3rd LA PIETRA WEEK IN PROBABILITY

Stochastic Models in Physics
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SLE and conformal invariance for critical Ising model

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1D Ising model

0 1 .... N N+1 ....

+	−	+	+	+	−	−	+	−	−	+	−	+	+	··

$Z = \Sigma_{conf.}\, x^{\#\{(+)(-)\text{neighbors}\}}$, $P[\text{conf.}] \sim x^{\#\{(+)(-)\}}$

$0 \leq x = e^{-J/kT} \leq 1.$ Let $\sigma(0) = \text{“}+\text{”}$.

$P[\sigma(N) = \text{“}+\text{”}] = ?$

1D Ising model

0 1 .... N N+1 ....

+	–	+	+	+	–	–	+	–	–	+	–	+	+	..

$\mathbf{Z=\Sigma_{conf.}\, x^{\#\{(+)(-)neighbors\}}}$, $\mathbf{P}[\text{conf.}]\mathbf{\sim x^{\#\{(+)(-)\}}}$

$\mathbf{0 \leq x = e^{-J/kT} \leq 1.}$ Let $\sigma(0)=$"+".

$\mathbf{P}[\sigma(N)=\text{"+"}] = \mathbf{?}$

$\mathbf{=\frac{1}{2}(1+y^N)}$, $y=(1-x)/(1+x)$.

$$Z_{n+1;\sigma(n+1)=\text{"+"}} = Z_{n;\sigma(n)=\text{"+"}} + xZ_{n;\sigma(n)=\text{"–"}}$$

$$Z_{n+1;\sigma(n+1)=\text{"–"}} = xZ_{n;\sigma(n)=\text{"+"}} + Z_{n;\sigma(n)=\text{"–"}}$$

EXERCISE: Do the same in the magnetic field:

$$\mathbf{P}[\text{configuration}] \sim \mathbf{x}^{\#\{(+)(-)\}}\, \mathbf{b}^{\#\{(-)\}}, \quad b>0,$$

$\sigma(0)=$"+". $\mathbf{P}[\sigma(N)=\text{"+"}]=?$

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<u>EXERCISE:</u> Let $\sigma(0)=\text{"+"}=\sigma(N+M)$. $\mathbf{P}[\sigma(N)=\text{"+"}]=?$

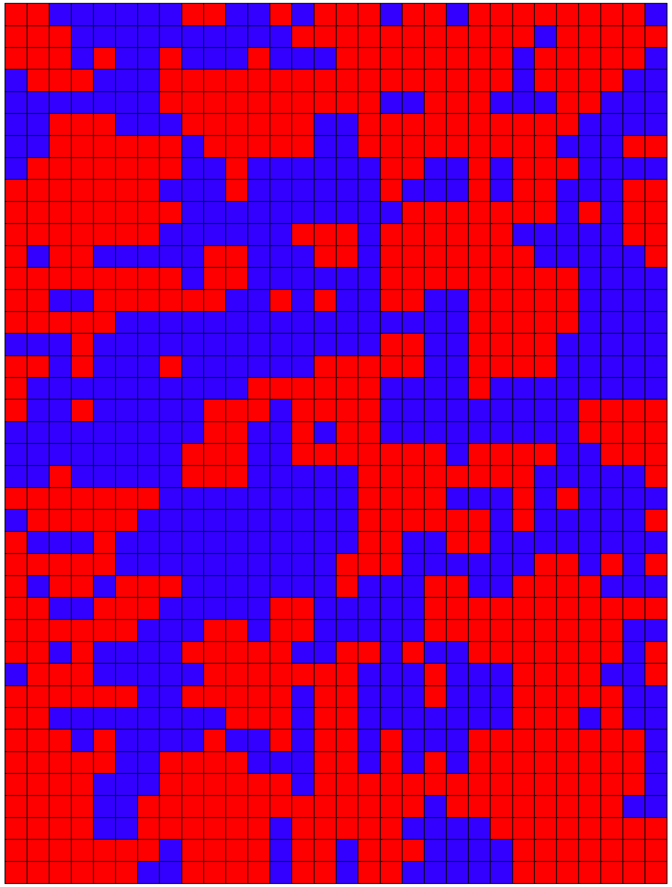
0 .... N .... N+M

+	−	+	+	+	−	−	+	−	+	−	+	+	−	+

Check $\mathbf{P}[\sigma(N)=\text{"+"}] \to \frac{1}{2}$ (if $N/M \to \text{const} \neq 0,1$).

[Ising '25]: ***<u>NO PHASE TRANSITION</u>*** *AT X≠0 (1D)*

2D (spin) Ising model



Squares of two colors, representing spins +,–

Nearby spins tend to be the same:

P[conf.] **~ x**$^{\#\{(+)(-)\text{neighbors}\}}$

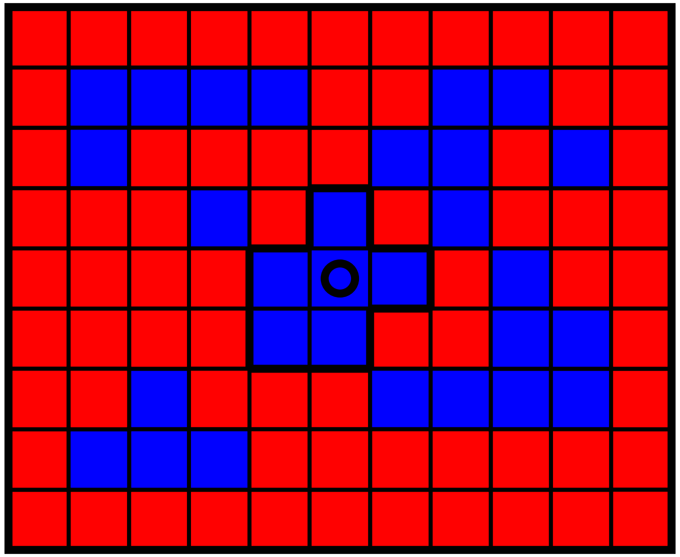
[Peierls '36]:

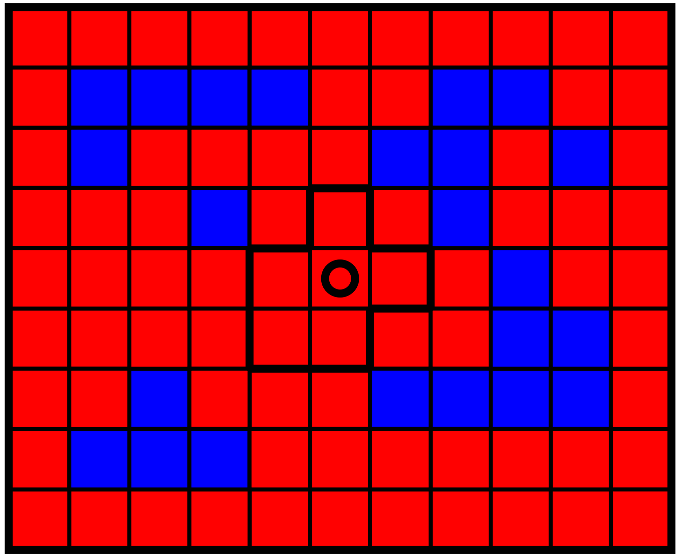
PHASE TRANSITION (2D)

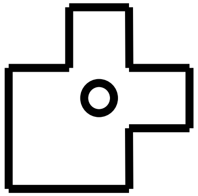
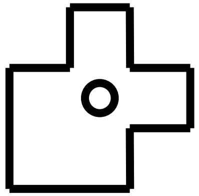
[Kramers-Wannier '41]:

$$x_{crit} = 1/(1+\sqrt{2})$$

σ(boundary of $(2N+1)\mathrm{x}(2N+1)$="+") $\mathbf{P}[\sigma\,(0)=\text{“+”}]=?$





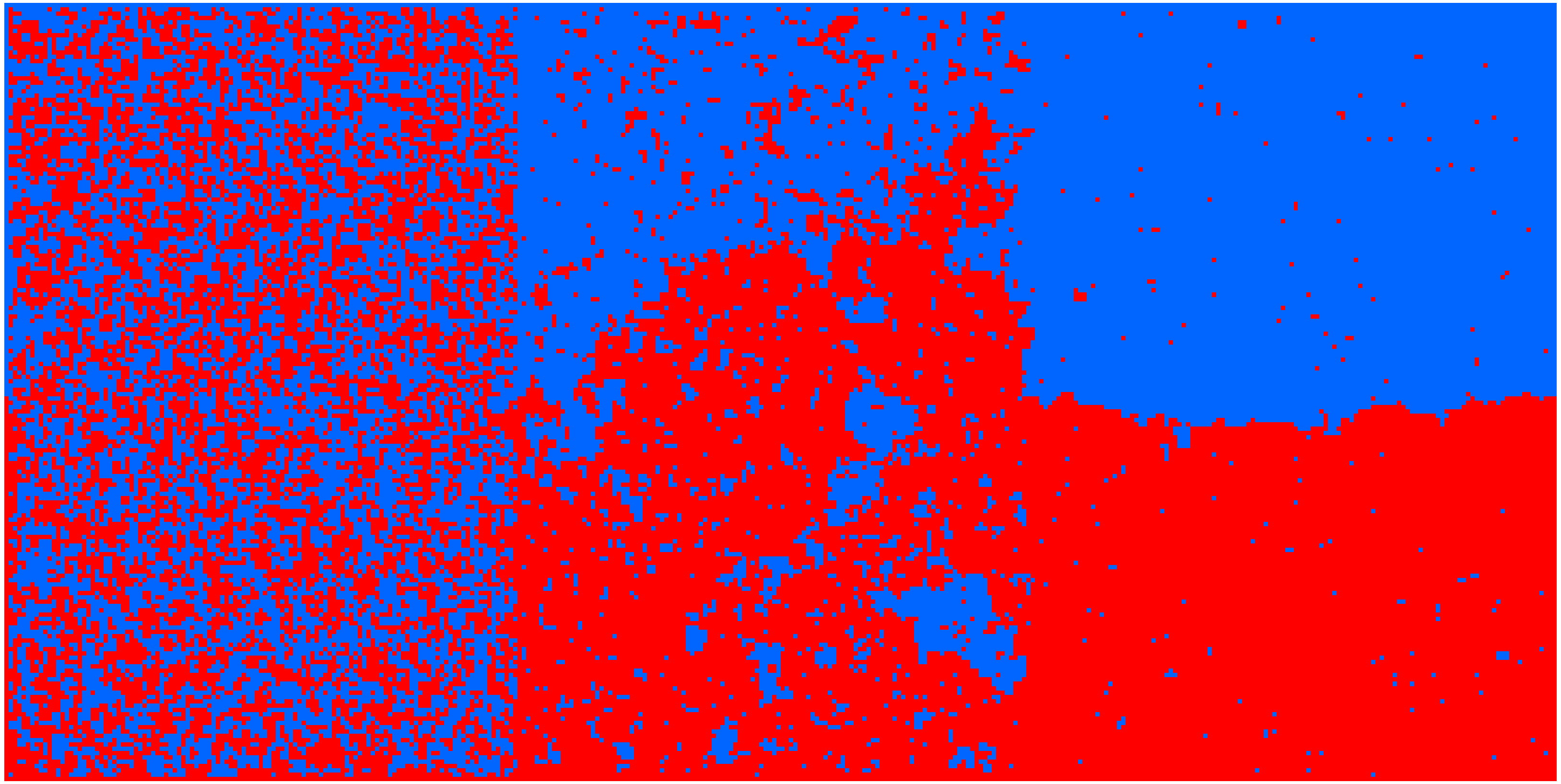
$\mathbf{P}\Big[$  $\Big] \le x^L/(1+x^L) \le x^L$, L=Length of 

$$\mathbf{P}[\sigma\,(0)=\text{“–”}] \le \textstyle\sum_{j=1,..,N} \sum_{L\geq 2j+2} 3^L x^L$$

$$\le (3x)^4/(1-(3x)^2)(1-3x) \le 1/6, \qquad \text{if } x \le 1/6.$$

2D: Phase transition

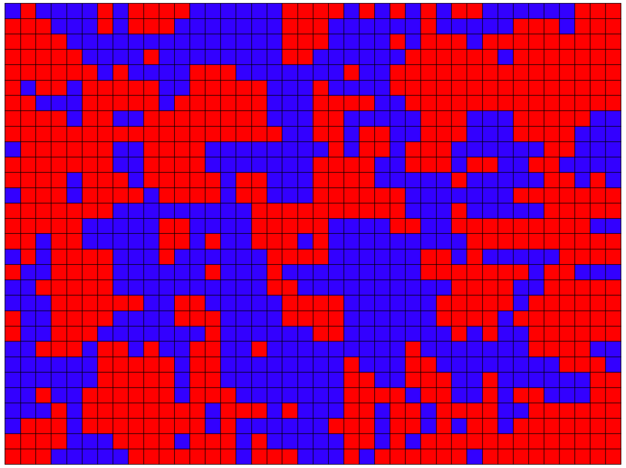
$x \to 1$ $(T \to \infty)$ $x = x_{crit}$ $x \to 0$ $(T \to 0)$



