Critical 2D Ising model: convergence of interfaces and universality

Dmitry Chelkak, PDMI (Steklov Institute) & St.Petersburg University

joint work with *Stanislav Smirnov* (Geneva)

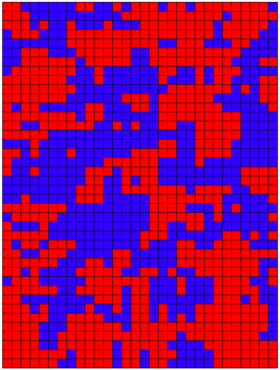
`arXiv:0810.2188`: "Discrete complex analysis on isoradial graphs"

`arXiv:0910.2045`: "Universality in the 2D Ising model and conformal invariance of fermionic observables"

`arXiv:10??`: "Conformal invariance of the 2D Ising model at criticality"

IMS Meeting 2010, Gothenburg, August 12

2D Ising model: (square grid)



Spins $\sigma_i = +1$ or -1.

Hamiltonian:

$$H = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \,.$$

Partition function:

$$\mathbb{P}(\text{conf.}) \sim e^{-\beta H} \sim x^{\#\,\langle +- \rangle},$$

where

$$x = e^{-2\beta} \in [0,1]\,.$$

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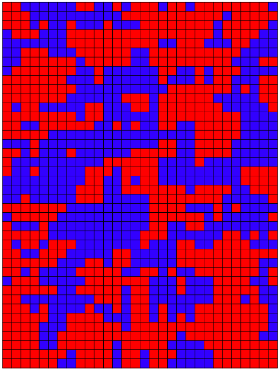

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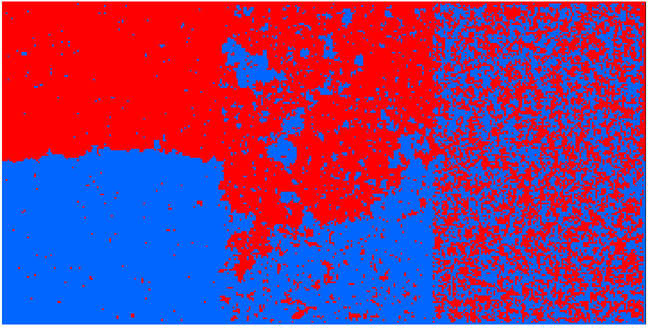
Critical value (cf. *Vincent Beffara* talk): $x_{\text{crit}} = 1/(\sqrt{2}+1)$

[Kramers-Wannier '41, Onsager'44]

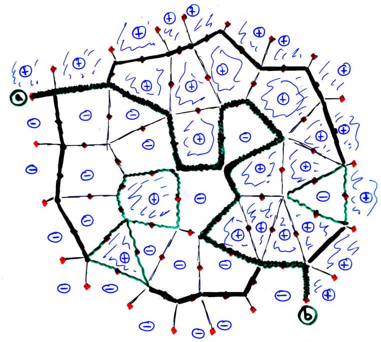
Phase transition:



$x < x_{\text{crit}}$ $\quad$ $x = x_{\text{crit}}$ $\quad$ $x > x_{\text{crit}}$

(Dobrushin boundary values: two marked points a, b on the boundary; -1 on the arc (ab), $+1$ on the opposite arc (ba))

Universality. **Critical Ising model on other planar graphs:**



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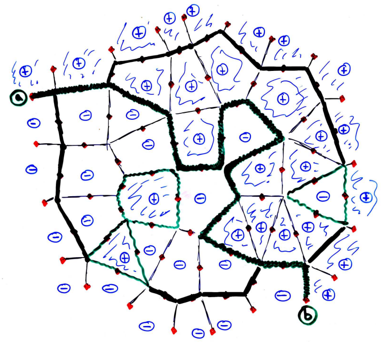

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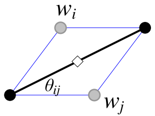
Isoradial graphs/rhombic lattices:

[Baxter'78: Z-invariant graphs;

Mercat'01; Kenyon'02; Costa-Santos'06;

Riva-Cardy'06; Boutillier-de Tilière'08-10; ...]



