# From 2D lattice models to conformal geometry and CFT

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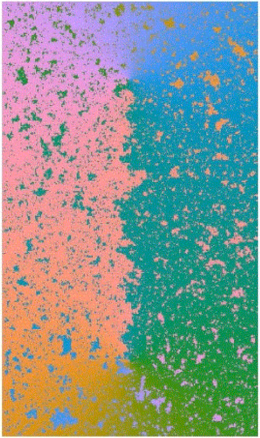
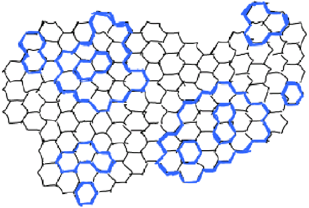
Outline:

- Loop O(n) model:
  phase transition and *conjectural* conformal invariance in the limit
- *Predictions* on scaling limits:
  correlations and loop ensembles
- Recent *results* on conformal invariance for the Ising model
- Research *routes*



## Loop O(n) model: definition and [conjectural] critical point

- $\Omega_\delta$: *discrete domain*, i.e. a (simply connected) subset of the honeycomb grid with mesh size $\delta$;
- $\mathrm{Conf}^{\varnothing}_{\Omega_\delta}$: set of *configurations*, i.e. collections of loops in $\Omega_\delta$;



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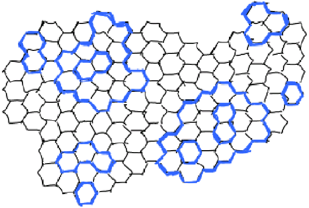

- Probabilities: $\mathbb{P}(\omega) = [\mathcal{Z}^{\varnothing}_{\Omega_\delta}]^{-1} \cdot x^{\#\mathrm{edges}(\omega)} n^{\#\mathrm{loops}(\omega)}$, where
$$\mathcal{Z}^{\varnothing}_{\Omega_\delta} = \sum_{\omega \in \mathrm{Conf}^{\varnothing}_{\Omega_\delta}} x^{\#\mathrm{edges}(\omega)} n^{\#\mathrm{loops}(\omega)}$$
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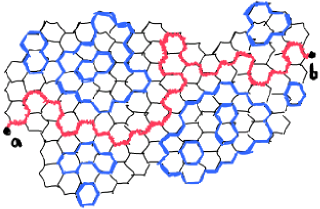


collections of loops plus a *path* $\gamma^{a,b}_{\Omega_\delta}$ linking two boundary points $a$ and $b$ in $\Omega_\delta$. Similarly, the partition function $\mathcal{Z}^{a,b}_{\Omega_\delta}$ is

$$\sum_{\omega\in\mathrm{Conf}^{a,b}_{\Omega_\delta}} x^{\#\mathrm{edges}(\omega)} n^{\#\mathrm{loops}(\omega)} .$$

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**Phase transition:** given $n \in [0,2]$ and $x \in (0,+\infty)$, how the ratio

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Informally: how expensive is to have a long path (or a long loop)?

More rigorously: assume that $\Omega_\delta$ are discrete approximations to a given (smooth) domain $\Omega \subset \mathbb{C}$ with two marked points $a, b \in \partial\Omega$.

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**Physicists prediction (Nienhuis, 1980s):** $x_{\mathrm{crit}} = 1/\sqrt{2+\sqrt{2-n}}$.

- For $x < x_{\mathrm{crit}}$: $(\star)$ decays exponentially as $\delta \to 0$;
- For $x = x_{\mathrm{crit}}$: $(\star) \sim \mathrm{const} \cdot \delta^{2\alpha}$ with $\alpha = \frac{1}{4} + \frac{3}{4\pi} \arccos \frac{n}{2}$;
- For $x > x_{\mathrm{crit}}$: $(\star) \sim \mathrm{const} \cdot \delta^{2\widetilde{\alpha}}$ with $\widetilde{\alpha} = \frac{1}{4} - \frac{3}{4\pi} \arccos \frac{n}{2}$.

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Rigorously known *only* in the following two particular cases:

- Ising model ($n=1$):
  phase transition at $x_{\mathrm{crit}}$ (back to 1940s) and $\alpha = \frac{1}{2}$;
- Self-Avoiding Walk ($n=0$):
  phase transition at $x_{\mathrm{crit}}$ (Duminil-Copin – Smirnov, 2010).

