MAGNETIZATION IN THE ZIG-ZAG LAYERED ISING MODEL AND ORTHOGONAL POLYNOMIALS

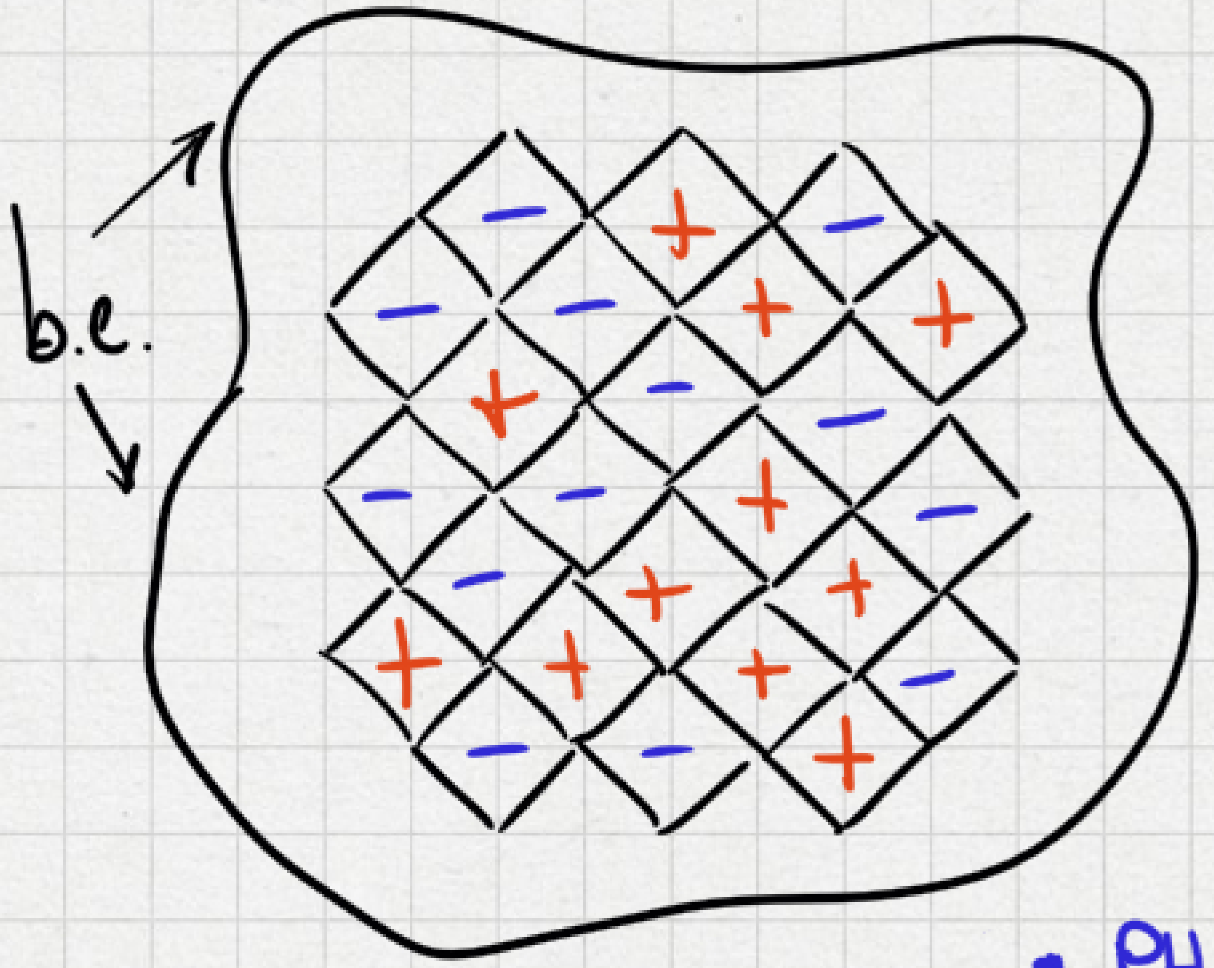
[D. Chelkak, ISRMT @ U(M) seminar, 03/27/23]

[arXiv: 1904.09168 w/ C. Hongler & R. Mahfouf]
(partly back to 2012/13
goal: "synthesis" of discrete complex analysis ideas and orthogonal polynomials)

Outline:

- HOMOGENEOUS ISING MODEL ON $\mathbb{Z}^2$
- FERMIONIC OBSERVABLES
- SHORT PROOFS OF ONSAGER & T.T. WU FORMULAE
- DÉTOUR: UNIVERSALITY FOR THE Z-INVARIANT MODEL [arXiv: 2104.12858 w/ K. Izyurov & R. Mahfouf]
- ZIG-ZAG LAYERED MODEL IN THE HALF-PLANE: FORMULA FOR M_m VIA HANKEL DET'S
  $\updownarrow$
  [SPECTRAL MEASURE OF A JACOBI MATRIX]
- POSSIBLE SETUPS OF INTEREST:
  - WETTING PHASE TRANSITION
  - PERIODIC INTERACTIONS
  - MCCOY-WU MODEL: RANDOM I.I.D. INTERACTIONS

HOMOGENEOUS ISING MODEL ON $\mathbb{Z}^2$:

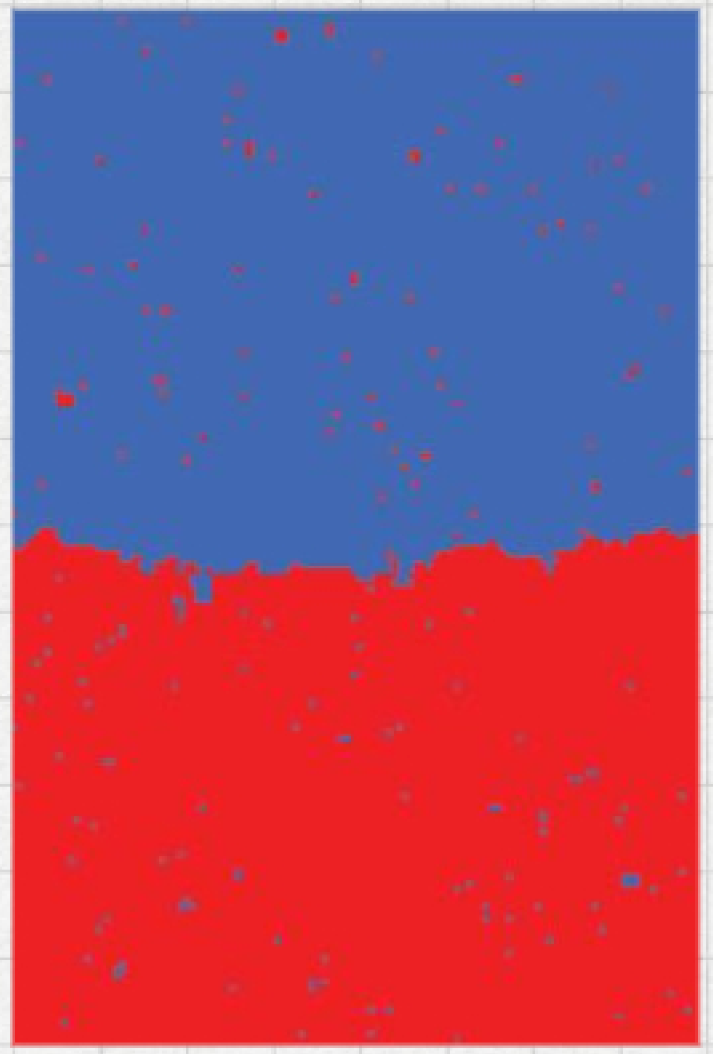


- spins $\sigma_u \in \{\pm 1\}$ on vertices $\Longleftrightarrow$ faces
- INFINITE-VOLUME LIMIT:

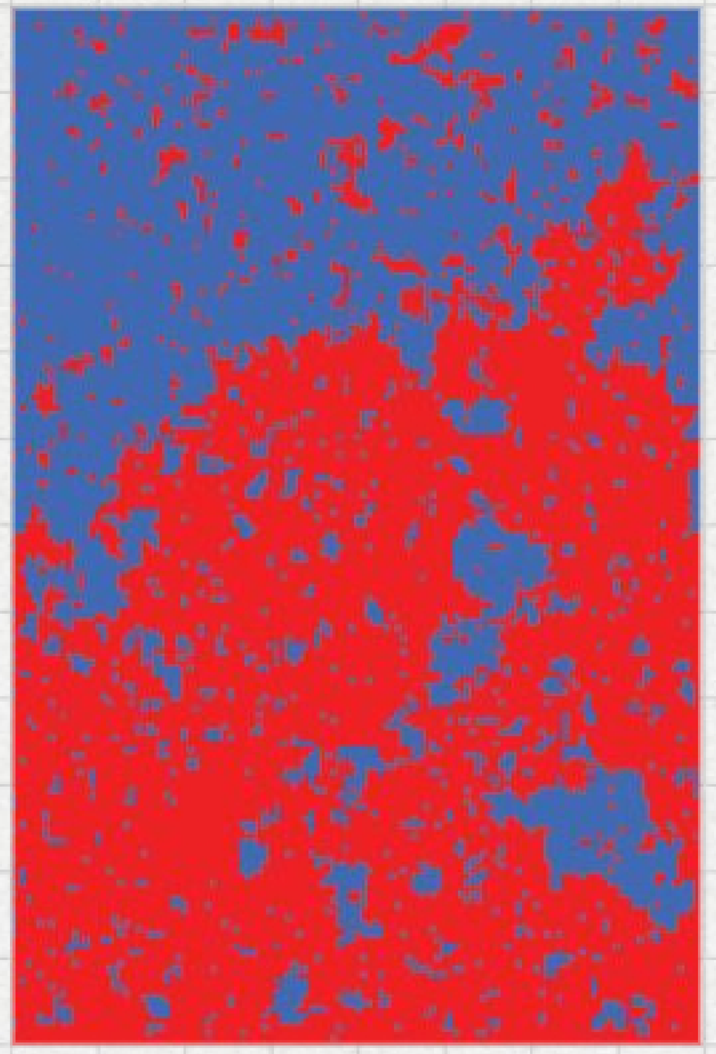
$$\left\{ \begin{matrix} \text{Gibbs} \\ \text{measures} \end{matrix} \right\} = \left\{ t \mu^+ + (1-t) \mu^- \right\}_{t \in [0,1]} \quad (*)$$

