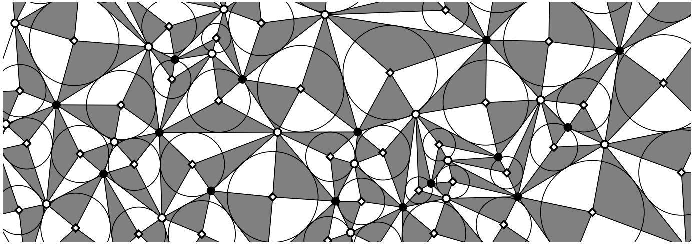
Planar Ising model:

convergence results on regular grids and s-embeddings of irregular graphs into $\mathbb{R}^{2,1}$



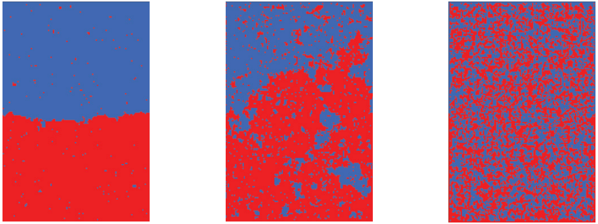
Dmitry Chelkak, ÉNS Paris

UCLA, January 6, 2022

Ferromagnetic, w/o external field 2d nearest-neighbor Lenz–Ising model (1920)

Given a piece of the **square grid** and a parameter $x \in (0,1)$ one assigns random spins $\sigma_u = \pm 1$ to its vertices so that the probability to get a configuration (σ_u) is proportional to $x^{\#\{u \sim u' : \sigma_u \neq \sigma_{u'}\}}$.

Archetypical example of a phase transition:



$x < x_{\mathrm{crit}}$ $\quad$ $x_{\mathrm{crit}} = \tan \frac{\pi}{8}$ $\quad$ $x > x_{\mathrm{crit}}$

[samples with $+1/-1$ (Dobrushin) boundary conditions]

Boltzmann–Gibbs:

▷ energy [external field $h = 0$]

$$H = -\sum_{u \sim u'} \sigma_u \sigma_{u'} - h \sum \sigma_u$$

▷ probability of a configuration (σ_u) is proportional to

$$\exp\left(-H[(\sigma_u)]/kT\right),$$

where T is the temperature

▷ $\sigma_u \sigma_{u'} = \pm 1 \rightsquigarrow x = e^{-2/kT}$.

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Ernst Ising
Wilhelm Lenz

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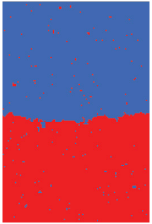

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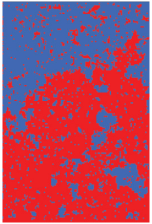
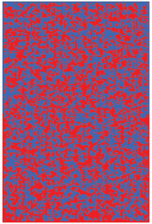


$x_{\mathrm{crit}} = \tan \frac{\pi}{8}$



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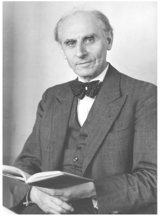

Pierre Curie (1895): metals lost ferromagnetic properties if $T \geqslant T_{\mathrm{crit}}$ [$T_{\mathrm{crit}} = 1043\mathrm{K}$ for iron]

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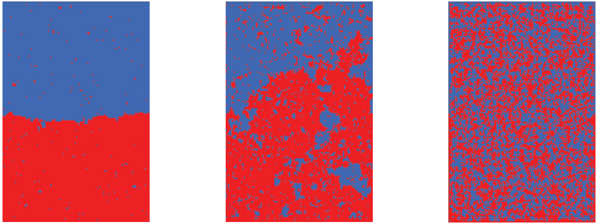

$x < x_{\text{crit}}$ $\quad$ $x_{\text{crit}} = \tan \frac{\pi}{8}$ $\quad$ $x > x_{\text{crit}}$

[Peierls'36] $\exists$ phase transition; [Kramers–Wannier'41] x_{crit}

Ernst Ising:

..I discussed the result of my paper widely with Professor Lenz and with Dr. Wolfgang Pauli, who at that time was teaching in Hamburg. There was some disappointment the linear model did not show the expected ferromagnetic properties.

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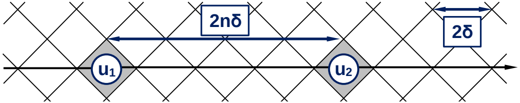
Archetypical example of a phase transition:

Power laws [e.g., infinite-volume 'diagonal correlations' Kaufman–Onsager'49, Yang'52, McCoy–Wu'66+]



(i) If $x < x_{\mathrm{crit}}$, then $M(x) := \lim_{n\to\infty}(\mathbb{E}[\sigma_{u_1}\sigma_{u_2}])^{\frac{1}{2}} > 0$;

moreover, $M(x) \sim \mathrm{cst}\cdot(x_{\mathrm{crit}} - x)^{\frac{1}{8}}$ as $x \uparrow x_{\mathrm{crit}}$;

(ii) If $x = x_{\mathrm{crit}}$, then $\delta^{-\frac{1}{4}} \cdot \mathbb{E}[\sigma_{u_1}\sigma_{u_2}] \sim \mathrm{C}_\sigma^2 \cdot (2n\delta)^{-\frac{1}{4}}$.