## Loop O(n) model: [conjectural] conformal invariance at $\mathbf{x_{crit}}$

**I. Quantities:** Let $\Omega_\delta$ be a discrete approximation to a (smooth) domain $\Omega \subset \mathbb{C}$ and $a, b \in \partial\Omega$. Denote [ wishful thinking ]

$$f_\Omega(a,b) \;:=\; \lim_{\delta\to 0} \delta^{-2\alpha} \cdot \mathcal{Z}^{a,b}_{\Omega_\delta} / \mathcal{Z}^{\varnothing}_{\Omega_\delta}\,.$$

[Conjectural] invariance under conformal transforms $\varphi : \Omega \to \Omega'$:

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Another example: for an edge $e_\delta$, let $\varepsilon(e_\delta) := \mathbf{1}[e_\delta \in \omega]$ .
Let $e_{1,\delta}, ..., e_{m,\delta}$ approximate a collection of points $e_1, ..., e_m \in \Omega$. Denote

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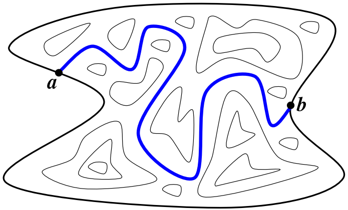
## Loop O(n) model: [conjectural] conformal invariance at $x_{\mathrm{crit}}$

## II. Interfaces, loop ensembles:

For Dobrushin boundary conditions, one expects that

$$\gamma_{\Omega_\delta}^{a,b} \underset{\delta\to 0}{\longrightarrow} \gamma_{\Omega}^{a,b}\,.$$




The limit (*random curve* linking $a$ and $b$ inside $\Omega$) is [conjecturally] *conformally invariant*:

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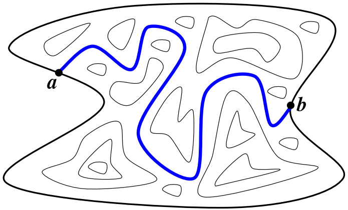


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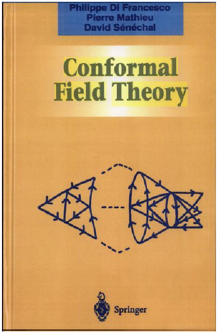
For $\varnothing$ boundary conditions: the limit as $\delta \to 0$ of the whole collection of loops in $\Omega_\delta$ (i.e., *random loop ensemble* in $\Omega$) is [conjecturally] invariant under conformal maps $\varphi : \Omega \to \Omega'$.

NB: topology of convergence – ?: random curves/loop ensembles
= measures on the (metric) set of curves/loop ensembles

## Predictions on scaling limits: correlations and loop ensembles

### I. Correlations

General idea (in 2D): conformal covariance of scaling limits + further assumptions on their singularities (fusion rules, null-vectors, ...) $\Rightarrow$ one of the conformal field theories parameterized by a central charge $c \in [0, 1]$.



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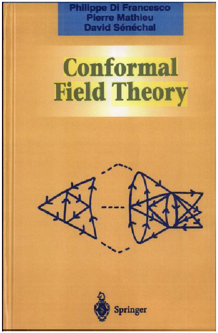


$$c = 13 - 6(t + t^{-1}), \quad t = \frac{4}{\kappa} = 1 + \frac{1}{\pi} \arccos \frac{n}{2}$$

is identified (Nienhuis, 1980s), one has:

- the set of scaling exponents (e.g., $\alpha = h_{2,1}$, $\beta = 2h_{1,3}$);
- PDEs for the correlation functions;
- explicit formulae ('small configurations' or particular theories).

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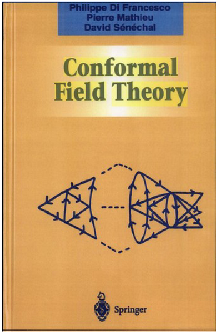


NB: there are two setups

- full-plane $\mathbb{C}$;
- general domains $\Omega \subset \mathbb{C}$, conformally equivalent to the upper half-plane $\mathbb{H}$.

