DISCRETE HARMONIC AND DISCRETE HOLOMORPHIC FUNCTIONS [La Pietra '2008]

- DISCRETE HARMONIC FUNCTIONS [ON ISORADIAL GRAPHS]

  CONVERGENCE AS MESH $\to 0$

- DISCRETE HOLOMORPHIC AND "STRONG - HOLOMORPHIC" FUNCTIONS

SQUARE LATTICE

[HAVING ISORADIAL GRAPHS IN MIND]

Harmonic measure

$$\omega^\delta(u, b^\delta a^\delta, \Omega^\delta)$$

$$\| $$

$$\mathbb{P}^u \{ RW^\delta_\tau \in b^\delta a^\delta \}$$

$\uparrow$ first hit to $\partial\Omega^\delta$

Continuous setup:

$$\omega(u, ba, \Omega)$$

$$\|$$

$$\mathbb{P}^u \{ B_\tau \in ba \}$$

If $(\Omega^\delta, a^\delta, b^\delta) \to (\Omega, a, b)$

then $\omega^\delta(u, b^\delta a^\delta, \Omega^\delta)$

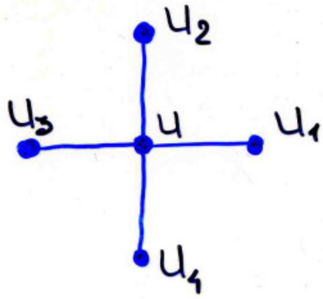
$\to \omega(u, ba, \Omega)$

- **WHY ?**

[• universal proof for different "lattices"

[• what does "$\to$" mean ?]

Definition: [square lattice]



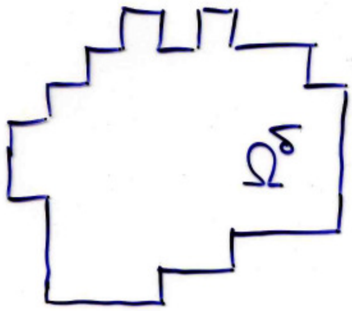
H defined on $\Gamma = \Gamma^\delta$

- $(\Delta^\delta H)(u) := \frac{1}{\delta^2} \sum_{k=1}^{4} (H(u_k) - H(u))$

and $m^\delta(u) := \delta^2$

- H is discrete harmonic iff $\Delta^\delta H \equiv 0$

EXERCISE: (Dirichlet boundary value problem)



$? H : \begin{cases} \Delta^\delta H \equiv 0 & \text{inside } \Omega^\delta \\ H \equiv F & \text{on the boundary } \Omega^\delta \end{cases}$

Prove that

there exists unique solution

Hint:

Minimize discrete Dirichlet energy: $\sum_{\langle u_j u_k \rangle} (H(u_j) - H(u_k))^2$

Note:

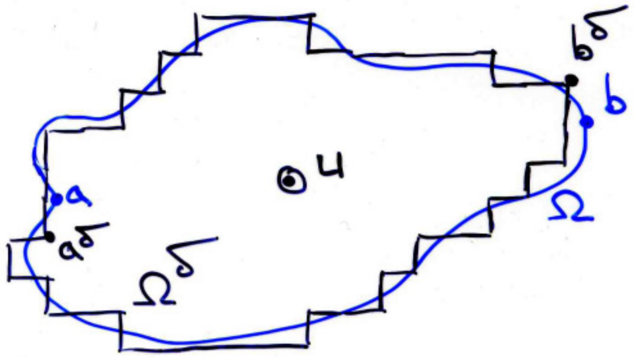
$\omega^\delta(\cdot, b^\delta a^\delta, \Omega^\delta)$ is the solution of Dirichlet boundary value problem:

$$\begin{cases} \omega^\delta(\cdot, b^\delta a^\delta, \Omega^\delta) \equiv 0 & \text{on } a^\delta b^\delta \\ \omega^\delta(\cdot, b^\delta a^\delta, \Omega^\delta) \equiv 1 & \text{on } b^\delta a^\delta \end{cases}$$

Start with some "nice" (Hausdorf) convergence $(\Omega^\delta, a^\delta, b^\delta) \to (\Omega, a, b)$

Let

$$\underline{h^\delta(\cdot) := \omega^\delta(\cdot, b^\delta a^\delta, \Omega^\delta)}$$



① **HARNACK'S ESTIMATE**

$$\left|\frac{H(u_k)-H(u)}{\delta}\right| \leq \frac{\text{const}}{\text{dist}(u,\partial\Omega^\delta)} \cdot \underbrace{\max_{\Omega^\delta}|H|}_{\leq 1}$$