(Dobrushin boundary conditions:

the upper arc is blue, the lower is red)

2D Ising model at criticality is considered a classical example of **conformal invariance** in statistical mechanics, which is used in deriving many of its properties.

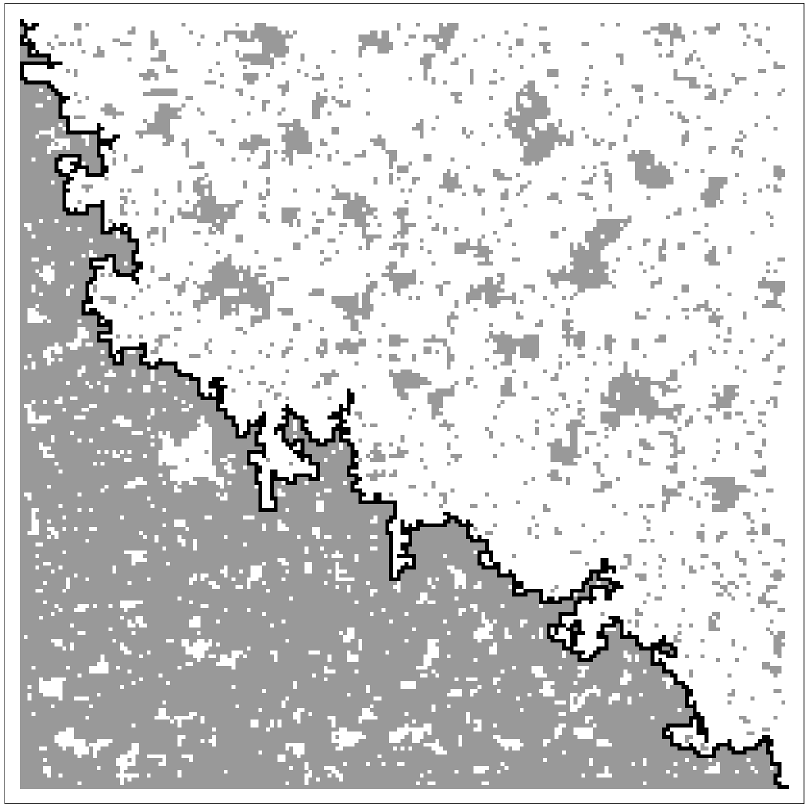


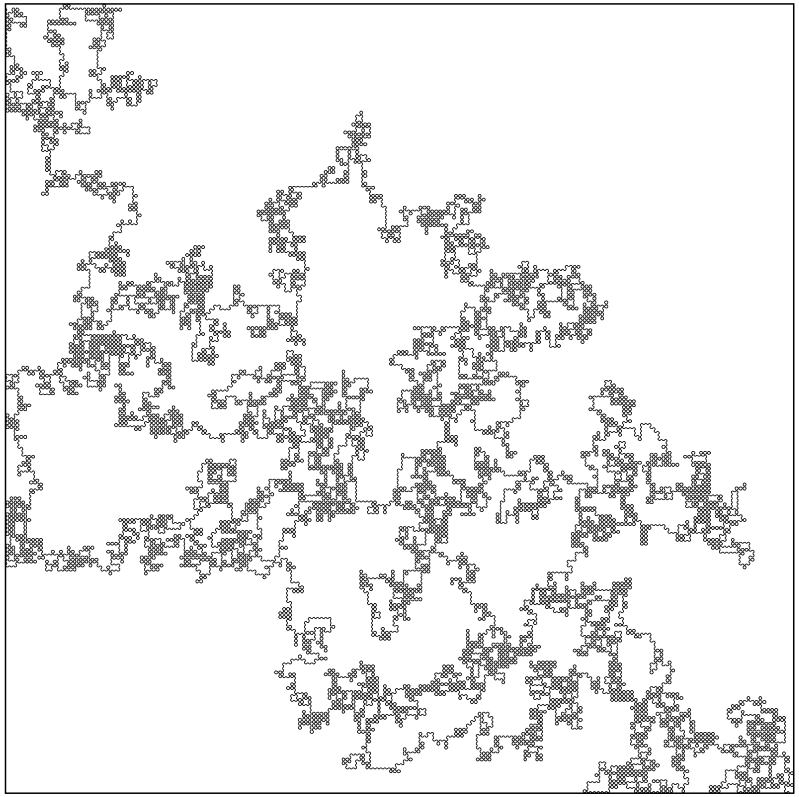
However,

- No mathematical proof has ever been given.
- Most of the physics arguments concern nice domains only or do not take boundary conditions into account, and thus only give evidence of the (weaker!) Mobius invariance of the scaling limit.
- Only conformal invariance of correlations is usually discussed, we ultimately discuss the full picture.