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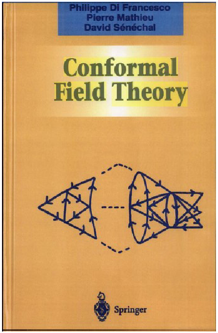


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Example: for the scaling limits of spin correlations in the Ising model as $\delta \to 0$, one [conjecturally] has $\langle \sigma_{z_1} \ldots \sigma_{z_m} \rangle_\Omega = \langle \sigma_{\varphi(z_1)} \ldots \sigma_{\varphi(z_k)} \rangle_{\Omega'} \cdot \prod_{s=1}^{k} |\varphi'(z_s)|^{\frac{1}{8}}$, with

$$\big[\, \langle \sigma_{z_1} \ldots \sigma_{z_k} \rangle_{\mathbb{C}} \,\big]^2 = \mathcal{C}^k \cdot \sum_{\mu \in \{\pm 1\}^k : \mu_1 + \ldots + \mu_k = 0} \; \prod_{1 \leqslant s < m \leqslant k} |z_s - z_m|^{\frac{\mu_s \mu_m}{2}}$$

$$\big[\, \langle \sigma_{z_1} \ldots \sigma_{z_k} \rangle_{\mathbb{H}}^{+} \,\big]^2 = \mathcal{C}^k \cdot \prod_{1 \leqslant s \leqslant k} (2 \,\mathrm{Im}\, z_s)^{-\frac{1}{4}} \times \sum_{\mu \in \{\pm 1\}^k} \prod_{s < m} \left| \frac{z_s - z_m}{z_s - \overline{z}_m} \right|^{\frac{\mu_s \mu_m}{2}}$$

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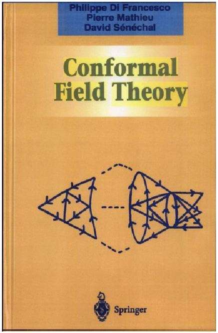


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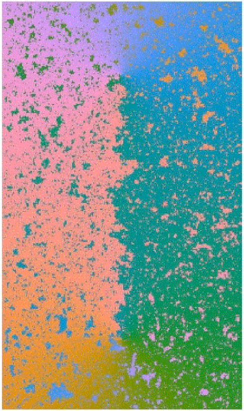
**Convergence Theorem:** Ch.–Hongler–Izyurov, 2012

## Predictions on scaling limits: correlations and loop ensembles

### II. Interfaces, loop ensembles

Question: What could be a good candidate for the scaling limit of interfaces and loop ensembles as $\delta \to 0$?

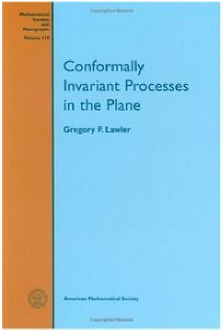


# Predictions on scaling limits: correlations and loop ensembles

## II. Interfaces, loop ensembles

Question: What could be a good candidate for the scaling limit of interfaces and loop ensembles as $\delta \to 0$?

- Interfaces (e.g., generated by Dobrushin boundary conditions): $\mathrm{SLE}_\kappa$ curves $[c = 13 - 6(\frac{\kappa}{4} + \frac{4}{\kappa})]$




In one line: non-self-intersecting 2D curves, *introduced by Schramm in 2000*, are defined dynamically via the classical Loewner evolution *[1923]* with a 1D Brownian motion input, can be analyzed combining *geometrical complex analysis* and *stochastic calculus*.

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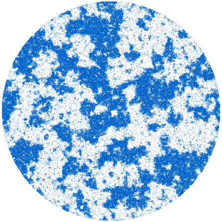
- Interfaces (e.g., Dobrushin b.c.);
- Loop ensembles (e.g., the collection of all outermost loops for $\varnothing$ b.c.):



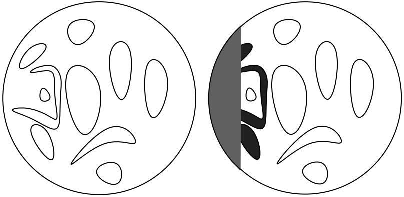
Ising model sample with **free b.c.**



Intuition: Distribution of loops should be conformally invariant and satisfy the domain Markov property:

Question: What could be a good candidate for the scaling limit of the collection of all outermost loops for $\varnothing$ b.c.? Intuition: should be conformally invariant and satisfy the domain Markov property:

Let $D_1 \subset D_2$. Given the set of loops from the CLE in $D_2$ that intersect $D_2 \setminus D_1$, *the conditional law of the remaining loops is an independent CLE* in each component of the (interior of the) complement of this set.