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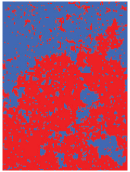

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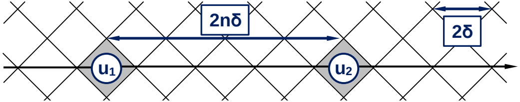

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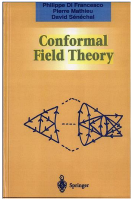
▷ Since then, many lattice models of phase transition were suggested and studied.

▷ Theoretical physics [Belavin–Polyakov–Zamolodchikov'84+]:

the small mesh size limit $\delta \to 0$ at the critical point $x = x_{\mathrm{crit}}$ should be

conformally invariant.



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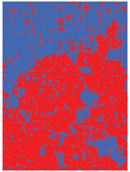
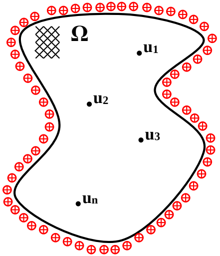


- **Theorem:** [w/ Hongler & Izyurov, *Ann. Math.'15*]

Let $x = x_{\mathrm{crit}}$ and $\Omega \subset \mathbb{C}$ be a simply connected domain. Let Ω^δ approximate Ω on the square grids $\delta\mathbb{Z}^2$ with $\delta \to 0$. Then,

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1} \dots \sigma_{u_n}] \to \mathcal{C}^n_\sigma \cdot \langle \sigma_{u_1} \dots \sigma_{u_n} \rangle^{+}_{\Omega}.$$



If $\varphi : \Omega \to \Omega'$ is conformal, then

$$\langle \sigma_{u_1} \dots \sigma_{u_n} \rangle^{+}_{\Omega} = \langle \sigma_{\varphi(u_1)} \dots \sigma_{\varphi(u_n)} \rangle^{+}_{\Omega'} \cdot \prod_{s=1}^{n} |\varphi'(u_s)|^{\frac{1}{8}}.$$

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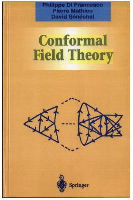

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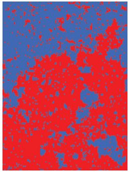

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Q: What makes the planar Ising model so special?

A: an important structure behind: 'discrete free fermions'

⇝ many intrinsic links with various subjects from orthogonal polynomials to cluster algebras

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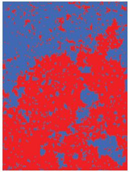

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Outline:

- ▷ background:
  - ▷ 'free fermions' and the propagation equation ;
  - ▷ discrete holomorphic functions on $\mathbb{Z}^2$ at x_{crit} ;
- ▷ CFT description at x_{crit} on regular grids as $\delta \to 0$;
- ▷ universality in the bi-periodic case and beyond ;
- ▷ embeddings of (irregular) planar graphs into $\mathbb{R}^{2,1}$.

▷ Since then, many lattice models of phase transition were suggested and studied.

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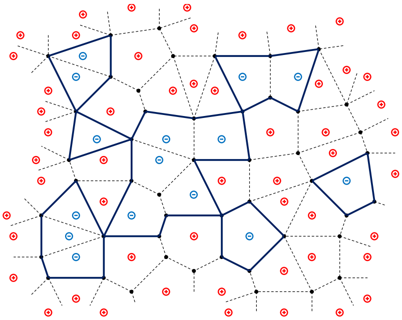
$\rightsquigarrow$ many intrinsic links with various subjects from orthogonal polynomials to cluster algebras

Ising model on a planar graph: definition and free fermions

▷ **Boltzmann–Gibbs:** given a weighted graph (G°, J) one assigns ± 1 spins to its vertices ($\Leftrightarrow$ faces of the dual graph $G^\bullet$) so that the probability of (σ_u)

is proportional to $\exp\big[-\frac{1}{kT}\sum_{e=\langle uu'\rangle}(-J_e\sigma_u\sigma_{u'})\big]$,

where $J_e > 0$ are called interaction constants.



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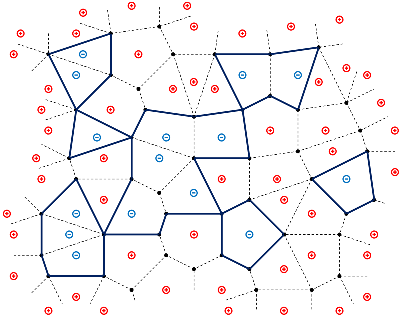

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▷ This can be written as

$$\mathbb{P}\big[\,\text{sample } (\sigma_u)\,\big] \;=\; \mathcal{Z}^{-1}\prod_{e=\langle uu'\rangle:\,\sigma_u\neq\sigma_{u'}} x_e\,,$$

where $x_e := \exp[-2J_e/kT] \in (0,1)$. The normalizing factor $\mathcal{Z} = \mathcal{Z}(G^\circ, x)$ is called the partition function.