- PHASE TRANSITION: $\mu^+ = \mu^- \Longleftrightarrow x \geq x_{crit}$

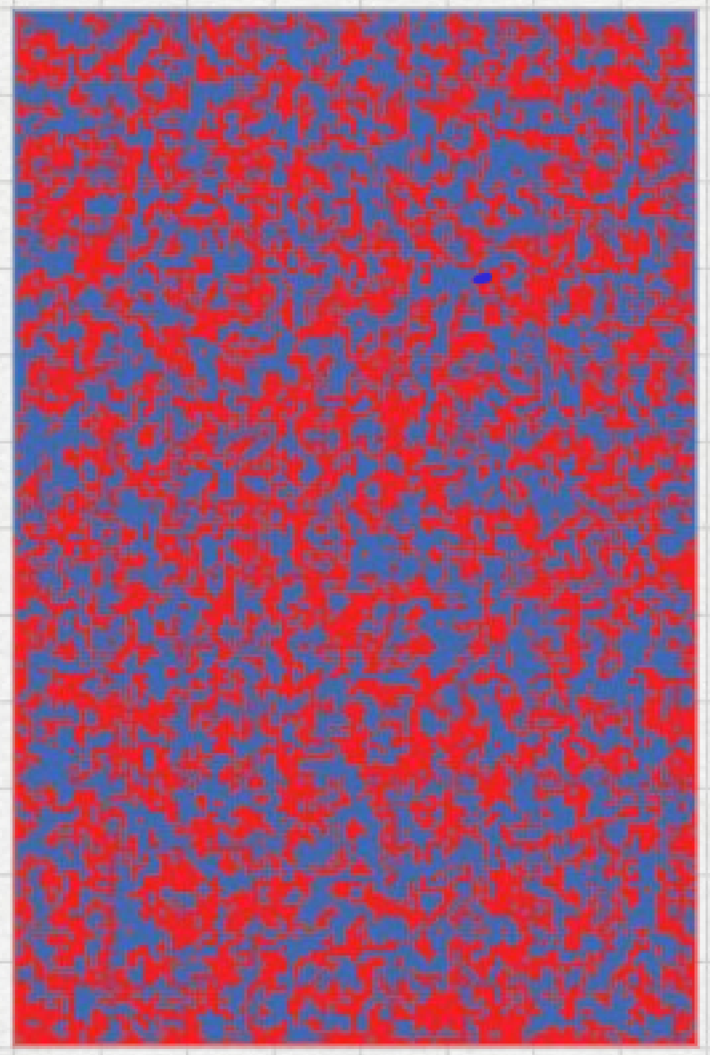
$x < x_{crit}$



$x = x_{crit} = \tan \frac{\pi}{8}$



$x > x_{crit}$



IN FINITE DOMAINS:

$$\mathbb{P} \propto \exp \left[\beta \sum_{u \sim u'} J_{uu'} \sigma_u \sigma_{u'} + \cancel{h \sum \sigma_u} \right]$$

$$\propto x^{\#\{(u,u') \,:\, u \sim u',\ \sigma_u \neq \sigma_{u'}\}}$$

Parametrization:

$$x = e^{-2\beta J} =: \tan \frac{\theta}{2}$$

$x \in (0,1)$; $\theta \in (0, \frac{\pi}{2})$

(the same monotonicity as the temperature)

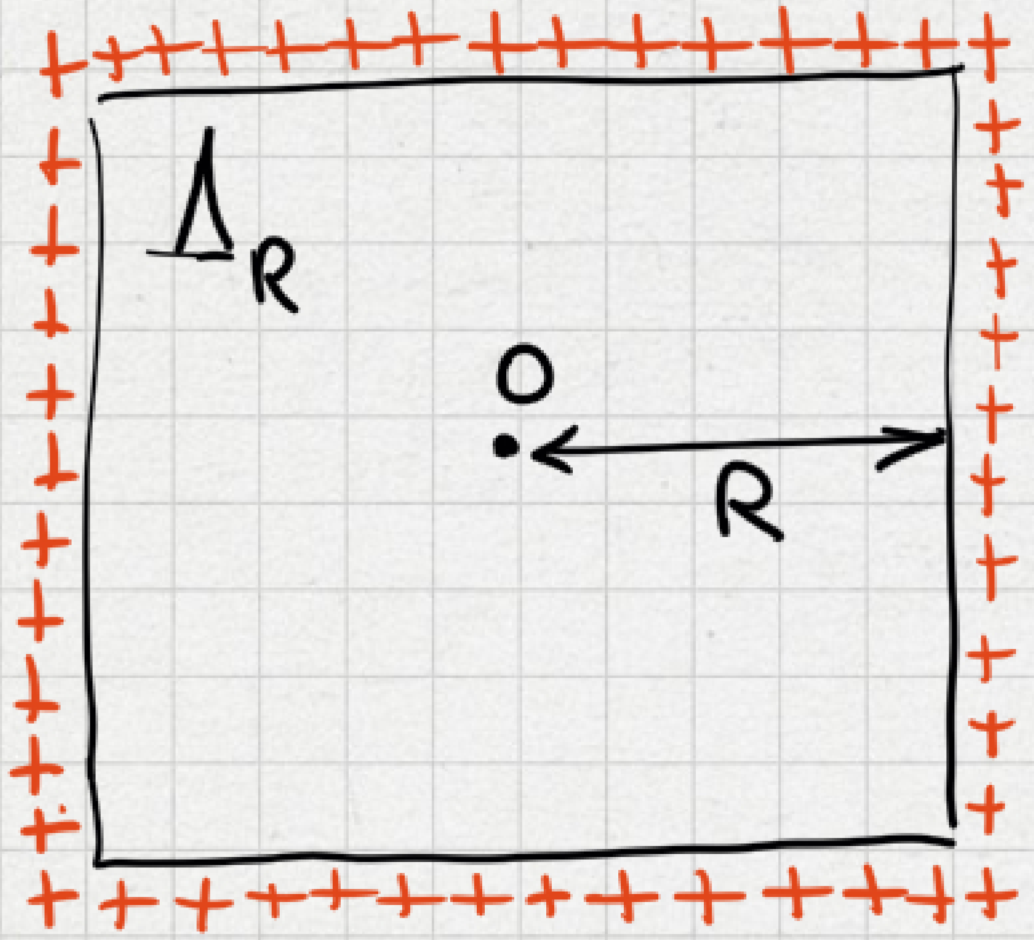
(*) in 3d: for $x < x_{crit}$ $\exists$ non-translation-invariant Gibbs measures

HOMOGENEOUS ISING MODEL ON $\mathbb{Z}^2$:

MAGNETIZATION:

$$M := \lim_{R\to\infty} \underbrace{\mathbb{E}^+_{\Lambda_R}[\sigma_0]}$$

Lemma: decreasing fct of R



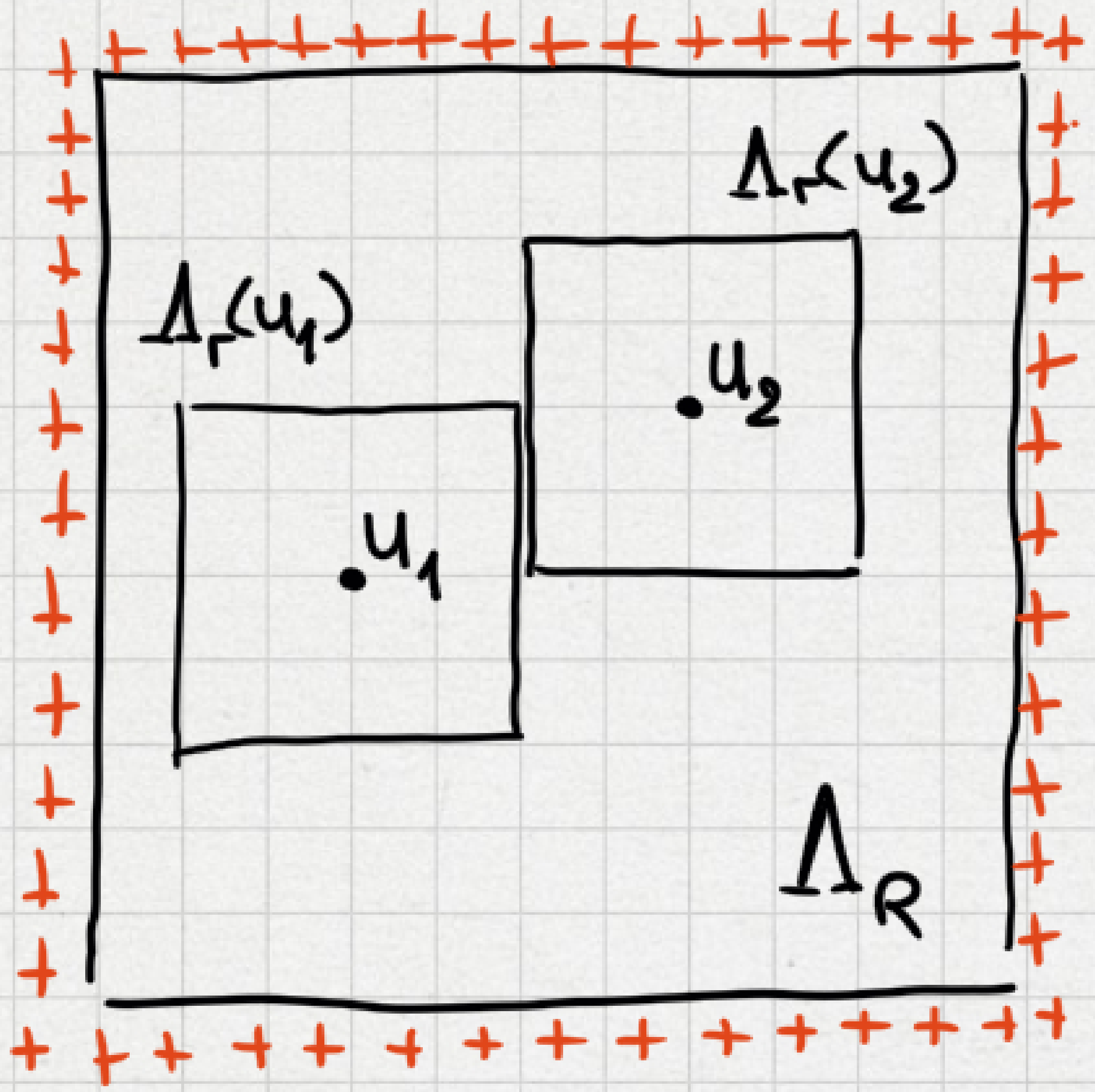
Lemma: $\lim_{|u-v|\to\infty} \mathbb{E}[\sigma_{u_1}\sigma_{u_2}] = M^2$