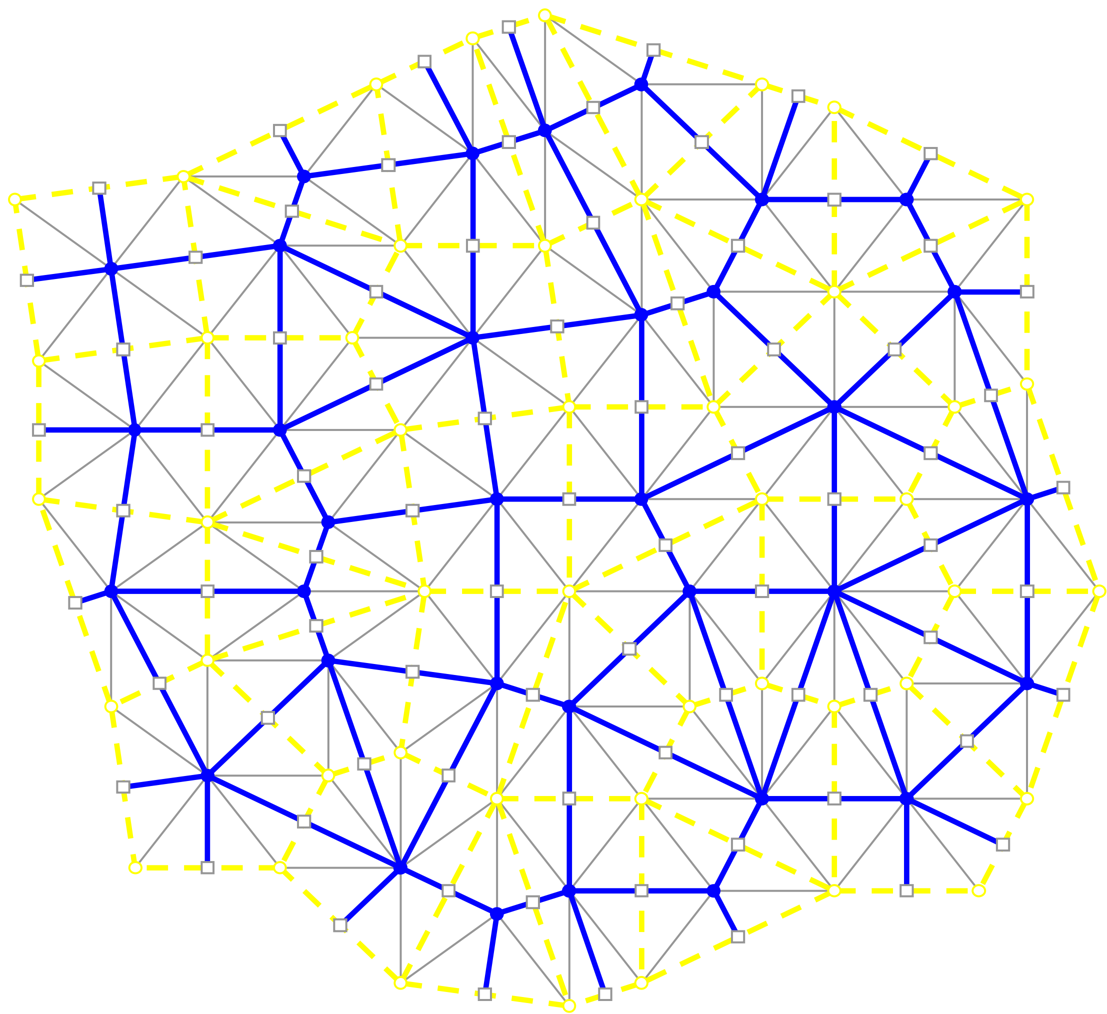
Theorem 1 [Smirnov]. Critical spin-Ising and FK-Ising models on the square lattice have **conformally invariant scaling limits** as the lattice mesh → 0. Interfaces converge to SLE(3) and SLE(16/3), respectively (*and corresponding loop soups*).

Theorem 2 [Chelkak-Smirnov]. The convergence holds true on arbitrary isoradial graphs (**universality** for these models).





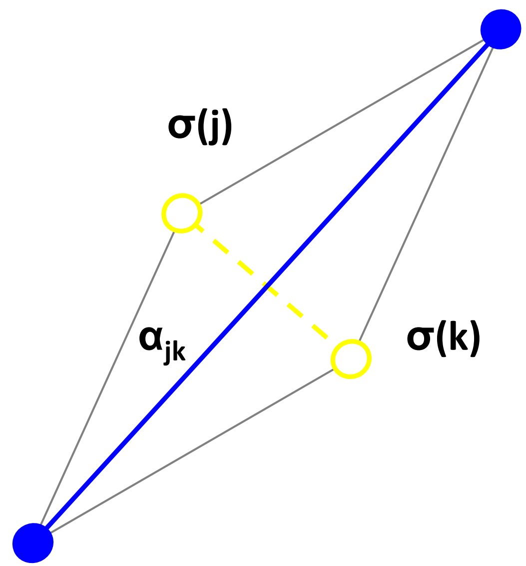
Ising model on isoradial graphs:



$$\mathbf{P}[\text{conf.}] \sim$$

$$\prod_{\langle jk\rangle:\sigma(j)\neq\sigma(k)} X_{jk}$$

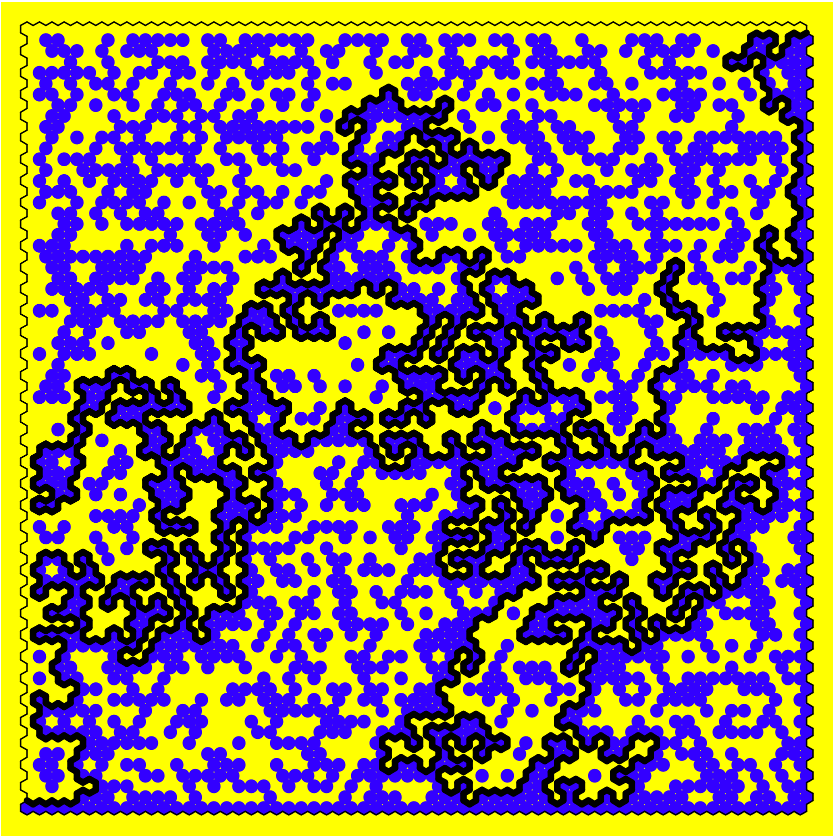
$$X_{jk} = \tan(\alpha_{jk}/2)$$

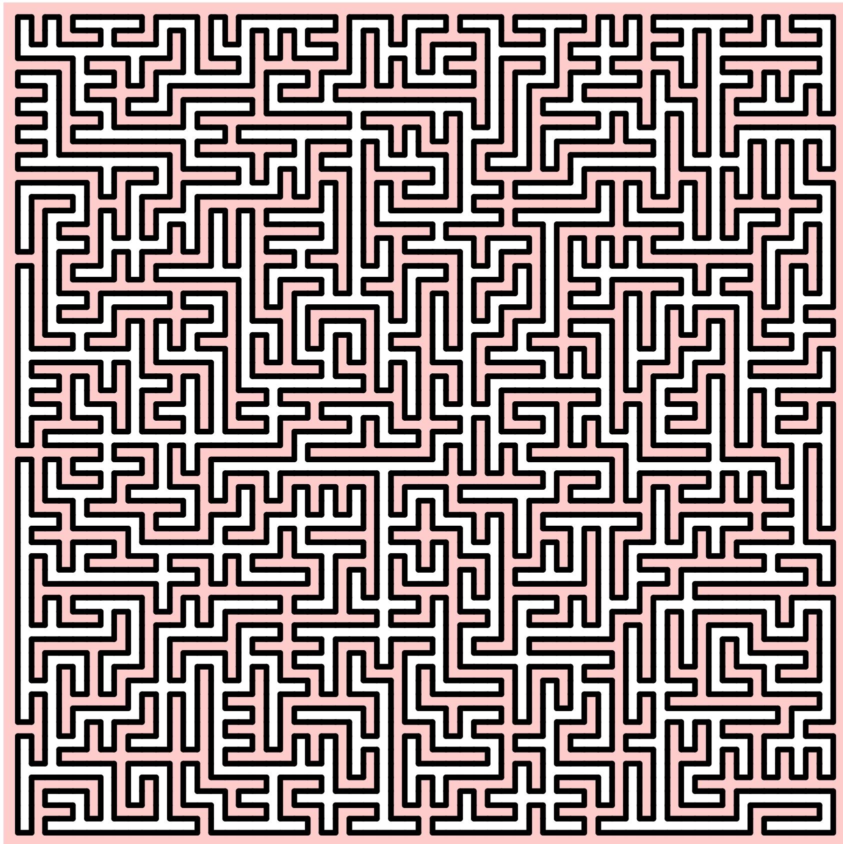


Some earlier results:

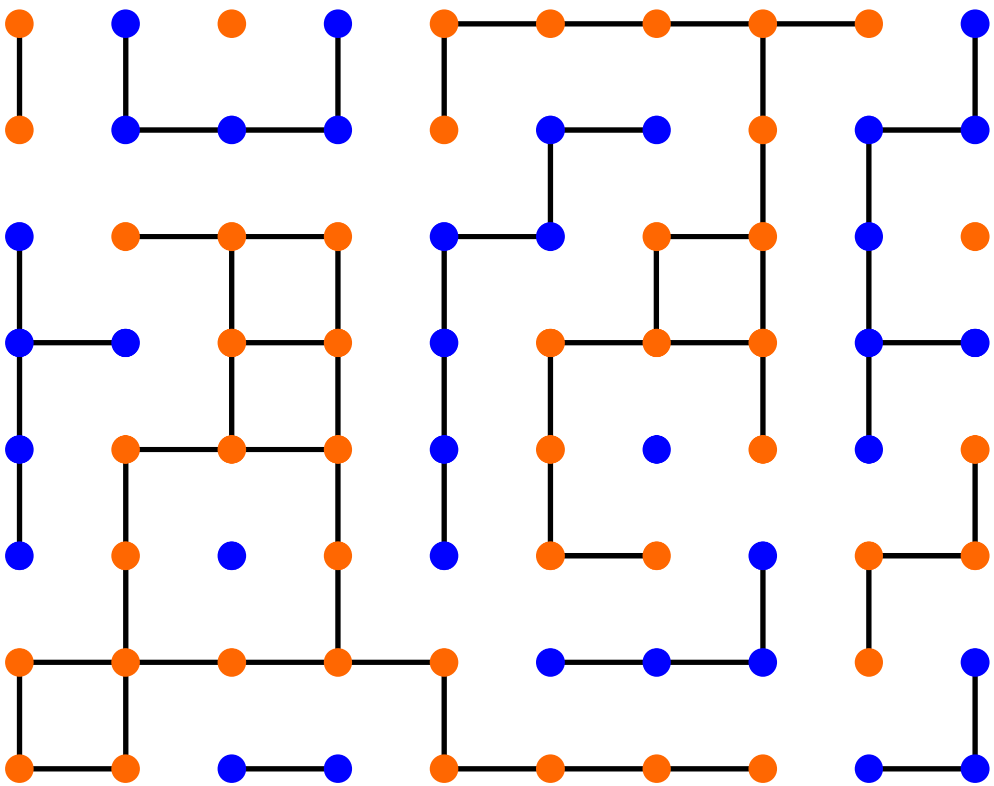
LERW → SLE(2)
UST → **SLE(8)**
[Lawler-Schramm -Werner, 2001]

Percolation → **SLE(6)**
[Smirnov, 2001]