$x_{ij} = \tan \frac{1}{2}\theta_{ij}$. Why?:

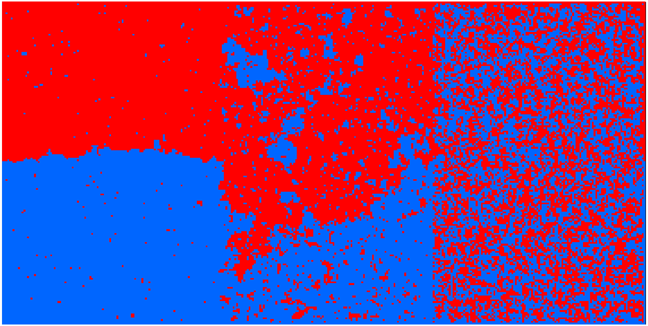
- *locality;*
- *self-duality;*
- *$Y - \Delta$ invariance.*

Conformal invariance:

Observables (partition functions; spin correlations; crossing probabilities)

$\updownarrow$

Geometry (interfaces, loop ensembles/soups)



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"$\Uparrow$": SLE computations

"$\Downarrow$": Conformal martingale principle

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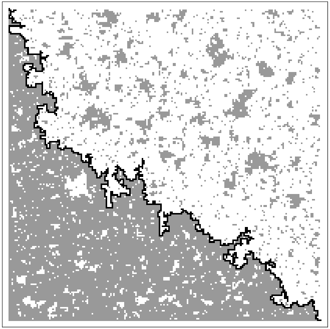

THEOREM 1 (Ch.-Smirnov): *Interfaces of the critical Ising model on arbitrary isoradial graphs converge to SLE_3 as mesh size tends to* 0.

Remark: See `arXiv:0910.2045` for the main ingredient of the proof – uniform convergence of the discrete martingale observable.

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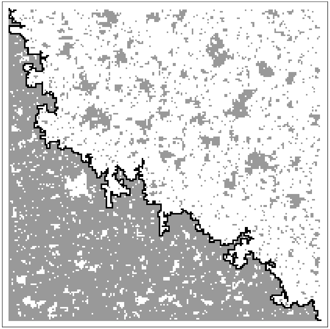
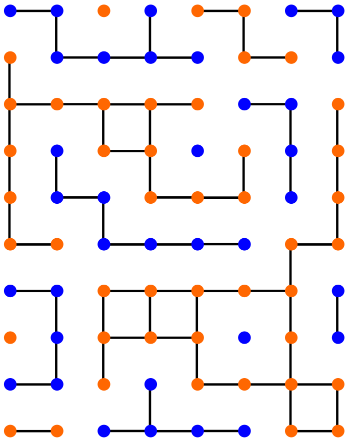


THEOREM 1 (Ch.-Smirnov): *Interfaces of the critical Ising model on arbitrary isoradial graphs converge to SLE_3 as mesh size tends to* 0.

Remark: The crucial step – construction of the discrete martingale observable (aka holomorphic fermion) – was done by S.Smirnov (on a square grid).

FK-Ising model (random cluster representation):



$$\mathbb{P}(\text{spins conf.}) \sim x^{\#\langle +- \rangle}$$

$$= \prod_{<ij>} \left[x + (1-x) \cdot \mathbb{1}_{s(i)=s(j)} \right]$$

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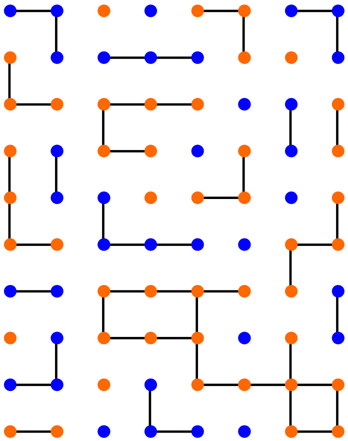