Pf: $\mathbb{E}^+_{\Lambda_R}[\sigma_{u_1}] \times \mathbb{E}^+_{\Lambda_R}[\sigma_{u_2}]$

$\leq \mathbb{E}^+_{\Lambda_R}[\sigma_{u_1}\sigma_{u_2}]$

$\leq \mathbb{E}^+_{\Lambda_r(u_1)}[\sigma_{u_1}] \times \mathbb{E}^+_{\Lambda_r(u_2)}[\sigma_{u_2}]$

(where $r := \frac{1}{2}|u_2-u_1|$)



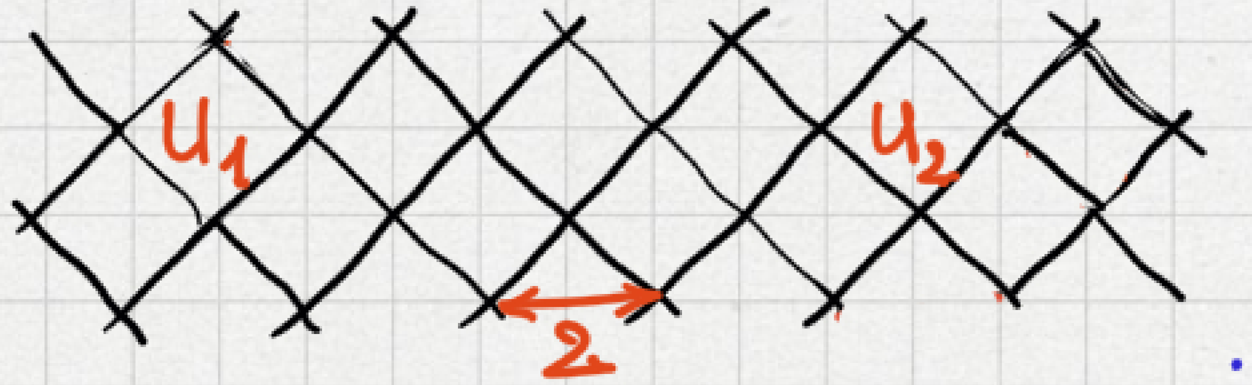
Spin-spin expectations $\mathbb{E}[\sigma_u\sigma_v]$ are well defined (= do not depend on the Gibbs measure) & translation-invariant

Interactions $e^{-2\beta J} = x = \tan\frac{\theta}{2}$

Theorem: (Kaufman-Onsager; Yang)

for $x < x_{crit}$, $M = (1-\tan^4\theta)^{1/8}$

45°-rotated $\mathbb{Z}^2$:



Theorem: (T.T. Wu, "diagonal" correlations)

for $x = x_{crit}$ and $u_2 - u_1 = 2n$

$$\mathbb{E}[\sigma_{u_1}\sigma_{u_2}] = \left(\frac{2}{\pi}\right)^n \cdot \prod_{k=1}^{n-1}\left(1-\frac{1}{4k^2}\right)^{k-n} \sim \frac{\mathcal{C}_\sigma^2}{|u_2-u_1|^{1/4}}$$

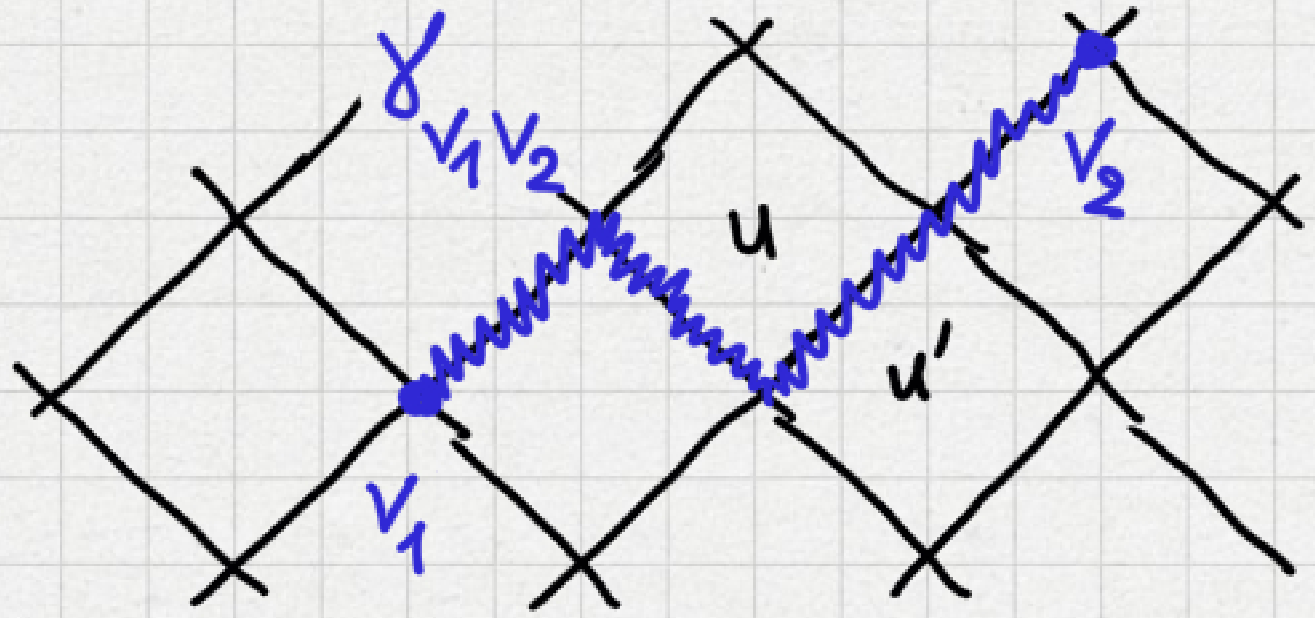
FERMIONIC OBSERVABLES:

- Kaufman (-Onsager)'49 } transfer matrices
- Schultz - Mattias - Lieb'64 }
- Kac-Ward'52 } combinatorics, reduction to dimers
- Potts-Ward'55
- Fisher'66, Kasteleyn'67
- Kadanoff - Ceva'71 } "disorder operators"
- McCoy - Perk - Wu'80
- Dotsenko - Dotsenko'83
- Mercat'01 } discrete complex analysis
- Smirnov et al'03+

"Planar Ising model is a free fermion"

[partition function = Pfaffian]

- "disorder lines"



$(\mu_{v_1}\mu_{v_2}) := \prod_{(uu')\cap\gamma_{v_1v_2}\neq\emptyset} x^{\sigma_u\sigma_{u'}}$ replaces $1\to x$ if $\sigma_u=\sigma_{u'}$, $x\to 1$ if $\sigma_u\neq\sigma_{u'}$