$\Rightarrow \{h^\delta\}$ are uniform Lip's inside

$\Rightarrow$ PRECOMPACTNESS IN $C(K)$, $K \subset \Omega$

$\Rightarrow$ $\exists$ subsequential limit

$$h^{\delta_k} \overset{K}{\rightrightarrows} h, \quad h : \Omega \to [0,1]$$

② **APPROXIMATION PROPERTY**

$\forall \varphi : \mathbb{R}^2 \to \mathbb{R}$ - smooth

$$\Delta^\delta(\varphi|_{\Gamma^\delta}) = (\Delta\varphi)|_{\Gamma^\delta} + O(\delta)$$

Then:

$\forall \varphi$: supp $\varphi \subset \Omega$

$$\iint_{\Omega} h(u)(\Delta\varphi)(u)\, dm(u) = \lim_{\delta \downarrow 0} \sum_{\Omega^\delta} h(u)(\Delta\varphi)(u)\, \overbrace{m^\delta(u)}^{= 1/\delta^2}$$

$$= \lim_{\delta\downarrow 0} \underbrace{\sum_{\Omega^\delta} h^\delta(u)(\Delta^\delta \varphi)(u)\, m^\delta(u)}_{\substack{\| \\ \sum_{\Omega^\delta} (\Delta^\delta h^\delta)(u)\varphi(u) m^\delta(u) \\ \| \\ 0}}$$

[discrete integration by parts]

$\Rightarrow$ h IS HARMONIC INSIDE Ω

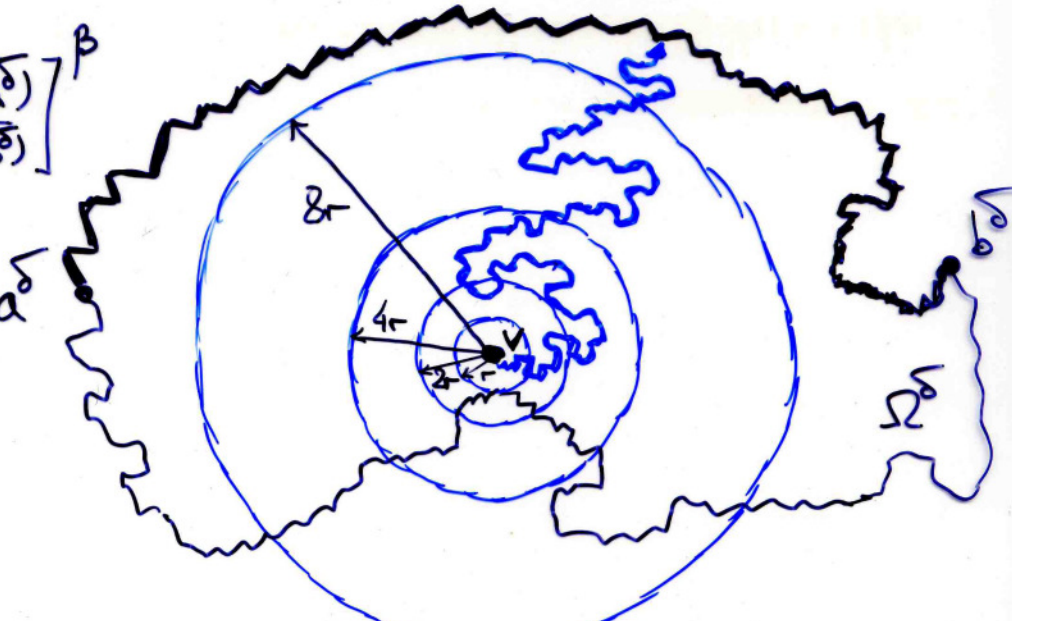
③ WEAK BEUERLING-TYPE ESTIMATE

$$h^\delta(v) \leq \text{const} \cdot \left[\frac{\text{dist}(v, a^\delta b^\delta)}{\text{dist}(v, a^\delta b^\delta)}\right]^\beta$$

for some $\beta > 0$.

