martingale

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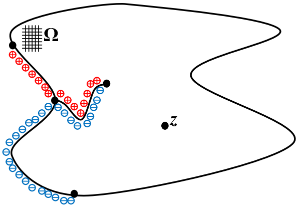


- RSW-type crossing estimates ⇒ tightness of the family $(\gamma_\delta)_{\delta\to 0}$:
  [Aizenmann – Burchard (1999), Kemppainen – Smirnov '12];

◦ Crossings in rectangles: [Duminil-Copin – Hongler –Nolin '09];
◦ Rough domains: [Ch. '12 ⇝ Ch. – Duminil-Copin – Hongler '13]

- Identification of subsequential limits: for each $\gamma = \lim_{\delta_k \to 0} \gamma_{\delta_k}$, the quantities $M_{\Omega\setminus\gamma[0,t]}(z)$ are martingales wrt $\mathcal{F}_t := \sigma(\gamma[0,t])$.

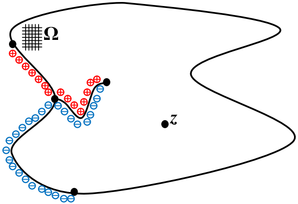
- **conformal covariance of M_Ω ⇒ conformal invariance of γ**

Convergence of correlations ⇝ convergence of interfaces

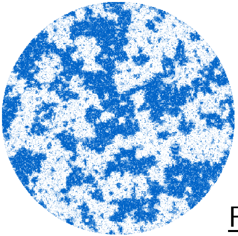
[see Ch. – Duminil-Copin – Hongler – Kemppainen – Smirnov '13]

- "Martingale observables"
- Convergence of observables
- **Uniform RSW-type estimates**

⇝ control of "pinning points" arising along the exploration



Convergence and conformal invariance of the loop ensemble

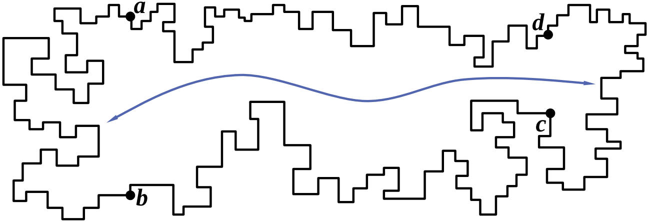


- "Exploration" [Hongler – Kytölä'11; Benoist – Duminil-Copin – Hongler'14; Benoist – Hongler'16] iteratively switching between spin- and FK(=random-cluster)-representations of the Ising model.

Related work: [Kempainnen – Smirnov '15–'16]

“Strong” RSW-type theory for the critical (FK-)Ising model

[“toolbox” arXiv:1212.6205 & Duminil-Copin – Hongler – Nolin’09
$\rightsquigarrow$ Ch. – Duminil-Copin – Hongler’13]



Thm: [Ch – DC – H] Uniformly wrt Ω and boundary conditions,

$$\mathbb{P}_{\Omega}^{\mathrm{FK}}[(ab) \leftrightarrow (cd)] \in [\eta(\mathrm{L}), 1 - \eta(\mathrm{L})],$$

where L is the effective resistance of $(\Omega; (ab), (cd))$.

FK-representation of the Ising model: sample a Bernoulli percolation with parameter $1 - x_{\mathrm{crit}}$ on edges of spin clusters.

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[“toolbox” arXiv:1212.6205 & Duminil-Copin – Hongler – Nolin’09
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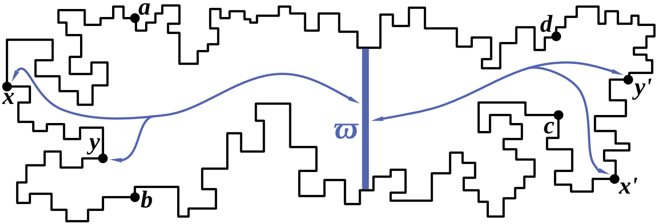
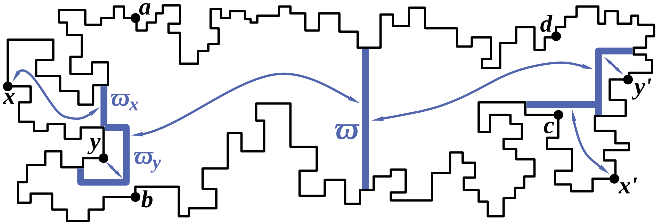


- **Basic ingredients:** second moment method, FKG inequality and estimates of point-to-wired arc connection events via fermionic observables and then discrete harmonic functions.