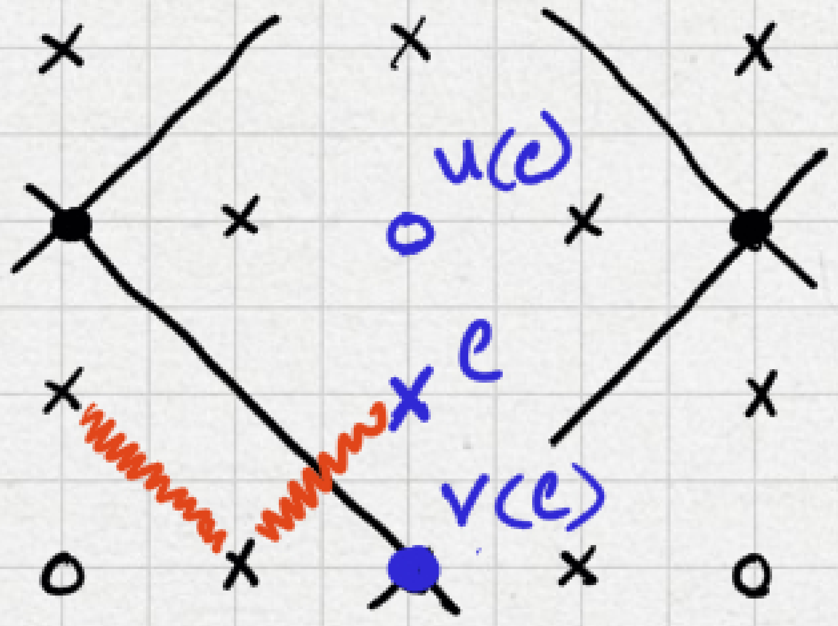
- mixed correlations

$\mathbb{E}[\mu_{v_1}\dots\mu_{v_{2m}}\sigma_{u_1}\dots\sigma_{u_{2n}}]$ are defined up to $\pm$

depends on the choice of γ's

- Kadanoff - Ceva fermions:

$\psi_e := \mu_{v(e)}\sigma_{u(e)}$



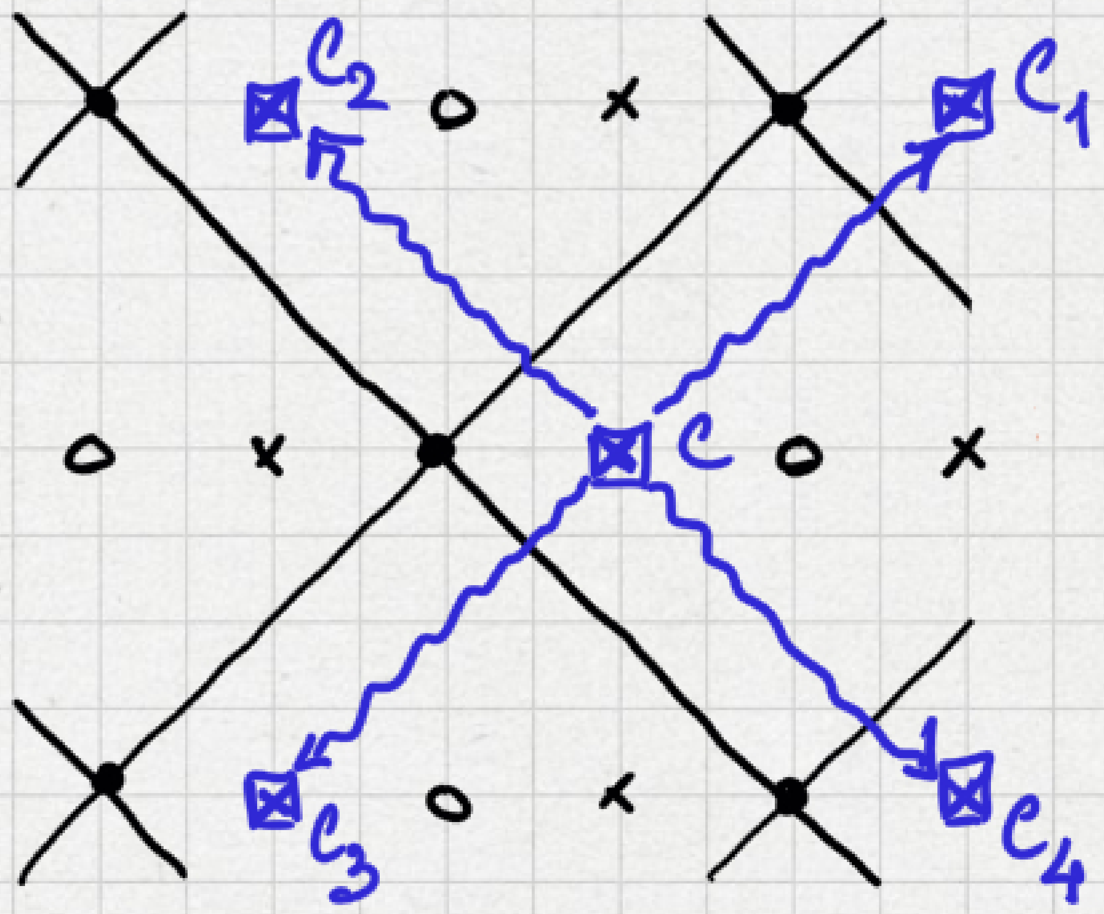
Functions

$e\mapsto \mathbb{E}[\chi_e\mu_{v_1}\dots\mu_{v_{2m-1}}\sigma_{u_1}\dots\sigma_{u_{2n-1}}]$

satisfy a simple three-term linear propagation equation

FERMIONIC OBSERVABLES IN THE HOMOGENEOUS ISING MODEL ON $\mathbb{Z}^2$:

On each of the four types of 'corners' the values of $F(c)$ satisfy

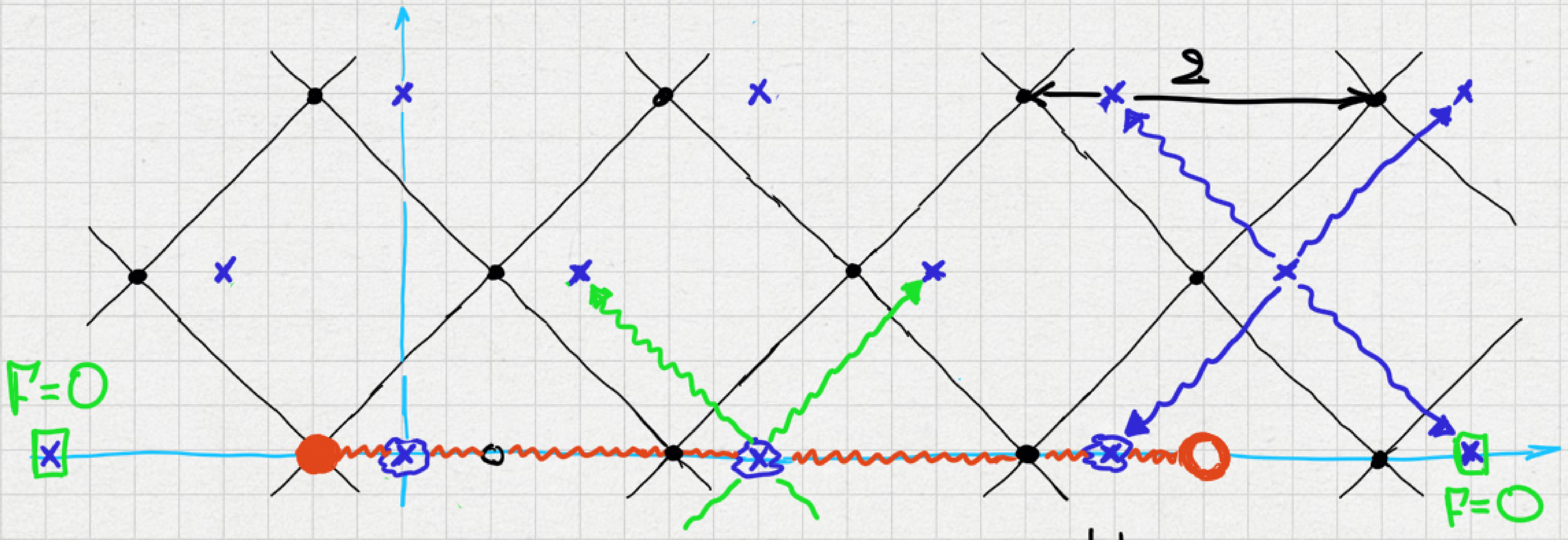


$$\boxed{X = \tan\frac{\theta}{2}}$$

$$F(c) = \frac{\sin 2\theta}{4} \sum_{k=1}^{4} F(c_k)$$

Key observation: [Section 3, arXiv: 1605.09035]

consider $F(c) := \mathbb{E}[\, X_c \, \mu_{(-\frac{1}{2},0)} \, \sigma_{(2n+\frac{1}{2},0)} \,]$



Denote $f_s(e^{it}) := \sum_{k\in\mathbb{Z}:\, k+s\in 2\mathbb{Z}} e^{ikt} F(k,s)$

$$f_s = \frac{\sin 2\theta \cdot \cos t}{2}\,(f_{s-1} + f_{s+1}) \Rightarrow f_1 = \frac{1 - \sqrt{1 - \sin^2 2\theta \cos^2 t}}{\sin 2\theta \cos t}\, f_0$$

$$\Rightarrow \begin{cases} f_0 = * + * e^{2it} + \dots + * e^{2int} \text{ is a polynomial} \\ \sin 2\theta \cos t \cdot f_1 - f_0 = \sqrt{1 - \sin^2 2\theta \cos^2 t} \cdot f_0 = \dots + * e^{-2it} + * + 0 + * e^{2int} + \dots \end{cases}$$