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Remark: Open edges connect equal spins (but not all of them)

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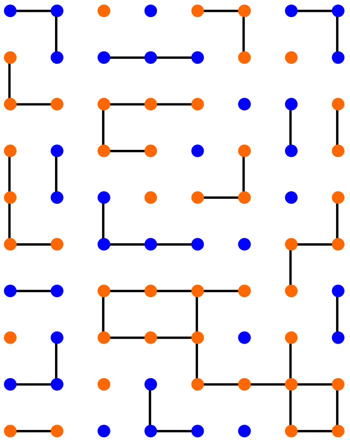

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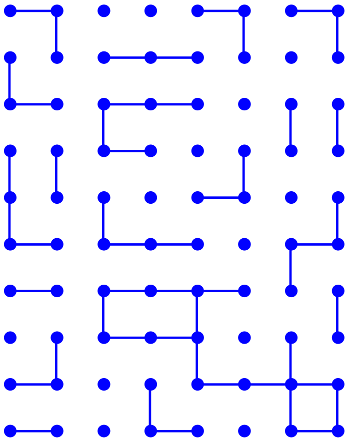

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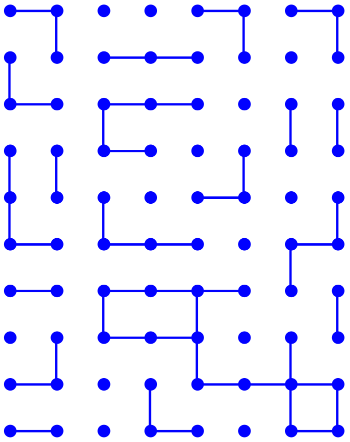

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since

$\#closed + \#open\ edges = const$

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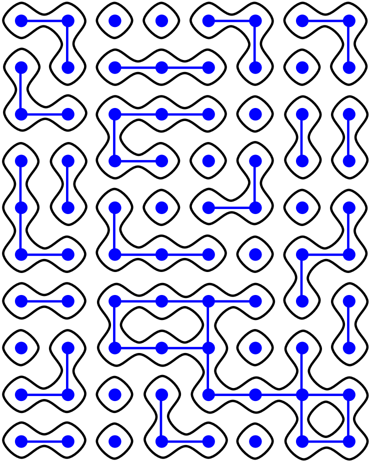

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$\sim \sqrt{2}^{\#loops}[(1-x)/(x\sqrt{2})]^{\#open}$

since $\#loops - \#open\ edges$
$= 2\#clusters + const$

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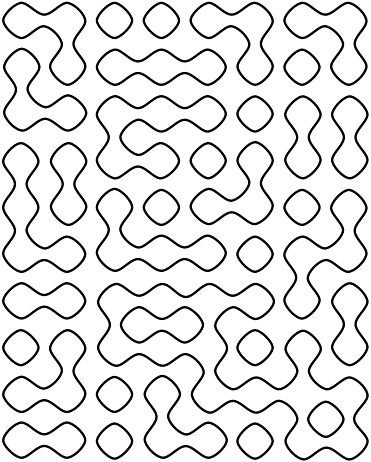

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$\mathbf{x} = \mathbf{x}_{\mathrm{crit}}$: $= \sqrt{2}^{\#loops}$

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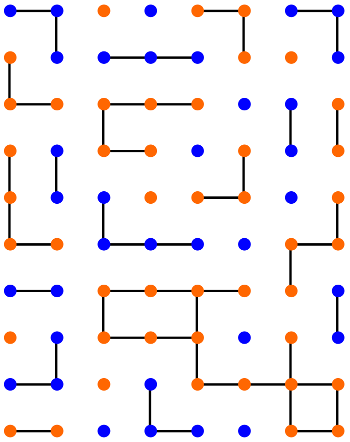

- Spin to FK:

  Run bond percolation ($p = 1 - x_{\mathrm{crit}}$) on spin-clusters;