▷ Equivalently: choose an even subgraph $\mathcal{C}$ of $G^\bullet$ with probability $\mathcal{Z}^{-1}x(\mathcal{C})$

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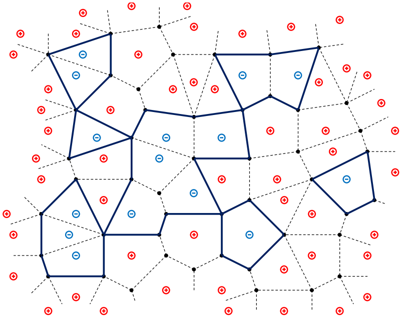

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[!] If G° is planar, then there is an important structure behind: $\mathcal{Z}=\mathrm{Pf}\,\mathcal{A}=(\det\mathcal{A})^{1/2}$, where $\mathcal{A}=-\mathcal{A}^\top$ is constructed out of (G°,x); the entries of $\mathcal{A}^{-1}$ are sometimes called *'fermionic observables'*.

Colloquial saying:

w/o external magnetic field, the planar Ising model is a free fermion model

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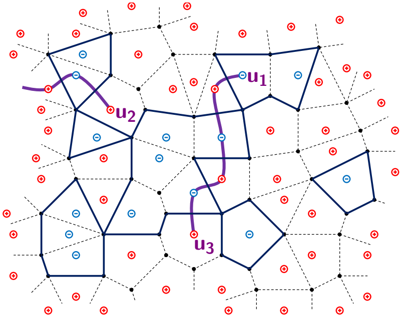
Q: Why is it important to compute $\mathcal{Z} = \mathcal{Z}(G^\circ, x)$?

A: e.g., one has

$$\mathbb{E}[\sigma_{u_1}\ldots\sigma_{u_n}] = \frac{\mathrm{Pf}\,\mathcal{A}_{[u_1,\ldots,u_n]}}{\mathrm{Pf}\,\mathcal{A}}$$



$\mathcal{A}_{[u_1,\ldots,u_n]}$: $x_e \mapsto -x_e$ **along cuts**

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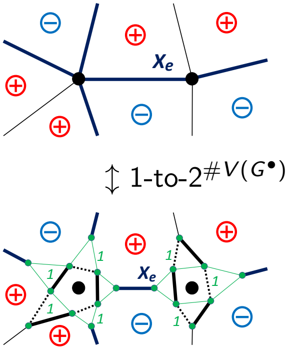

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"Proof": [not today ...] mappings onto (non-bipartite) dimer models, classical refs: [Kasteleyn'60s, Fisher'60s, ...]

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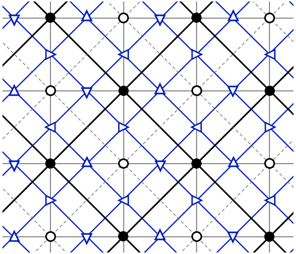

[if $G^\circ=\mathbb{Z}^2$, then there are four 'types' $\triangle,\triangleleft,\triangledown,\triangleright$ of vertices $c\in\Upsilon$]

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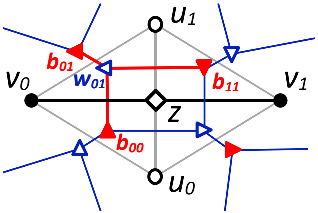
Ising model on a planar graph: definition and free fermions

$\Upsilon^\bullet \cup \Upsilon^\circ$



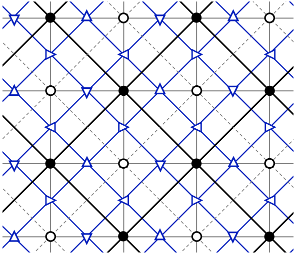
$\mathrm{Ker}\,\mathcal{A}$ **:**

functions on $\Upsilon^\bullet$ satisfying the equation

$$\mathbf{X}(\mathbf{b_{01}}) = \pm\mathbf{X}(\mathbf{b_{00}})\cos\theta_{\mathbf{z}} \pm \mathbf{X}(\mathbf{b_{11}})\sin\theta_{\mathbf{z}}$$

for $b_{00}, b_{01}, b_{11} \sim w_{01}$

▷ 'combinatorial bosonization': correspondence with a bipartite dimer model [Wu–Lin'75, Dubédat'11]



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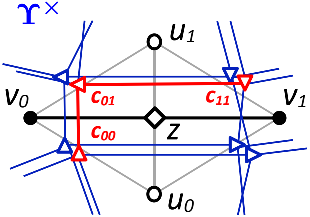
Notation: it is convenient to use the parametrization

$$x_e = \tan\tfrac{1}{2}\theta_e, \quad \theta_e \in (0, \tfrac{\pi}{2})$$

[recall that $x_{\mathrm{crit}} = \tan\frac{\pi}{8}$ for $\mathbb{Z}^2$]