(Spin) Ising model



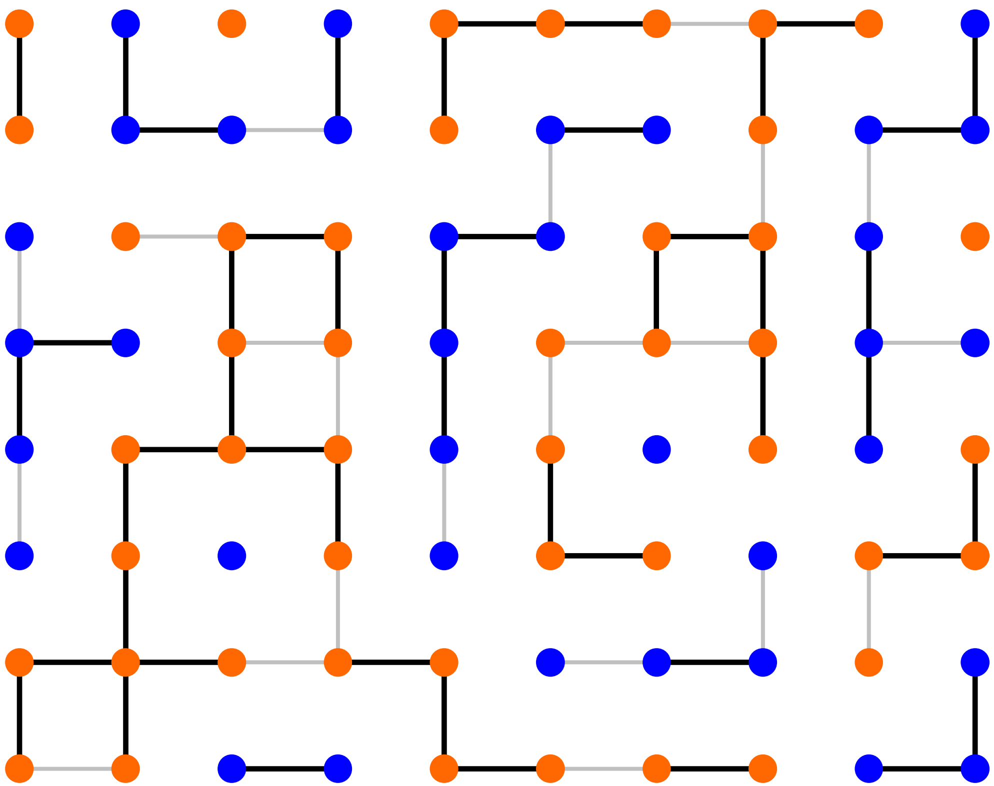
Conigurations:

spins +/−

$\mathbf{P} \sim x^{\#\{(+)(-)\text{neighbors}\}}$ =

$\Pi_{<jk>}[(1\text{-}x)+x\delta_{s(j)=s(k)}]$

(Spin) Ising model



Conigurations:

spins +/–

$\mathbf{P} \sim x^{\#\{(+)(-)\text{neighbors}\}} =$

$\prod_{<jk>}[(1-x)+x\delta_{s(j)=s(k)}]$

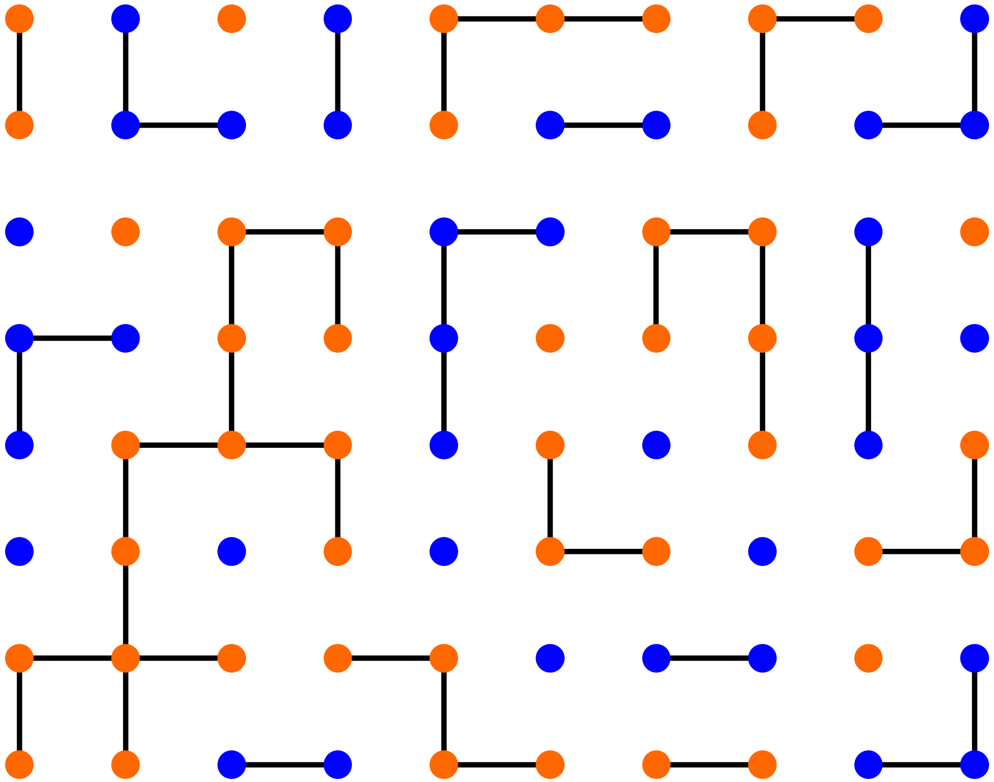
Expand, for each term prescribe an edge configuration:

x : edge is **open**

1-x : edge is closed

open edges connect the same spins (but not all!)