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  Toss a fair coin for each FK-cluster;

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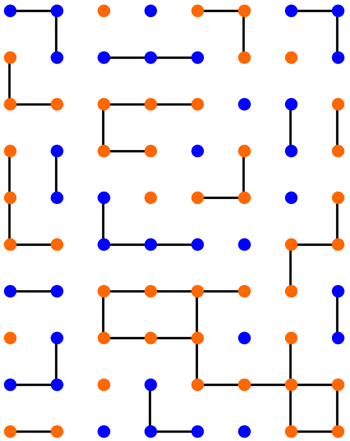

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- correlations $\leftrightarrow$ connectivity:

  $\mathbb{E}_{\mathrm{spin}}[\sigma_i \sigma_j] = \mathbb{P}_{\mathrm{FK}}[i \leftrightarrow j]$

  since $\mathbb{P}_{\mathrm{spin}}[\sigma_i = \sigma_j] = \mathbb{P}_{\mathrm{FK}}[i \leftrightarrow j] + \frac{1}{2}\mathbb{P}_{\mathrm{FK}}[i \not\leftrightarrow j]$

Conformal invariance:

Observables (partition functions; spin correlations; crossing probabilities)

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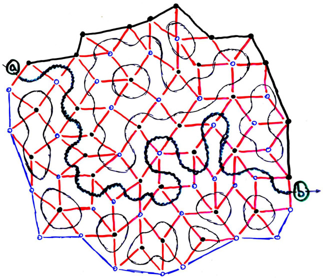
Geometry (interfaces, loop ensembles/soups)



THEOREM 2 (Ch.-Smirnov): *Interfaces of the FK-Ising model on arbitrary rhombic lattice converge to $SLE_{16/3}$ as mesh size tends to* 0.

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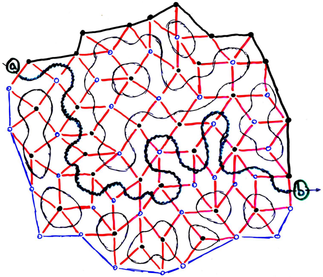

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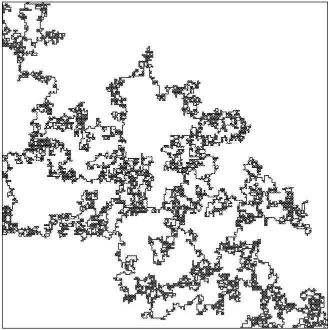

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Remark: On a square grid was done by S. Smirnov: see *"Towards conformal invariance of 2D lattice models", Proceedings of the ICM, Madrid 2006*, and further papers.

Topology of convergence and crossing probabilities:

- Convergence of discrete martingale observables immediately give *convergence of driving forces* in Loewner equation to $\sqrt{3}\, B_t$ (spin-Ising model) and $\sqrt{16/3}\, B_t$ (FK-Ising model).

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  Easy answer: Schramm-Loewner Evolution.

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- Convergence of discrete martingale observables immediately give *convergence of driving forces* in Loewner equation to $\sqrt{3}\,B_t$ (spin-Ising model) and $\sqrt{16/3}\,B_t$ (FK-Ising model).
- To deduce the *convergence of curves* themselves it's sufficient

  [Aizenmann-Burchard'99; Kemppainen-Smirnov'09-10]

  to (uniformly) estimate the *probability of crossing events* in quadrilaterals with alternating boundary conditions:

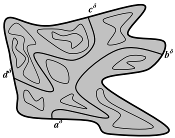
  "+"/"-"/"+"/"-" for spin model;

  wired/free/wired/free for FK-model.

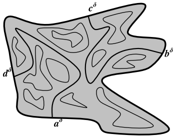
Topology of convergence and crossing probabilities:

<u>THEOREM 3</u> (arXiv:0910.2045): *Let discrete domains $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)$ with alternating (wired/free/wired/free) boundary conditions on four sides approximate some continuous topological quadrilateral $(\Omega; a, b, c, d)$.*

Then the probability of an FK cluster crossing between two wired sides has a scaling limit, which depends only on the conformal modulus of the limiting quadrilateral.