Ising model on a planar graph: definition and free fermions



Ker $\mathcal{A}$:

spinors on $\Upsilon^\times$

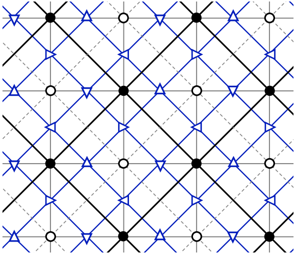
satisfying the equation

$$\mathbf{X}(\mathbf{c}_{01}) = \mathbf{X}(\mathbf{c}_{00})\cos\theta_z + \mathbf{X}(\mathbf{c}_{11})\sin\theta_z$$

for $c_{00} \sim c_{01} \sim c_{11}$

$\triangleright$ $\Upsilon^\times$ branches over all $z \in \diamondsuit$, $v \in G^\bullet$, $u \in G^\circ$

[down-to-earth: fix a section of $\Upsilon^\times$ & add signs]



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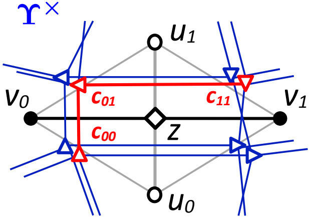
‘Interpretation’ of $\mathcal{A}$ for the homogeneous model on the square grid



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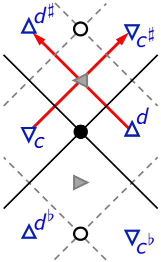
Critical model on $\mathbb{Z}^2$: $\theta = \frac{\pi}{4}$



$\rightsquigarrow$ **discrete CR equations**

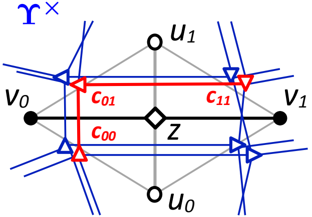
$$X(d^\sharp) - X(d) = X(c^\sharp) - X(c)$$

$$X(d) - X(d^\flat) = X(c^\flat) - X(c)$$

$\rightsquigarrow$ classical discrete harmonic functions on each ‘type’ of $c \in \Upsilon$

‘Fonctions preharmoniques et fonctions preholomorphes’ [Jacqueline Ferrand, 1944]

‘Interpretation’ of $\mathcal{A}$ for the homogeneous model on the square grid



$\mathrm{Ker}\,\mathcal{A}$ **:**

spinors on $\Upsilon^\times$

satisfying the equation

$$\mathbf{X}(\mathbf{c_{01}}) = \mathbf{X}(\mathbf{c_{00}})\cos\theta_z + \mathbf{X}(\mathbf{c_{11}})\sin\theta_z$$

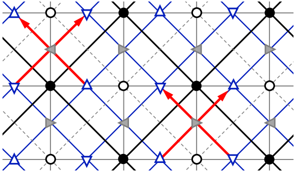
for $c_{00} \sim c_{01} \sim c_{11}$

▷ $\Upsilon^\times$ branches over all $z \in \diamondsuit$, $v \in G^\bullet$, $u \in G^\circ$

[down-to-earth: fix a section of $\Upsilon^\times$ & add signs]

Critical model on $\mathbb{Z}^2$: $\theta = \frac{\pi}{4}$

⇝ **discrete CR equations**



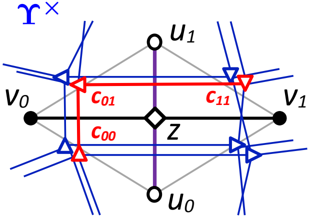
⇝ classical discrete harmonic functions on each ‘type’ of $c \in \Upsilon$

[!] If G° is planar, then there is an important structure behind: $\mathcal{Z} = \mathrm{Pf}\,\mathcal{A} = (\det \mathcal{A})^{1/2}$, where $\mathcal{A} = -\mathcal{A}^\top : \mathbb{R}^\Upsilon \to \mathbb{R}^\Upsilon$ and $\Upsilon :=$ **the medial graph of $\Lambda := G^\circ \cup G^\bullet$**

Warning: for general planar graphs (G°, x), $\mathcal{A}$ <u>cannot</u> be written as a usual Laplacian.

Not quite random walks [!]

‘Interpretation’ of $\mathcal{A}$ for the homogeneous model on the square grid



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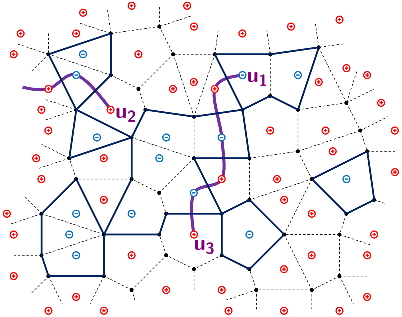


$\mathcal{A}_{[u_1,\ldots,u_n]}$: $x_e \mapsto -x_e$ **along cuts**

Spin correlations:

$$\mathbb{E}[\sigma_{u_1}\ldots\sigma_{u_n}] = \frac{\mathrm{Pf}\,\mathcal{A}_{[u_1,\ldots,u_n]}}{\mathrm{Pf}\,\mathcal{A}}$$

$\mathcal{A}_{[u_1,\ldots,u_n]}$ acts similarly to $\mathcal{A}$ on functions/spinors that have (additional) **branchings over $u_1,\ldots,u_n$.**