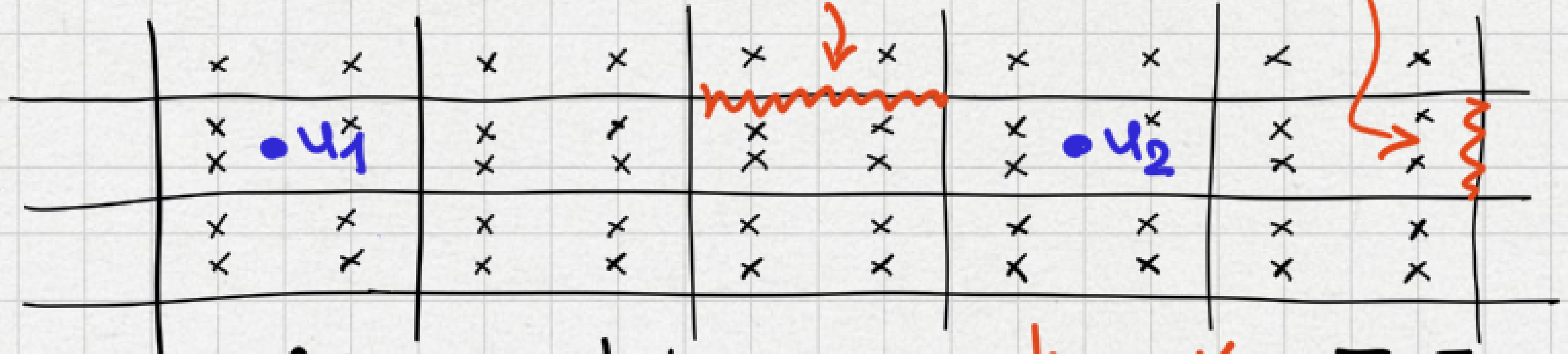
⇝ "SYNTHETIC" PROOF OF KAUFMAN-ONSAGER-YANG & T.T.WU THEOREMS :

For the homogeneous model on $\mathbb{Z}^2$:

$x = \tan\frac{\theta}{2} < x_{crit}$: $\quad M^2 = \lim_{n\to\infty} D_n = (1-\tan^4\theta)^{1/4}$

$x = \tan\frac{\pi}{8} = x_{crit}$: $\quad D_n = \left(\frac{2}{\pi}\right)^n \prod_{k=1}^{n-1}\left(1-\frac{1}{4k^2}\right)^{k-n}$

Similar results hold for "horizontal" correlations with $x^h = \tan\frac{\theta^h}{2}$, $x^v = \tan\frac{\theta^v}{2}$



[criticality condition : $\theta^h + \theta^v = \frac{\pi}{2}$]

$$f_0(e^{it}) = D_n + * e^{2it} + \cdots + * e^{2i(n-1)t} + D_n^\star e^{2int}$$

$$w(e^{it}) f_0(e^{it}) =$$

$$\cdots + * e^{-2it} + D_{n+1}\cos^2\theta + 0 + D^\star_{n+1}\sin^2\theta \, e^{2int} + \cdots,$$

where $\quad w(e^{it}) = \sqrt{1 - \sin^2 2\theta \cos^2 t}$,

$D_n = \mathbb{E}[\sigma_0\sigma_{2n}]$, $\quad D_n^\star = \mathbb{E}[\mu_0\mu_{2n}] = \mathbb{E}^\star[\sigma_0\sigma_{2n}]$

↑ for the dual parameter $x^\star := \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

- $x = x_{crit}$ ⇝ Legendre polynomials
- $x < x_{crit}$ ⇝ basics in OPUC ⊕ "Szegő" thm. for $w(t)$

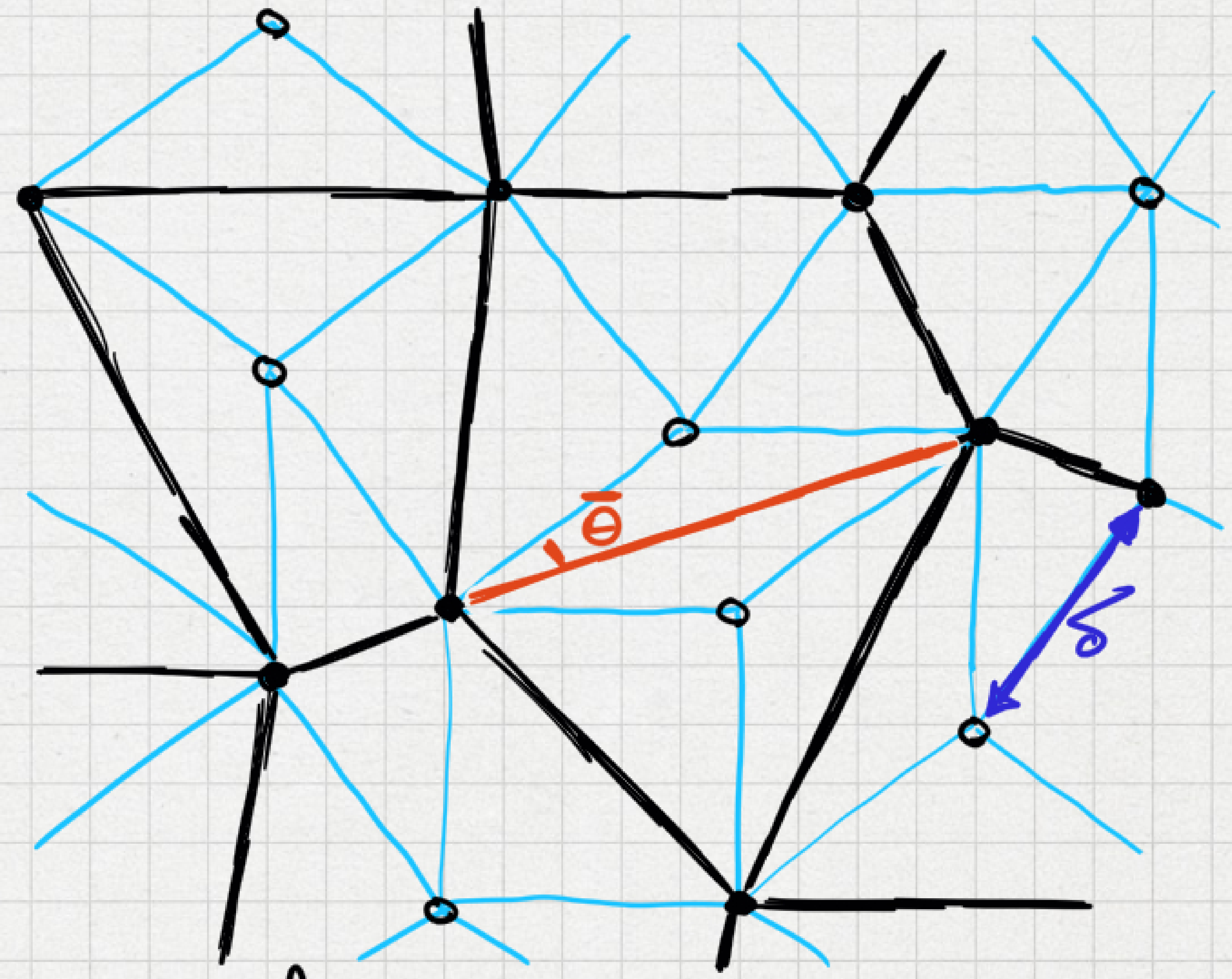
⇝ weights $\quad w(e^{it}) = \sqrt{(1-\sin\theta^h\cos\theta^v\cos t)^2 - (\cos\theta^h\sin\theta^v)^2}$

and $\quad w^\#(e^{it}) = (w(e^{it}))^{-1}$ on $\mathbb{T}$

(at criticality: $w(x) = \sqrt{1-\sin^2\theta\cdot x^2}$ and $w^\#(x) = 1/w(x)$ on $[-1,1]$)

DIGRESSION: UNIVERSALITY FOR BAXTER'S Z-INVARIANT WEIGHTS

[arXiv: 2104.12858 w/ K. Izyurov & R. Mahfouf ("discrete complex analysis", no orthogonal polynomials)]



Subcritical: $M = (k^*)^{1/4}$ (Baxter's formula)

Critical: $\delta^{-\frac{1}{4}}\mathbb{E}[\sigma_{u_1}\sigma_{u_2}] \to \mathcal{C}_\sigma^2 \cdot |u_2-u_1|^{-1/4}$ as $\delta \to 0$

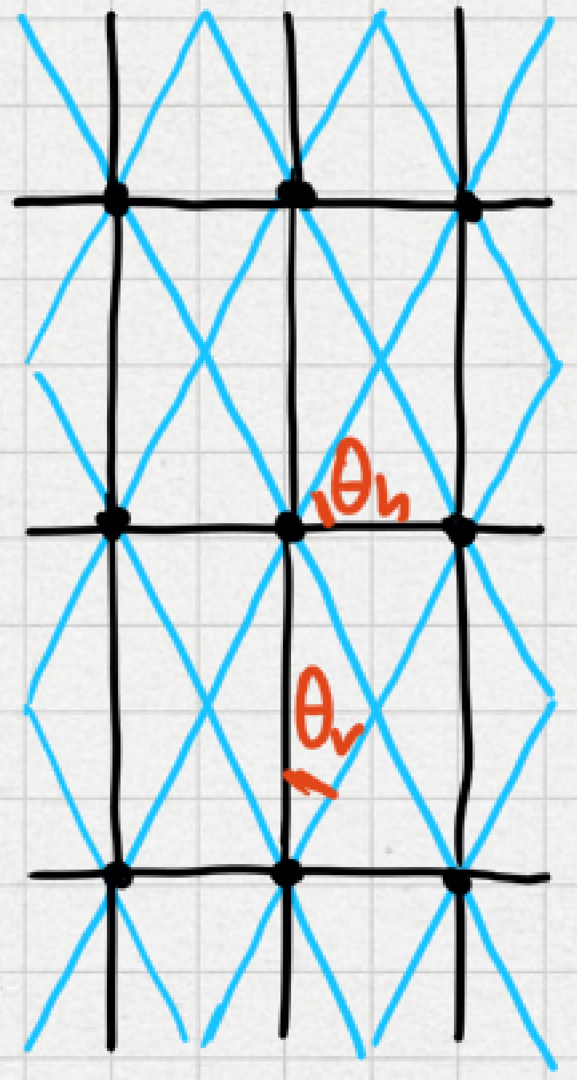
Massive: $\delta^{-\frac{1}{4}}\mathbb{E}[\sigma_{u_1}\sigma_{u_2}] \to \mathcal{C}_\sigma^2 \cdot \Xi(|u_2-u_1|, m)$