P^δ vs. Q^δ



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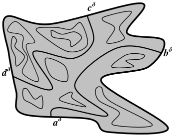

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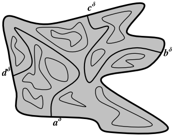
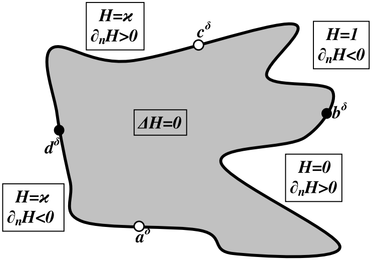


<u>Remark:</u> [predicted by Bauer-Bernard-Kytölä'05 using CFT arguments]

$$p(\mathbb{H}; 0, 1-u, 1, \infty) = \frac{\sqrt{1-\sqrt{1-u}}}{\sqrt{1-\sqrt{u}} + \sqrt{1-\sqrt{1-u}}}, \quad u \in [0,1].$$

Crossing probabilities (FK-Ising model). Core argument:

There exists an (almost) discrete harmonic function which solves the following *discrete boundary value problem*:



where $\varkappa^\delta$ is uniquely determined by the ratio of crossing probabilities $t^\delta = \mathrm{P}^\delta / \mathrm{Q}^\delta$.

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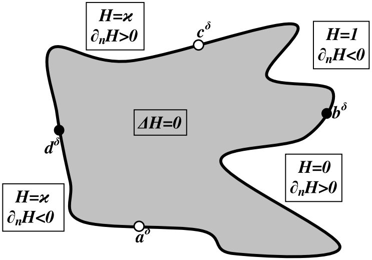

$$\varkappa^\delta = \left[\frac{(t^\delta)^2+\sqrt{2}t^\delta}{(t^\delta)^2+\sqrt{2}t^\delta+1}\right]^2$$

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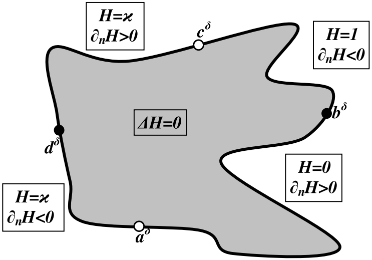
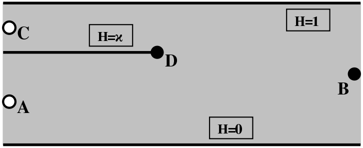


Uniformization (in the limit):



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$\varkappa$ is determined by the conformal modulus of (Ω, a, b, c, d)

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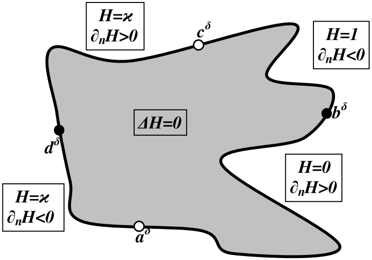

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Remark: This also allows to extract sharp two-sided estimates staying on a discrete level.

Crossing probabilities. From FK- to spin-Ising:

COROLLARY 4: *Let discrete domains* $(\Omega^\delta; a^\delta, b^\delta, c^\delta, d^\delta)$ *with alternating ("+"/"-"/"+"/"-") boundary conditions on four sides approximate some continuous nondegenerate topological quadrilateral* $(\Omega; a, b, c, d)$.

Then the probability of "+" crossing between two "+" sides remains uniformly bounded from 0 *and* 1 *as* $\delta \to 0$*, with bounds depending only on the conformal modulus of the limiting quadrilateral.*

Remark: Convergence to a scaling limit (known using CFT arguments, Bauer-Bernard-Kytölä'05) is still open.

[proof on a blackboard (2 min, time permitting)]

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THANK YOU!