Reminder: parametrization

$$x_e = \tan\tfrac{1}{2}\theta_e,\quad \theta_e \in (0,\tfrac{\pi}{2})$$

[recall that $x_{\mathrm{crit}} = \tan\frac{\pi}{8}$ for $\mathbb{Z}^2$]

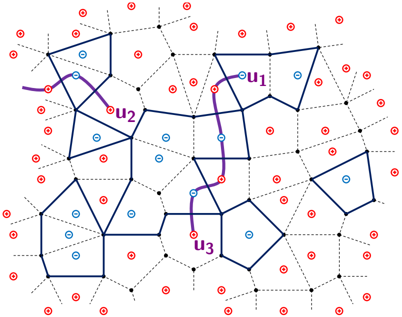
‘Interpretation’ of $\mathcal{A}$ for the homogeneous model on the square grid

Homogeneous model on $\delta\mathbb{Z}^2$ as $\delta \to 0$:

the matrix $\mathcal{A} = -\mathcal{A}^\top$ is a discretization of the (massive) Dirac operator $f \mapsto \partial\overline{f} + imf$,

$$m \asymp \delta^{-1} \cdot (x - x_{\mathrm{crit}}), \quad x_{\mathrm{crit}} = \tan\tfrac{\pi}{8}$$

$\rightsquigarrow$ **isomonodromic τ-functions** [Sato–Miwa–Jimbo’77, Wu–McCoy–Tracy–Barouch’76, . . . , Palmer’07]



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In finite $\Omega \subset \mathbb{C}$: **Riemann-type boundary conditions $\overline{f} = \tau f$,**

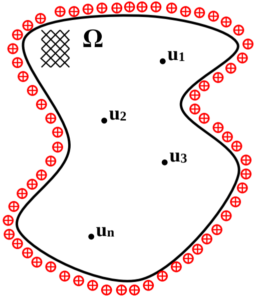
τ = ‘unit tangent vector to $\partial\Omega$’

Convergence of correlations in discrete domains at criticality: Ising CFT

- **Theorem:** [Ch.–Hongler–Izyurov, *Ann. Math.'15*]

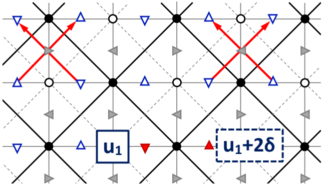
Let $x = x_{\mathrm{crit}}$, $\Omega \subset \mathbb{C}$ be a (bounded) simply connected domain and $\Omega^\delta \subset \delta\mathbb{Z}^2$ approximate Ω as $\delta \to 0$. Then,

$$\delta^{-\frac{n}{8}} \cdot \mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1} \dots \sigma_{u_n}] \;\to\; \mathcal{C}^n_\sigma \cdot \langle \boldsymbol{\sigma}_{u_1} \dots \boldsymbol{\sigma}_{u_n} \rangle^{+}_{\Omega}.$$



Idea: control $\dfrac{\mathbb{E}[\sigma_{u_1+2\delta} \dots \sigma_{u_n}]}{\mathbb{E}[\sigma_{u_1} \dots \sigma_{u_n}]} = \mathcal{A}^{-1}_{[u_1,\dots,u_n]}(u_1 + \tfrac{1}{2}\delta, u_1 + \tfrac{3}{2}\delta)$

up to $o(\delta)$ by viewing the kernel $\mathcal{A}^{-1}_{[u_1,\dots,u_n]}(u_1 + \tfrac{1}{2}\delta, \cdot\,)$ as a solution to an appropriate discrete Riemann-type b.v.p. Non-trivial technicalities at $\partial\Omega^\delta$ and near singularities.



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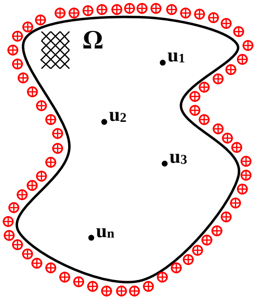

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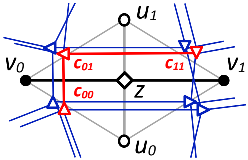
Important tool: function H_X [Smirnov'06+ for $(\mathbb{Z}^2, x_{\mathrm{crit}})$]

$$X(c_{01}) = X(c_{00}) \cos\theta_z + X(c_{11}) \sin\theta_z \quad (\text{on } \Upsilon^\times)$$

$\Rightarrow$ one can define $H_X(v_p^\bullet) - H_X(v_q^\circ) := (X(c_{pq}))^2$ on Λ.



Convergence of correlations in discrete domains at criticality: Ising CFT

$\langle \ldots \rangle$ [arXiv:2103.10263 w/ Hongler and Izyurov]

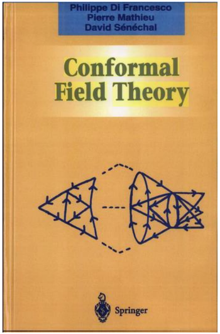
▷ convergence of mixed correlations: spins $\delta^{-\frac{1}{8}}\sigma_u$, fermions $\delta^{-\frac{1}{2}}\psi$, energy densities $\delta^{-1}(\sigma_u\sigma_{u'}-\sqrt{2}/2)$, $u\sim u'$, etc

in multiply connected domains, with mixed boundary conditions. No explicit formulae are available; the limits are defined via appropriate Riemann-type b.v.p.