- **But..** How to handle triple connections $x \leftrightarrow y \leftrightarrow \varpi$?

“Strong” RSW-type theory for the critical (FK-)Ising model

[“toolbox” arXiv:1212.6205 & Duminil-Copin – Hongler – Nolin’09
$\rightsquigarrow$ Ch. – Duminil-Copin – Hongler’13]



- **“Surgery”:** given x, y (and ϖ), to construct ϖ_x, ϖ_y such that

$$\mathrm{Z}_{\mathrm{RW}}[x \leftrightarrow \varpi] \asymp \mathrm{Z}_{\mathrm{RW}}[x \leftrightarrow \varpi_x] \cdot \mathrm{Z}_{\mathrm{RW}}[\varpi_x \leftrightarrow \varpi],$$
$$\mathrm{Z}_{\mathrm{RW}}[y \leftrightarrow \varpi] \asymp \mathrm{Z}_{\mathrm{RW}}[y \leftrightarrow \varpi_y] \cdot \mathrm{Z}_{\mathrm{RW}}[\varpi_y \leftrightarrow \varpi]$$

(with uniform wrt everything(!) constants in $\asymp$ estimates) and

$$\mathrm{Z}_{\mathrm{RW}}[\varpi_x \leftrightarrow \varpi] \asymp \mathrm{Z}_{\mathrm{RW}}[(xy) \leftrightarrow \varpi] \asymp \mathrm{Z}_{\mathrm{RW}}[\varpi_y \leftrightarrow \varpi].$$

“Strong” RSW-type theory for the critical (FK-)Ising model

[“toolbox” arXiv:1212.6205 & Duminil-Copin – Hongler – Nolin’09
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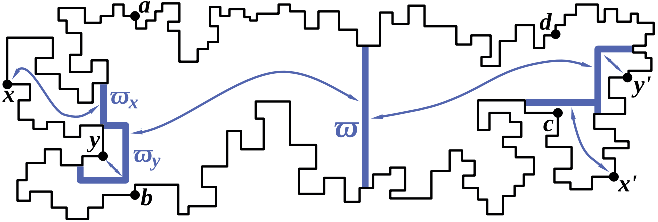

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$$\mathrm{Z_{RW}}[x \leftrightarrow \varpi] \asymp \mathrm{Z_{RW}}[x \leftrightarrow \varpi_x] \cdot \mathrm{Z_{RW}}[\varpi_x \leftrightarrow \varpi].$$

- **Remark.** Note that for the effective resistances one would have

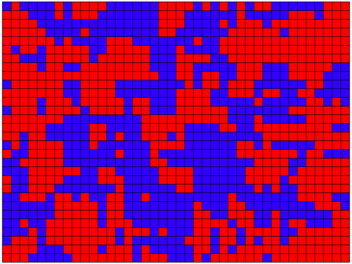
$$\mathrm{L}[x \leftrightarrow \varpi] \asymp \mathrm{L}[x \leftrightarrow \varpi_x] + \mathrm{L}[\varpi_x \leftrightarrow \varpi].$$

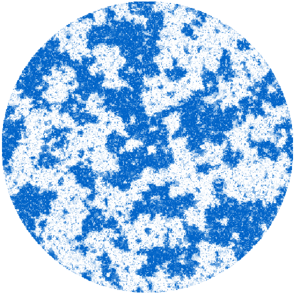
[see arXiv:1212.6205 for all that and more, e.g. $\mathrm{L} \asymp \log(1+\mathrm{Z_{RW}^{-1}})$]

Some important open questions

- **Spin field vs nested CLE(3):** is there a way to couple them so that one (of them) is a deterministic function of the other?