Edwards-Sokol covering '88

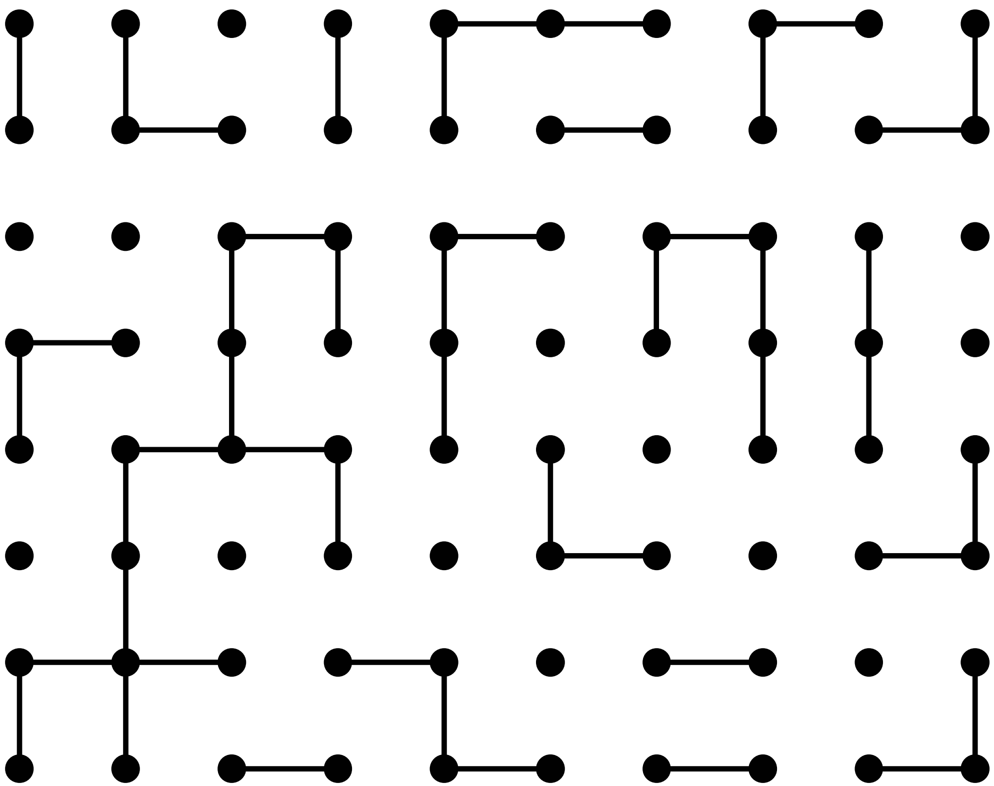


Conigurations:

spins +/–, open **edges** *connect the same spins (but not all of them!)*

$\mathbf{P} \sim (1\text{-}x)^{\#\text{open}} x^{\#\text{closed}}$

Fortuin-Kasteleyn (random cluster) model '72:



Conigurations:

spins +/–, open **edges** *connect the same spins (but not all of them!)*

P ~ $(1\text{-}x)^{\#open}x^{\#closed}$

Erase spins:

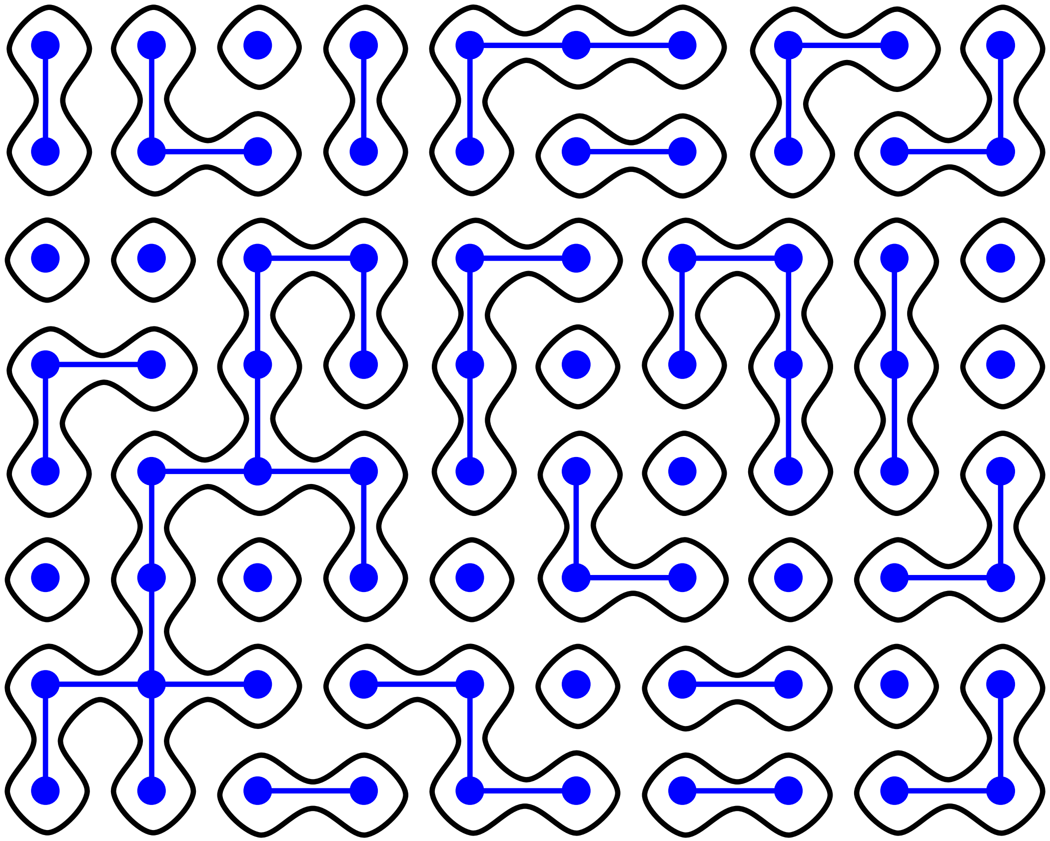
Probability of edge configurations is ~ to

$(1\text{-}x)^{\#open}x^{\#clos}2^{\#clusters}$

or

$((1\text{-}x)/x)^{\#open}2^{\#clusters}$

Loop gas representation:



Conigurations: dense loop collections. **P** ~ to

$((1-x)/x)^{\#open} 2^{\#clusters}$

or

$((1-x)/2^{½}x)^{\#open} 2^{½\#loops}$

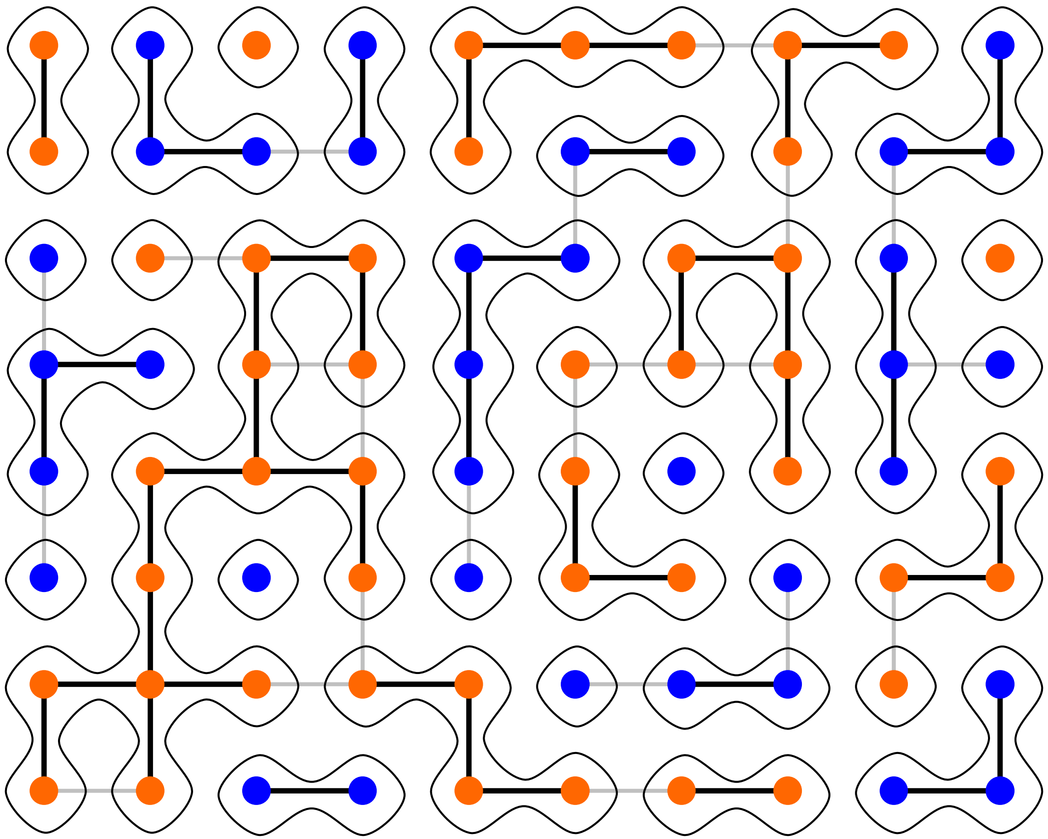
[#loops – #open = 2#clusters+const]

<u>Self-dual case</u>:

$(1-x)/x=2^{½}$, i.e.

$X=1/(1+ 2^{½})$

Spin, FK, Loop gas



EXERCISE:

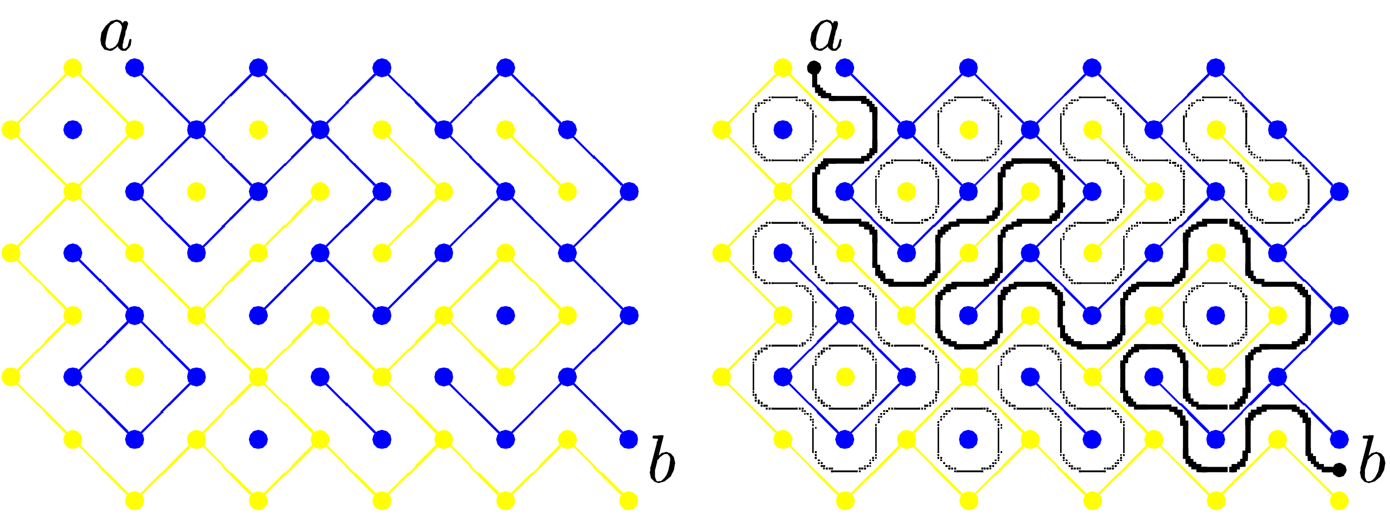
$$\mathbf{P}_{\mathbf{spin}}[\sigma(j)=\sigma(k)] = (1+\mathbf{P}_{\mathbf{FK}}[j\leftrightarrow k])/2$$

EXERCISE:

Start with Q different spins (Potts model).

Note: Loop gas is well-defined for all positive Q's!

Loop representation of the FK model



Configurations are dense loop collections on the medial lattice

Loops separate clusters from dual clusters

Dobrushin b.c.: besides loops an interface $\gamma : a \leftrightarrow b$

For $p = \sqrt{q}/(1+\sqrt{q})$ the probability $\mathbf{Prob} \asymp \left(\sqrt{\boldsymbol{q}}\right)^{\#\ \text{loops}}$

Outline:

- **Introduction**
- **Discrete harmonic/holomorphic functions**
- **Holomorphic observables in the Ising model**
- **SLE and the interfaces in the Ising model**
- **Further developments**

We will discuss how to

- *Find an discrete holomorphical observable with a conformally invariant scaling limit*
- *Using one observable, construct (conformally invariant) scaling limits of a domain wall*

Possible further topics:

- Retrieve needed a priori estimates from the observable
- Construct the full scaling limit
- Generalize to isoradial graphs (universality)
- Perturbation $p \to p_{crit}$ — no conformal invariance