! In this setup nothing depends on the lattice !

critical: $x = \tan\frac{\theta}{2}$, $\theta = \bar\theta$

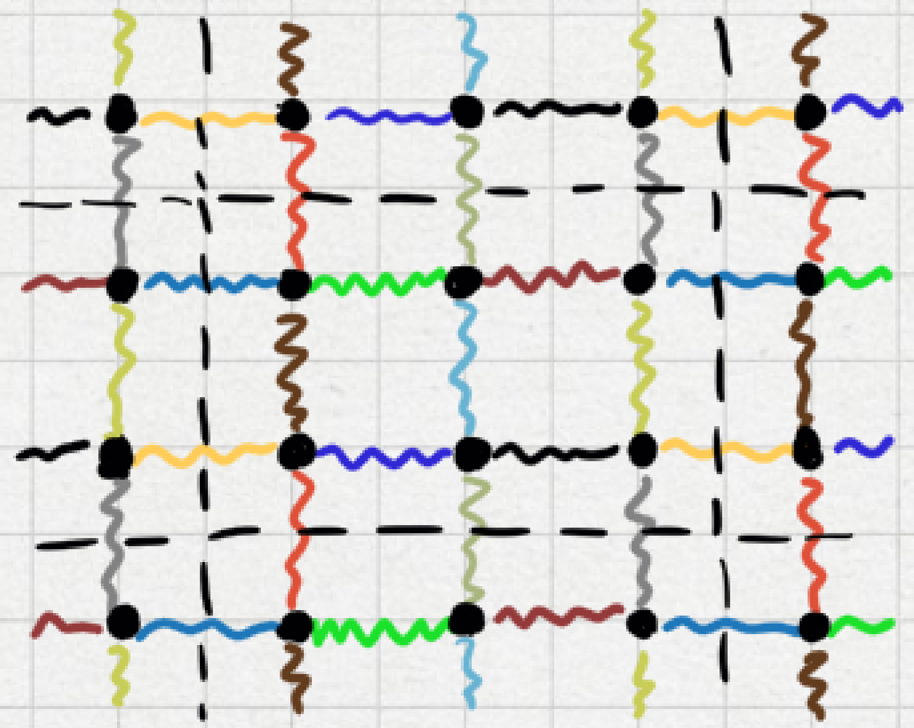
elliptic: $\sin\theta = \mathrm{sn}\left(\frac{2K}{\pi}\bar\theta \mid k\right)$

"massive": $q = \exp\left(-\frac{\pi K'}{K}\right) = \frac{1}{2}m\delta \to 0$

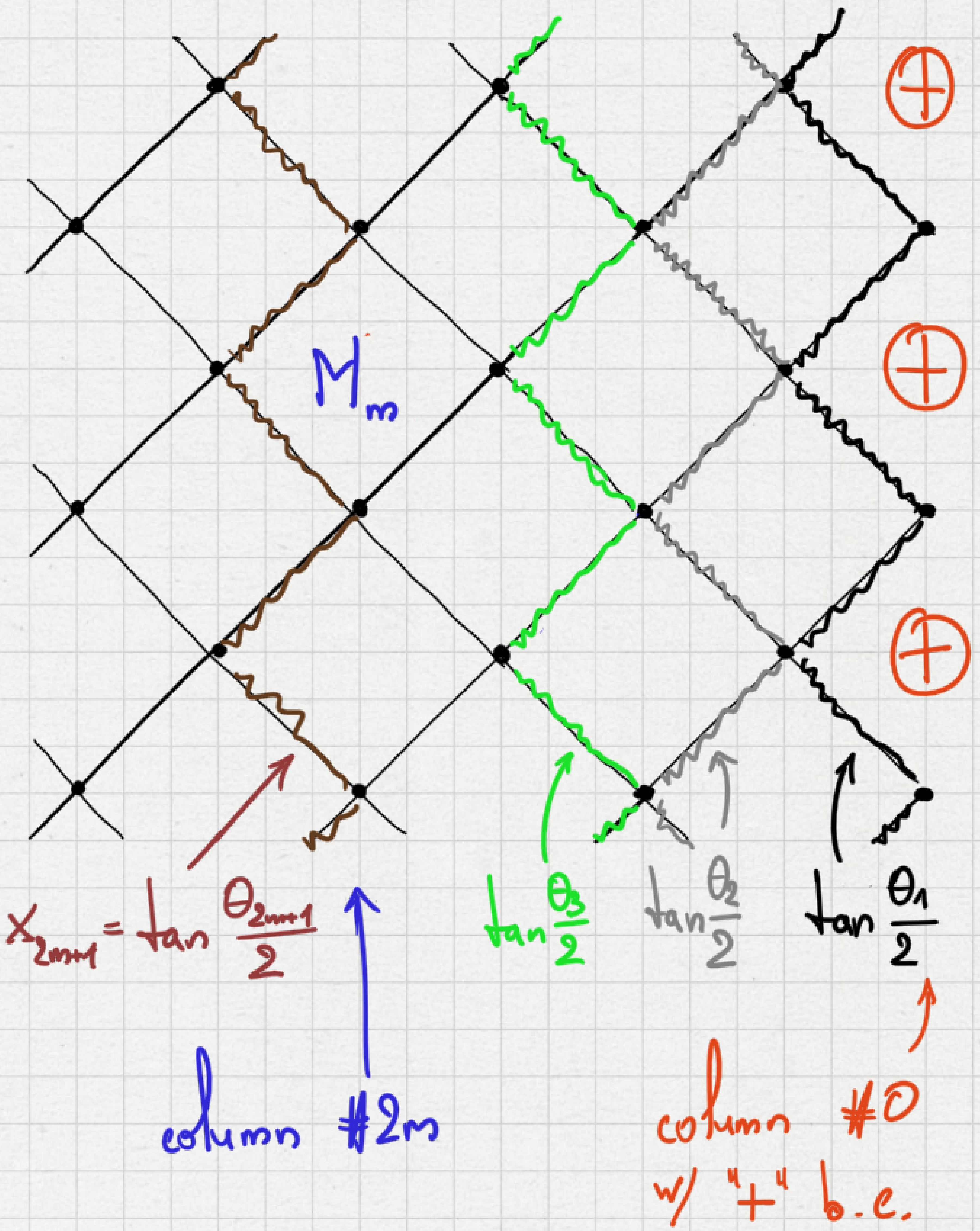


NOT UNDERSTOOD:

Dependence of $C(u)$ from u and from the fundamental domain for general double-periodic critical weights



MAGNETIZATION IN THE ZIG-ZAG LAYERED HALF-PLANE, MAIN RESULT:



$$B := \begin{bmatrix} \cos\theta_1 \cos\theta_2 & -\sin\theta_2 \sin\theta_3 & & 0 \\ & \cos\theta_3 \cos\theta_4 & -\sin\theta_4 \sin\theta_5 & \\ & 0 & \cos\theta_5 \cos\theta_6 & \ddots \\ & & & \ddots \end{bmatrix}$$

$\nu :=$ spectral measure of $Y := B^* B$ associated with e_1 ($\operatorname{supp} \nu \subset [0,1]$)

Theorem:

$$M_m = \frac{\det P_m Y^{1/2} P_m}{(\det P_m Y P_m)^{1/2}} = \frac{H_m[\lambda^{1/2} \nu]}{(H_m[\nu]\, H_m[\lambda \nu])^{1/2}}$$

where $H_m[\mu] := \det \left[\int_0^1 \lambda^{p+q} \mu(d\lambda) \right]_{p,q=0}^{m-1}$

(This holds for all $\theta_k \in (0, \pi/2)$, w/o any additional assumptions.)

MAGNETIZATION IN THE m-TH COLUMN OF THE ZIG-ZAG LAYERED HALF-PLANE

$$B := \begin{bmatrix} \cos\theta_1\cos\theta_2 & -\sin\theta_2\sin\theta_3 & & 0 \\ & \cos\theta_3\cos\theta_4 & -\sin\theta_4\sin\theta_5 & \\ 0 & & \cos\theta_5\cos\theta_6 & \\ & & & \ddots \end{bmatrix}$$