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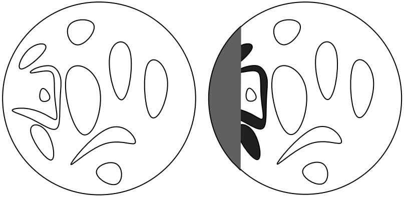

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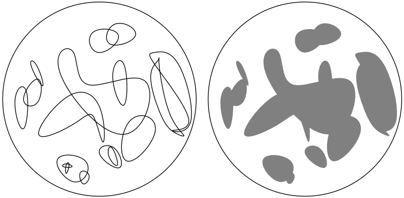

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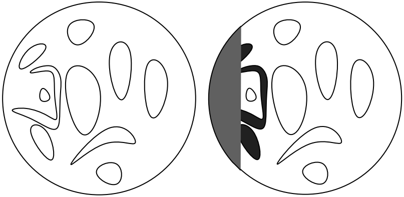
Loop-soup construction:

- sample a (countable) set of Brownian loops in $D$ using some conformally-friendly Poisson process of intensity $c \in [0, 1]$;
- fill the outermost clusters.

Question: What could be a good candidate for the scaling limit of the collection of all outermost loops for $\varnothing$ b.c.? Intuition: should be conformally invariant and satisfy the domain Markov property:

Thm (Sheffield–Werner, 2012):

Provided loops do not touch each other, *the loop-soup construction gives the only possibility*. This ensemble is called $\mathrm{CLE}_\kappa$ and consists of $\mathrm{SLE}_\kappa$-type bubbles, where $c = 13 - 6(\frac{\kappa}{4} + \frac{4}{\kappa})$.





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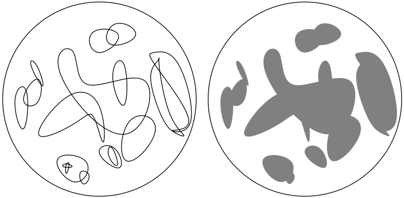
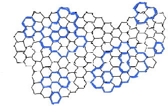


## Predictions on scaling limits: correlations and loop ensembles

### Conformal Field Theory

Assuming conformal covariance of correlation functions appearing in the limit, they should form one of "algebraic structures", parameterized by a central charge.

*Lattice models [e.g., loop $O(n)$]*



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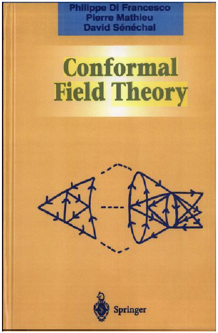


Deep interactions 'in continuum', cf.

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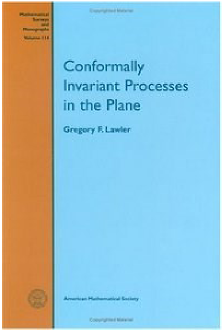

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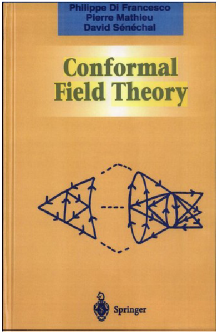


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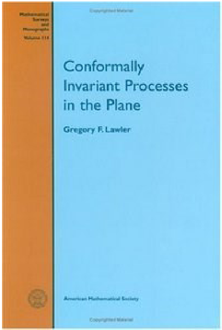


[..........]

But can one prove that these beautiful 'algebraic' and 'geometric' structures indeed arise in the limit of some lattice model as $\delta \to 0$ (e.g., the Ising model, which contains a lot of integrability inside)?

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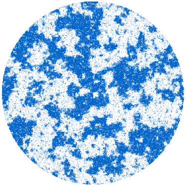

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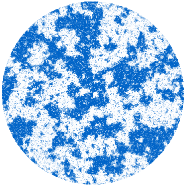

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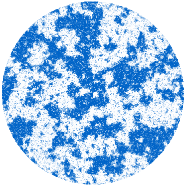

*Main tool*: discrete holomorphic functions

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*[Ch., Duminil-Copin, Hongler, Izyurov, Kemppainen, Kytölä, ...]*

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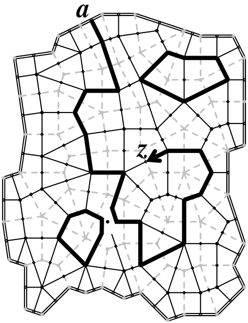
Combinatorial definition:

$$F_a^\delta(z) := \sum_{\omega \in \mathrm{Conf}_{\Omega_\delta}^{a,z}} x^{\#\mathrm{edges}(\omega)} e^{-\frac{i}{2}\mathrm{wind}(a \rightsquigarrow z)}$$