▷ *consistent definition* of Ising CFT correlations $\langle \mathcal{O}\rangle_\Omega^{\mathfrak{b}}$ in multiply connected domains + *fusion rules:* e.g.,

as $w\to z$ one has *both* $\langle \psi_w\psi_z^\star \mathcal{O}\rangle_\Omega^{\mathfrak{b}} = \frac{i}{2}\langle \varepsilon_z\mathcal{O}\rangle_\Omega^{\mathfrak{b}} + \ldots$

$$\langle \sigma_w\sigma_z \mathcal{O}\rangle_\Omega^{\mathfrak{b}} = |w-z|^{-\frac{1}{4}}\langle \mathcal{O}\rangle_\Omega^{\mathfrak{b}} + \tfrac{1}{2}|w-z|^{\frac{3}{4}}\langle \varepsilon_z\mathcal{O}\rangle_\Omega^{\mathfrak{b}} + \ldots$$



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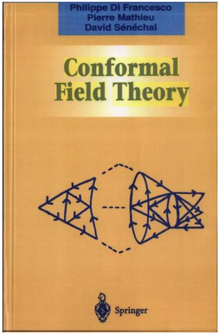

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▷ *consistent definition* of Ising CFT correlations $\langle \mathcal{O} \rangle_\Omega^{\mathfrak{b}}$ in multiply connected domains + *fusion rules.*

Moreover, similar results are now available for the **near-critical model** $x = x_{\mathrm{crit}} + m\delta$

The limits of correlation functions are **not conformally covariant** and defined via solutions of appropriate Riemann-type b.v.p.'s for $\partial\overline{f} + imf = 0$ ('massive' fermions).

[SC Park arXiv:1811.06636, 2103.04649 ; Ch.–Izyurov–Mahfouf arXiv:2104.12858 ; *in progress*]

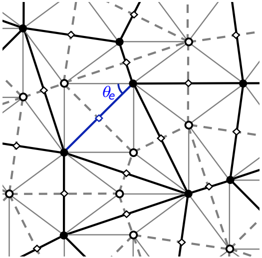
Universality on isoradial grids/rhombic tilings (Baxter's Z-invariant Ising model)

G°: each face is inscribed into a circle of common radius δ;
[equivalently, $\Lambda = G^\circ \cup G^\bullet$ form a tiling of the plane by rhombi]

special interaction parameters: $x_e = \tan \frac{1}{2}\theta_e$.

All the convergence results available on $\mathbb{Z}^2$ (correlations, interfaces, loop ensembles) hold within this class of models.

[w/ Smirnov, *Inv. Math.*'12] **"Universality[!?]** in the 2D Ising model and conformal invariance of fermionic observables"



"Proof": This setup still leads to a 'nice' notion of discrete holomorphic functions [Duffin'68], more-or-less the same ideas/techniques as for $\mathbb{Z}^2$ can be applied.

Particular cases:

triangular $x_{\mathrm{crit}} = \tan \frac{\pi}{6}$

hexagonal $x_{\mathrm{crit}} = \tan \frac{\pi}{12}$

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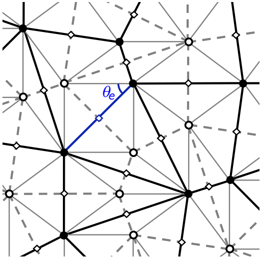
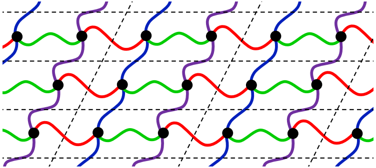


Problem: this framework is too rigid

e.g., consider a *'generic' bi-periodic Ising model:* the criticality condition is known [$x(\mathcal{E}_{00}) = x(\mathcal{E} \setminus \mathcal{E}_{00})$] but such models do <u>not</u> admit isoradial embeddings ...

Wanted: to draw (G°, x) so that the matrix $\mathcal{A}$ admits a 'discrete-complex-analysis' interpretation.



criticality condition: $1 + x_3x_4 = x_3 + x_4 + x_1x_2(1+x_3)(1+x_4)$

S-embeddings [arXiv:1712.04192, 2006.14559, ...]

The framework of rhombic tilings is too rigid:

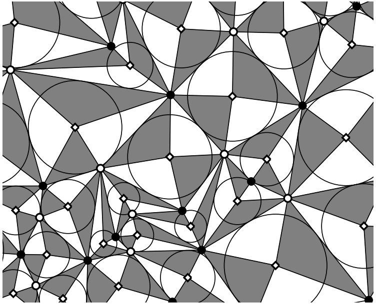
▷ it is even not general enough to be applied to ‘generic’ critical bi-periodic models

Not to mention really interesting setups:

▷ e.g., $\mathbb{Z}^2$ with random interaction constants x_e

▷ random planar maps carrying the Ising model

[?] ‘discrete-complex-analysis’ interpretation of $\mathcal{A}$



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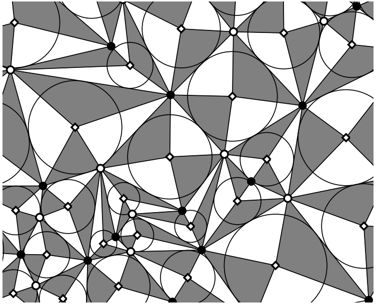
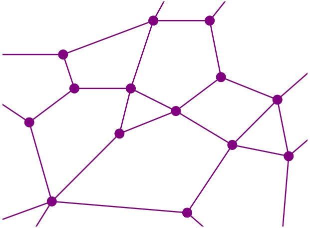




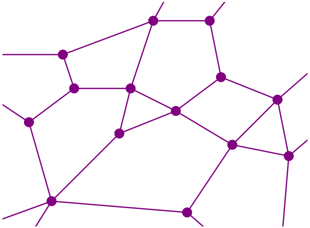
Analogy: Tutte's harmonic embedding $\mathcal{H} : G \to \mathbb{C}$

is a complex-valued (local) solution of $\Delta\mathcal{H} = 0$:

the position of each vertex is the (weighted) barycenter of the positions of its neighbors

[$\Rightarrow$ the random walk on $\mathcal{H}(G)$ is a martingale $\Rightarrow$ ...]

S-embeddings [arXiv:1712.04192, 2006.14559, ...]



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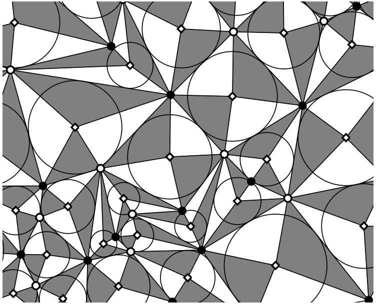

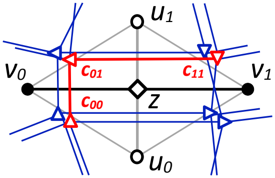

[!] S-embeddings into $\mathbb{R}^{2,1} \cong \mathbb{C}\times\mathbb{R}$: [tilings of the plane by tangential quads]

(local) $\mathbb{C}$-solution $\mathcal{X}(c_{01}) = \mathcal{X}(c_{00})\cos\theta_z + \mathcal{X}(c_{11})\sin\theta_z$

$\Updownarrow$

s-embedding

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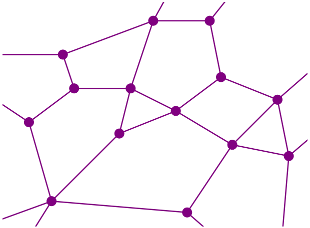

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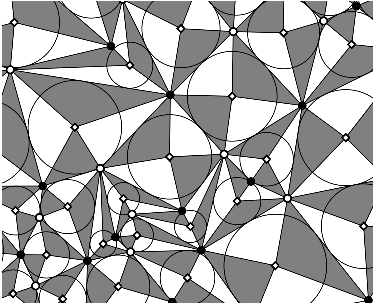

Particular case: for rhombic tilings of mesh size δ one has

$$\mathcal{Q}_\mathcal{X} = \pm\tfrac{1}{2}\delta.$$

The third coordinate disappears as $\delta \to 0$.

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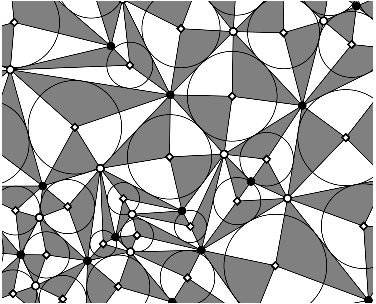

▷ s-holomorphic functions

$$\Pr[\,F(z);\overline{\mathcal{X}(c)}\,\mathbb{R}\,] \;=\; X(c)/\mathcal{X}(c)\,.$$

($X\in\mathbb{R}$ satisfies the 3-terms equation $\Leftrightarrow$ $F\in\mathbb{C}$ exists)

▷ $F(z)d\mathcal{S}_{\mathcal{X}}+\overline{F(z)}d\mathcal{Q}_{\mathcal{X}}$ is a closed form

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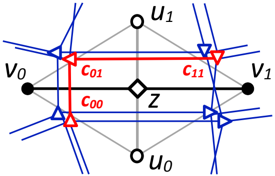

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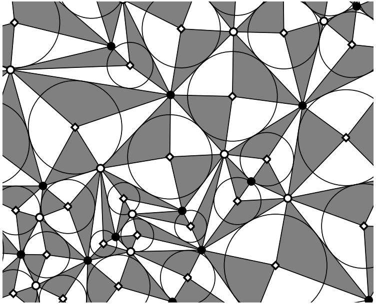

[cf. Ch.–Smirnov'12]:

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$\Rightarrow$ the third coordinate disappears as $\delta \to 0$ $\Rightarrow$

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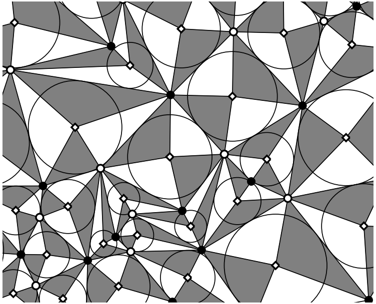

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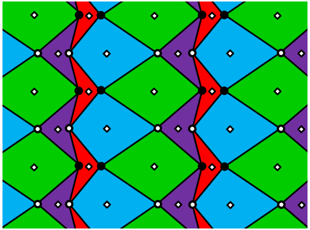
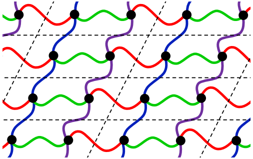


Theorem: conformal invariance and universality of the limit (of interfaces) for all critical bi-periodic models.