ν – spectral measure of B^*B

$$M_m = \frac{H_m[\lambda^{1/2}\nu]}{(H_m[\nu]\,H_m[\lambda\nu])^{1/2}} = \frac{\mathbb{E}_{\nu^{(m)}}[(\lambda_1\cdots\lambda_m)^{1/2}]}{(\mathbb{E}_{\nu^{(m)}}[\lambda_1\cdots\lambda_m])^{1/2}}$$

where

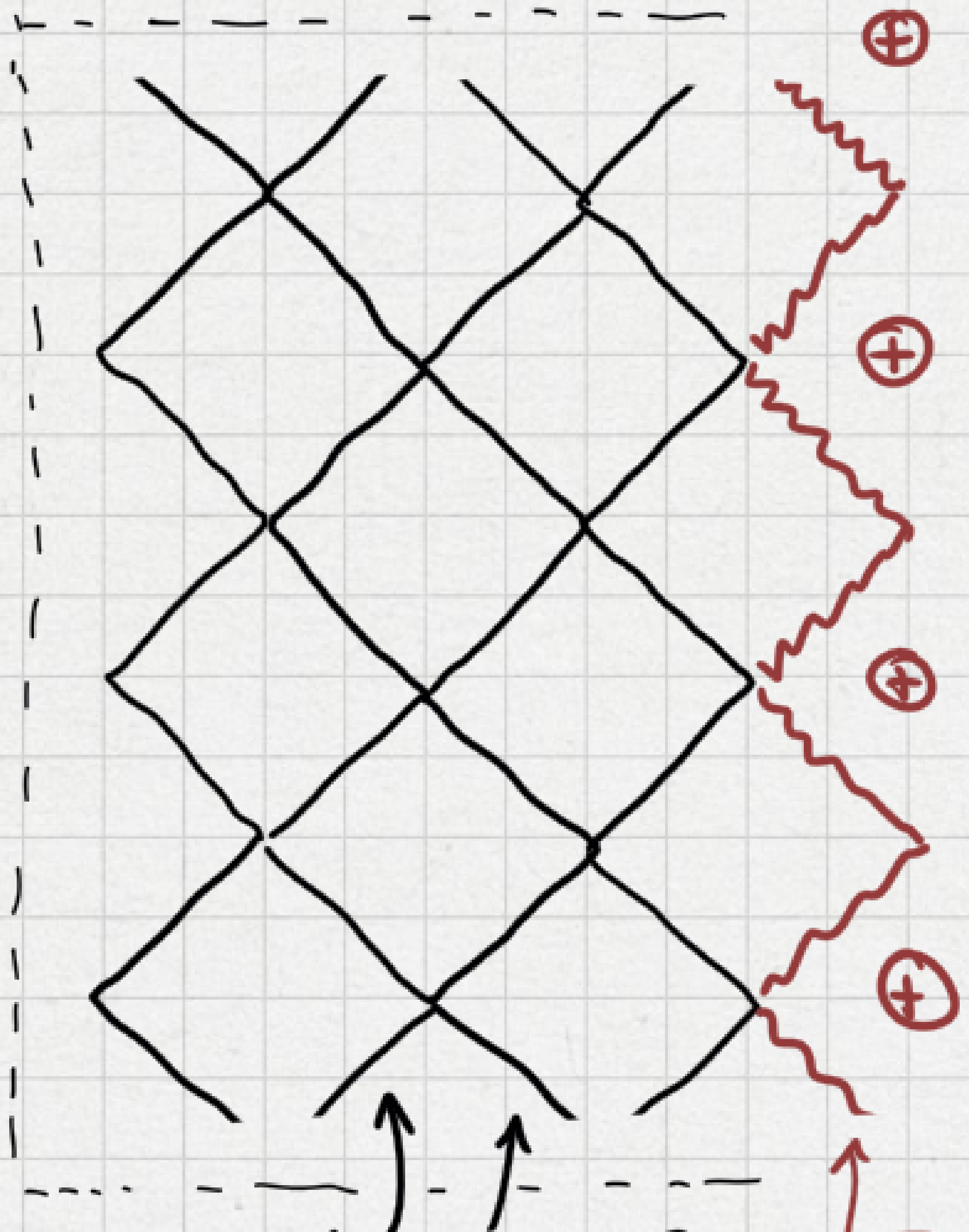
$$\frac{\nu^{(m)}(d\lambda_1,\dots,d\lambda_m)}{\nu(d\lambda_1)\cdots\nu(d\lambda_m)} \propto \prod_{1\le j<k\le m}(\lambda_k-\lambda_j)^2$$

Homogeneous case: ($\theta_k = \theta$ for all $k \ge 1$)

$$\operatorname{supp}\nu = \begin{cases} [\cos^2 2\theta,\, 1] & \text{if } \theta \le \pi/4 \\ \{0\} \cup [\cos^2 2\theta,\, 1] & \text{if } \theta > \pi/4 \end{cases}$$

Remark: $\Rightarrow$ exp. decay of M_m

$x < x_{crit}$ w/ boundary magnetic field $h > 0$



$e^{-2\beta J} = x = \tan\frac{\theta}{2}$

$e^{-h} = x_1 = \tan\frac{\theta_1}{2}$

$$\mathbb{P} \propto \exp\Big[\beta \sum_{u\sim u'} J\sigma_u\sigma_{u'} + h\sum_{u\sim \text{bdry}} \sigma_u\Big]$$

if $\cos^2\theta_1 < \cos 2\theta$ (small h)

then $\operatorname{supp}\nu = \{\lambda_0\} \cup [\cos^2 2\theta, 1]$, $\lambda_0 \in (0, \cos^2 2\theta)$

WETTING PHASE TRANSITION: ASYMPTOTIC ANALYSIS OF DETERMINANTS?

...... (explicit diagonalization of $\mathcal{J}$) ......

$x = \tan\frac{\theta}{2}$ $\quad x_1 = \frac{\tan\theta_1}{2}$

$$\boxed{M_m = (1-r)^{-3/2} \det\left[\alpha_{k-n} - \beta_{k+n} + (1-r)^{3/2}\gamma_{k+n}\right]_{k,n=0}^{m-1}}$$

$$\boxed{\begin{gathered} x < x_{crit} \\ 0 \le h < h^*(x) \end{gathered}}$$

Notation:

$q = \tan\theta < 1$

$r = 1 - \frac{\cos^2\theta_1}{\cos^2\theta} > q^2$

[h-small $\Longleftrightarrow \theta_1 \uparrow \pi$]

$$\alpha_s = \frac{1}{2\pi}\int_0^{2\pi} e^{-is\theta} w(e^{i\theta})\, d\theta$$

$$\beta_s = \frac{1}{2\pi}\int_0^{2\pi} e^{-is\theta} \xi(e^{i\theta}) w(e^{i\theta})\, d\theta$$

$$w(z) = |1 - q^2 z|$$

$$\xi(z) = \frac{(rz - q^2)(q^2 z - 1)}{(z - q^2)(q^2 z - r)}$$

$$\gamma_s = \frac{r^2 - q^4}{r^{3/2}(r - q^4)^{1/2}} \cdot \left(\frac{q^2}{r}\right)^s$$

Stat. phys $\Longrightarrow$ for all $h \ge 0$

$$M_m \xrightarrow[m\to\infty]{} M = (1 - \tan^4\theta)^{1/8}$$

Less understood:

analytic continuation to negative values of $(1-r)^{1/2}$, which correspond to $h < 0$

(recall that

$\operatorname{supp} \nu = \{\lambda_0(h)\} \cup [\cos^2 2\theta, 1]$).

WETTING PHASE TRANSITION :

STAT. PHYS. PICTURE (Pfister - Velenik '96)

$-h^* < -h < 0$

$-h \leq -h^*$

$$M_m = \pm(1-r)^{-\frac{3}{2}} \det\left[\alpha_{k-n} - \beta_{k+n} \pm (1-r)^{\frac{3}{2}} \gamma_{k+n}\right]_{k,n=0}^{m-1}$$

$$\gamma_j = \frac{r^2 - q^4}{r^{3/2}(r-q^4)^{1/2}} \left(\frac{q^2}{r}\right)^j$$

One should have

$$M_m(\theta,-h) \xrightarrow[m\to\infty]{} M(\theta) \quad \text{if } h < h^*$$

$$M_m(\theta,-h) \xrightarrow[m\to\infty]{} -M(\theta) \quad \text{if } h \geq h^*$$

It would be great to see this via asymptotic or spectral analysis !

$q = \tan\theta < 1$

$r = 1 - \dfrac{\cos^2\theta_1}{\cos^2\theta} > q^2$

$r = 1 \iff \theta_1 = \frac{\pi}{2} \iff h = 0$

$r = q^2 \iff h = h^*(\theta)$

[no γ_{k+j} term if $r \leq q^2$]