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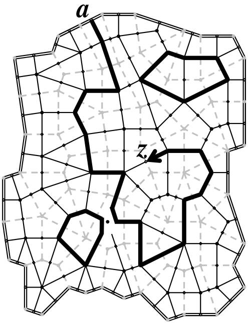


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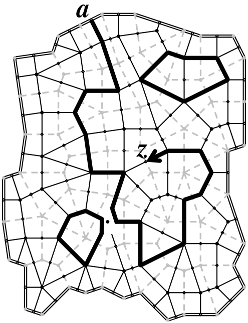


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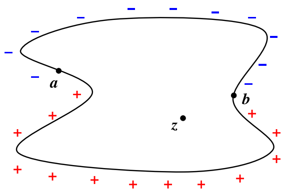


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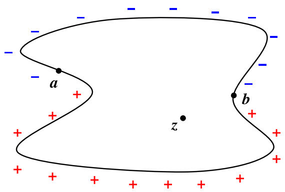


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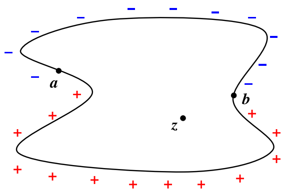


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• prove the convergence of $\gamma^{a,b}_{\Omega_\delta}$ and recover the limiting law using this family of martingales [some probabilistic techniques needed].

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**Example:** to handle $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_z]$, one should consider the following b.v.p.:

- $f(w^*) \equiv -f(w)$, branches around $z$;
- $\mathrm{Im}\left[ f(\zeta)\sqrt{n(\zeta)} \right] = 0$ for $\zeta \in \partial\Omega$;
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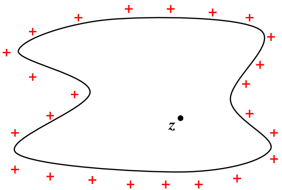

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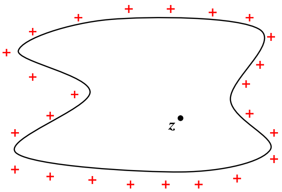


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**Claim:** For $\Omega_\delta \to \Omega$ as $\delta \to 0$,

- $\delta^{-1} \log \left[\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{z+\delta}] \,/\, \mathbb{E}^{+}_{\Omega_\delta}[\sigma_z]\right] \to \mathrm{Re}[\,\mathcal{A}_\Omega(z)\,]\,;$
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## Recent results on conformal invariance for the Ising model

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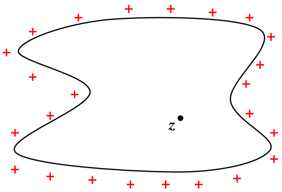


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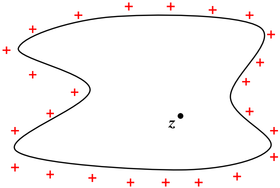


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**Conformal exponent $\frac{1}{8}$:** for any conformal map $\phi : \Omega \to \Omega'$,

- $f_{[\Omega,a]}(w) = f_{[\Omega',\phi(a)]}(\phi(w)) \cdot (\phi'(w))^{1/2}\,;$
- $\mathcal{A}_\Omega(z) = \mathcal{A}_{\Omega'}(\phi(z)) \cdot \phi'(z) + \frac{1}{8} \cdot \phi''(z)/\phi'(z)\,.$

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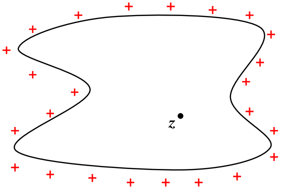


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**Steps of the proof:** • to find proper combinatorics in discrete;

- to handle tricky boundary conditions (Dirichlet for $\int \mathrm{Re}[f^2 dz]$);
- to prove convergence, incl. near singularities [complex analysis];
- to recover the normalization of $\mathbb{E}^{+}_{\Omega_\delta}[\sigma_z]$ [probabilistic techniques].

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Ising model:

• Better understanding at criticality: more CFT fields, Virasoro algebra at the lattice level, 'geometrical' observables and height functions, multiply connected domains and Riemann surfaces, etc.

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Other lattice models:

◦ E.g., convergence of the self-avoiding walk to $\mathrm{SLE}_{8/3}$