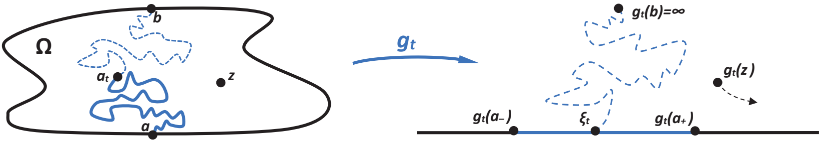
  Can one construct correlation functions of other CFT fields from CLE(3)? E.g., energy field $\leftrightsquigarrow$ "occupation density"?





Some important open questions

- **Massive SLE(3) curves:** fix $m \in \mathbb{R}$ and let $x = x_{\mathrm{crit}} + m\delta$.
  This breaks the conformal invariance ($\overline{\partial} f - im\overline{f} = 0$) but one can consider correlations and interfaces in a fixed domain as $\delta \to 0$.



Similarly to mLERW computations from [Makarov–Smirnov'09],

$$dg_t(z) = \frac{2dt}{g_t(z) - \xi_t}, \quad d\xi_t = \sqrt{3}\, dB_t + 3\, \frac{\partial}{\partial a_t} \log \mathcal{F}^{(m)}_{\Omega_t}(a_t, b)\, dt,$$

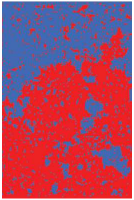
$$\mathcal{F}^{(m)}_{\Omega_t}(a_t, b) = [\, \langle \psi^{(m)}(a_t) \psi^{(m)}(b) \rangle_{\Omega_t} \,/\, \langle \psi(a_t) \psi(b) \rangle_{\Omega_t} \,]^{1/2}$$

$\rightsquigarrow 3\, \frac{\partial}{\partial a_t} \log \mathcal{F}^{(m)}_{\Omega_t}(a_t, b)$ is a quite non-trivial functional of $\xi[0, t]$.

A priori, even the existence of SDE solutions is unclear...

Some important open questions

- **Spin field vs nested CLE(3):** is there a way to couple them so that one (of them) is a deterministic function of the other?
- **Massive SLE(3) curves:** fix $m \in \mathbb{R}$ and let $x = x_{\text{crit}} + m\delta$. This breaks the conformal invariance ($\overline{\partial} f - im\overline{f} = 0$) but one can consider correlations and interfaces in a fixed domain as $\delta \to 0$.
- **Super-critical regime:** interfaces should converge to SLE(6)... Is it true that mSLE(3)$\to$ SLE(6) as $m \to +\infty$?



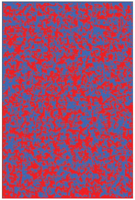
$x = x_{\text{crit}}$

- **Renormalization**

fixed $x > x_{\text{crit}}$, $\delta \to 0$

$$\longrightarrow$$

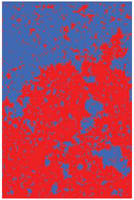
$(x - x_{\text{crit}}) \cdot \delta^{-1} \to \infty$



$x = 1$

Some important open questions

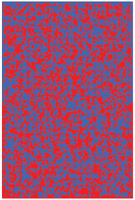
- **Spin field vs nested CLE(3):** is there a way to couple them so that one (of them) is a deterministic function of the other?
- **Massive SLE(3) curves:** fix $m \in \mathbb{R}$ and let $x = x_{\mathrm{crit}} + m\delta$. This breaks the conformal invariance ($\overline{\partial} f - im\overline{f} = 0$) but one can consider correlations and interfaces in a fixed domain as $\delta \to 0$.
- **Super-critical regime:** interfaces should converge to SLE(6)... Is it true that mSLE(3)$\to$ SLE(6) as $m \to +\infty$?



$x = x_{\mathrm{crit}}$

- **Renormalization**

$$\text{fixed } x > x_{\mathrm{crit}},\ \delta \to 0 \quad \longrightarrow \quad (x - x_{\mathrm{crit}}) \cdot \delta^{-1} \to \infty$$



$x = 1$

THANK YOU!