Rem: one can see a glimpse of this for the near-to-boundary magnetization

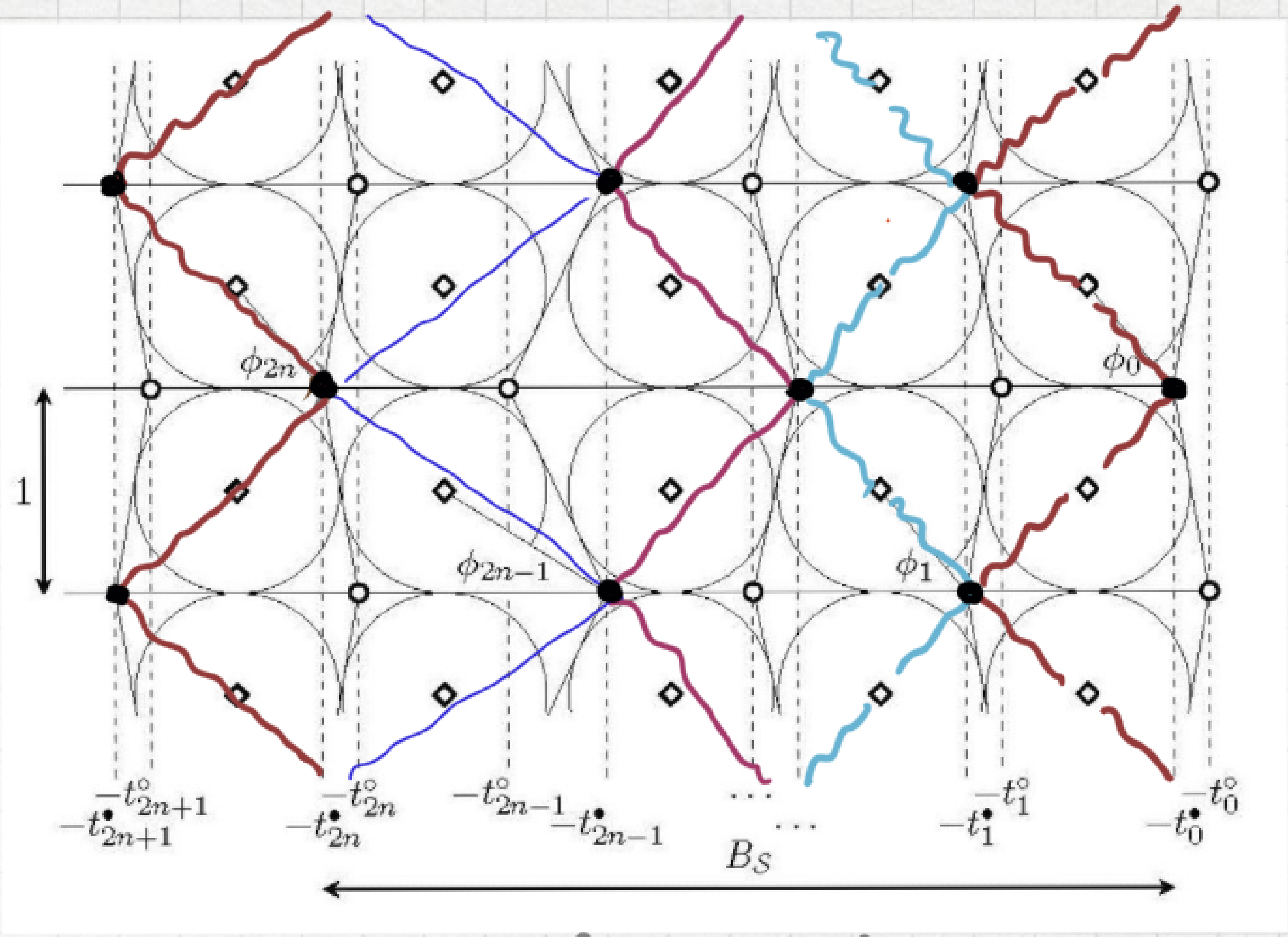
$$M_1(\theta,-h) = -|1-r|^{3/2}(\alpha_0 - \beta_0) + \gamma_0$$
$$= -M_1(\theta,h) + 2\gamma_0$$

disappears if $h \geq h^*(\theta)$

Rem: $\alpha_0 - \beta_0 = O((1-r)^2)$ as $r \to 1$ $\Rightarrow$ $M_1(\theta, h=0) = \gamma_0 = (1-\tan^4\theta)^{1/2}$ ["free" boundary conditions]

PERIODIC INTERACTIONS:

[TOWARDS UNIVERSALITY ON s-EMBEDDINGS]



[each critical double-periodic planar Ising model admits a canonical s-embeddings, rhombi $\rightsquigarrow$ tangential quads]

Reminder: in the critical model on $\mathbb{Z}^2$ and on all isoradial grids w/ critical Baxter's weights

$$\mathbb{E}[\sigma_{u_1}\sigma_{u_2}] \sim (\mathcal{C}_\sigma \delta^{1/8})^2 |u_2-u_1|^{-1/4} \quad \text{as } \delta^{-1}|u_2-u_1|\to\infty$$

$[\mathcal{C}_\sigma = 2^{1/6} e^{3/2\,\zeta'(-1)}]$

WANTED:

What is a generalization in the double-periodic case? $\left(\mathcal{C}(u_1)\mathcal{C}(u_2)|u_2-u_1|^{-1/4}\right)$

¿ "geometric" meaning ?

Periodic layered model:

$\theta_{k+n} = \theta_k \ \forall k \geq 1 \Rightarrow \mathrm{spec}(\mathcal{J}) = n$ segments

criticality $\iff$ $\mathrm{supp}\,\nu = [0, \ldots] \cup [\ldots, \ldots] \cup \ldots$

Observation:

aspect ratio of the fundamental domain under this drawing $= \mathcal{C}_\mathcal{J}$

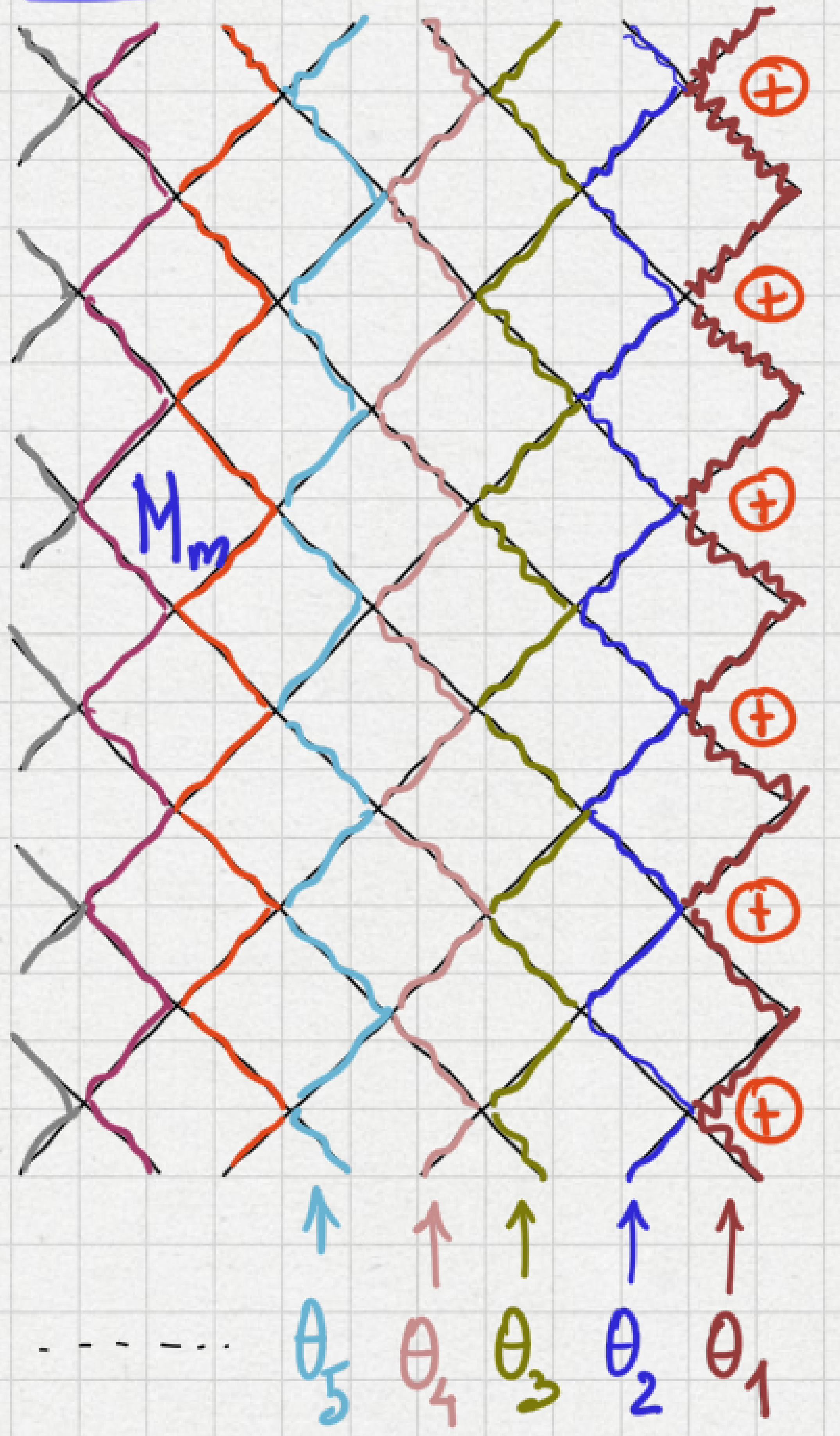
IDS($\mathcal{J}$) $\sim \mathcal{C}_\mathcal{J} \frac{1}{\pi}\sqrt{\lambda}$ as $\lambda \to 0$

TAKE-HOME MESSAGE:

$\exists$ a simple formula

$$\boxed{M_m = \frac{\det P_m \mathcal{Y}^{1/2} P_m}{(\det P_m \mathcal{Y} P_m)^{1/2}} = \frac{\mathbb{E}_{\nu^{(m)}}\left[(\lambda_1 \dots \lambda_m)^{1/2}\right]}{\left(\mathbb{E}_{\nu^{(m)}}[\lambda_1 \dots \lambda_m]\right)^{1/2}}}$$

$\leftarrow$ arXiv: 1904.09168



$$e^{-2p y_k} = x_k = \tan \frac{\theta_k}{2}$$

$$\frac{\nu^{(m)}(d\lambda_1, \dots, d\lambda_m)}{\nu(d\lambda_1) \dots \nu(d\lambda_m)} \propto \prod_{j<k} (\lambda_j - \lambda_k)^2$$

ν – spectral measure of $\mathcal{Y} = B^* B$

$$B = \begin{bmatrix} \cos\theta_1 \cos\theta_2 & -\sin\theta_2 \sin\theta_3 & & 0 \\ & \cos\theta_3 \cos\theta_4 & -\sin\theta_4 \sin\theta_5 & \\ & & \cos\theta_5 \cos\theta_6 & \ddots \\ 0 & & & \ddots \end{bmatrix}$$

Asymptotics of M_m as $m \to \infty$ $\longleftrightarrow$ properties of $\mathcal{Y}$ and ν

Several setups of interest for asymptotic analysis:

- wetting phase transition
- periodic interactions

– McCoy-Wu model: random i.i.d. θ_k

[Rem: works both ways: $\nu \longleftrightarrow \mathcal{Y} \longleftrightarrow \theta_k$]

Thank you!