"Proof:" there exists a canonical $\mathcal{S}$ with bi-periodic $\mathcal{Q}=\mathcal{Q}^\delta=O(\delta)$

$\rightsquigarrow$ holomorphic functions as $\delta\to 0$



S-embeddings [arXiv:1712.04192, 2006.14559, ...]

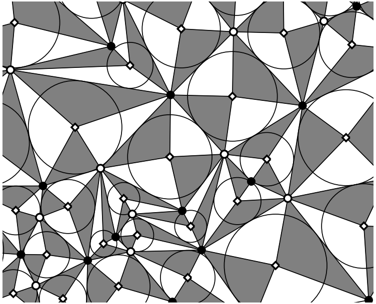
'discrete-complex-analysis' interpretation of $\mathcal{A}$:

▷ s-holomorphic functions

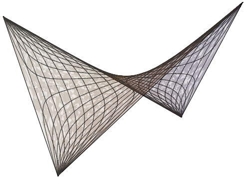
$$\mathrm{Pr}[\,F(z);\overline{\mathcal{X}(c)}\,\mathbb{R}\,] \;=\; X(c)/\mathcal{X}(c)\,.$$

($X\in\mathbb{R}$ satisfies the 3-terms equation $\Leftrightarrow$ $F\in\mathbb{C}$ exists)

▷ $F(z)d\mathcal{S}_{\mathcal{X}}+\overline{F(z)}d\mathcal{Q}_{\mathcal{X}}$ is a closed form



If $(\mathcal{S}^\delta,\mathcal{Q}^\delta)\to(z,t(z))$ **=: S $\subset\mathbb{R}^{2,1}$** $\Rightarrow$ subseq. limits of fermionic observables satisfy the condition $f(z)dz+\overline{f(z)}dt$ – closed form, which can be written as the conjugate Beltrami equation



$$\partial_\zeta\overline{f}=\overline{\nu}\cdot\partial_\zeta f \qquad \nu=-\partial_\zeta t/\partial_\zeta z$$

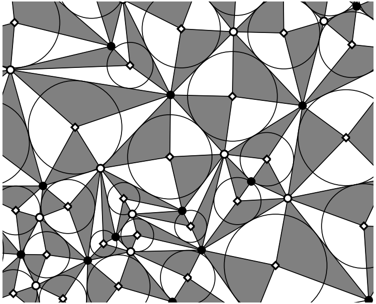
in the **conformal parametrization ζ of the surface S $\subset\mathbb{R}^{2,1}$**

S-embeddings [arXiv:1712.04192, 2006.14559, ...]

▷ **Assume that $(\mathcal{S}^\delta, \mathcal{Q}^\delta) \to$ smooth $\mathrm{S} \subset \mathbb{R}^{2,1}$.**

Then, the functions $\phi := z_\zeta^{1/2} \cdot f + \overline{z}_\zeta^{1/2} \cdot \overline{f}$ satisfy the equation $\partial_\zeta \overline{\phi} + im\phi = 0$, where

- ▷ ζ is a conformal parametrization of $\mathrm{S} \subset \mathbb{R}^{2,1}$,
- ▷ m is the **mean curvature of S** multiplied by its metric element in the parametrization ζ.



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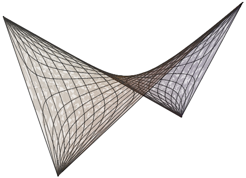

$$\partial_\zeta \overline{f} = \overline{\nu} \cdot \partial_\zeta f$$
$$\nu = -\partial_\zeta t / \partial_\zeta z$$

in the **conformal parametrization ζ of the surface $\mathrm{S} \subset \mathbb{R}^{2,1}$**

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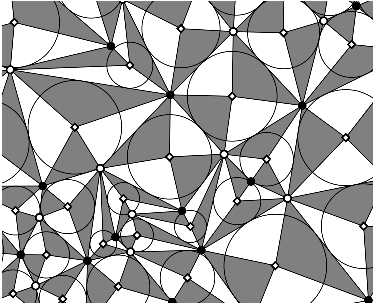

'Non-technical' open questions/research directions:

▷ to understand how these embeddings/surfaces behave in various setups of interest:
- random media (e.g., random interaction constants x_e on $\mathbb{Z}^2$);
- critical random planar maps weighted by the Ising model [? ⇝ Liouville CFT ?]: sounds like 'canonical fluctuations' near Lorentz-minimal surfaces should arise.

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▷ as well as many smaller projects at all levels to develop the theory further.