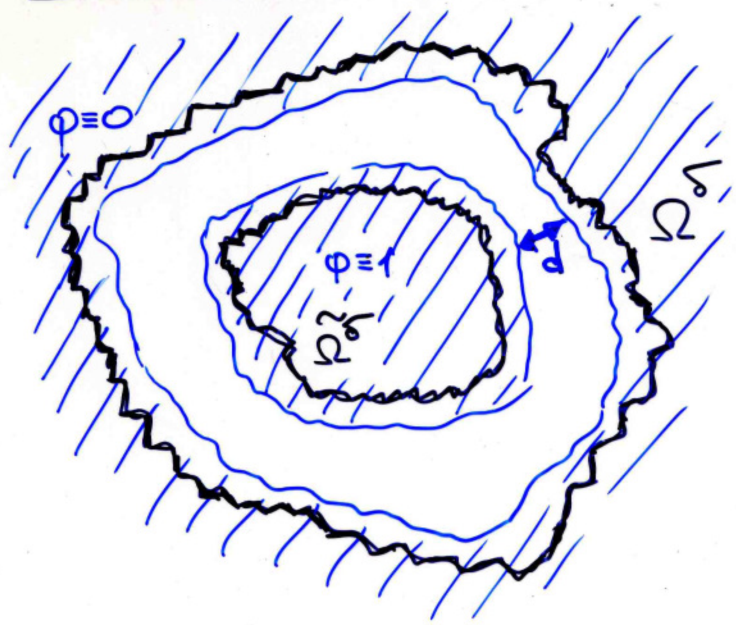
$\Uparrow$

probability to cross the annulus without touching the boundary $(a^\delta b^\delta)$ is bounded away from 1. $\Leftarrow$ APPROXIMATI[ON] PROPERT[Y]



THUS: $h(v) \leq \text{const} \cdot \left[\frac{\text{dist}(v, ab)}{\text{dist}(v, ba)}\right]^\beta \xrightarrow[v \to ab]{} 0$

HARNACK'S ESTIMATE: | La Pietr



Continuous case:

$$0 = \iint_\Omega \varphi h \underbrace{\Delta h}_{\equiv 0}$$

$$= - \iint_\Omega \nabla(\varphi h) \nabla h =$$

$$= - \iint_\Omega \varphi \|\nabla h\|^2 - \iint_\Omega \nabla\varphi \cdot \underbrace{h\nabla}_{\frac{1}{2}\nabla(}$$

$$\Rightarrow \iint_\Omega \varphi \|\nabla h\|^2 = \frac{1}{2} \iint_\Omega \underbrace{\Delta\varphi}_{\times\, 1/d^2} \cdot h^2$$

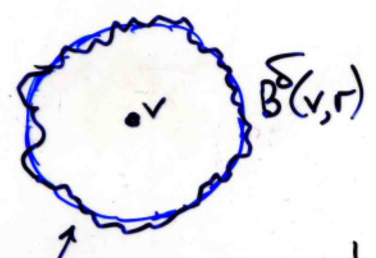
$$\Rightarrow \|\nabla h\|_{L^2(\tilde\Omega)} \leq \frac{\text{const}}{d} \|h\|_{L^2(\Omega)}$$

EXERCISE:

Do DISCRETIZATION of this proof

[Subtle point:
φ and ∇h are defined on different lattices

MEAN VALUE PROPERTY FOR DISCRETE HOLOMORP
(GRADIENTS OF DISCRETE HARMONIC) FUNCTIO



"nice" approximation of the disc $B(v,r)$

$$|F(v)| \leq \frac{\text{const}}{r^2} \sum_{B^\delta(v,r)} |F|$$

[Cauchy's formula together with asymptotics of Cauchy's kernel]

KENYON 'S
ON ISORADIAL GRAPH

La Pietra

So, if $(\Omega^\delta, a^\delta, b^\delta) \to (\Omega, a, b)$

[I say, in Hausdorf sense]

then $\omega^\delta(u, b^\delta a^\delta, \Omega^\delta) \to \omega(u, ba, \Omega)$

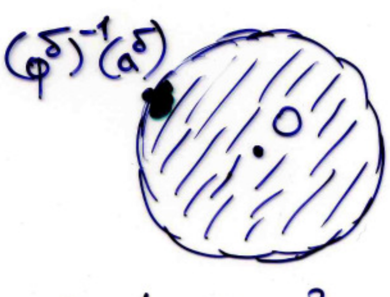
[or, equivalently,

$$|\omega^\delta(u, b^\delta a^\delta, \Omega^\delta) - \omega(u, b^\delta a^\delta, \Omega^\delta)| \to 0,$$

since the continuous harmonic measure is stable under $\Omega^\delta \to \Omega$]

WE NEED SOME <u>UNIFORM</u> ESTIMATE HERE !

<u>CARATHEODORY</u> <u>CONVERGENCE</u>:



$\mathbb{D} = \{z : |z| < 1\}$

<u>Normalization</u>:

$\varphi^\delta(0) = u$

$(\varphi^\delta)'(0) > 0$

[fiords beyond <u>thin</u> straits are OK]

- Set of all domains Ω:

  $$B(u,r) \subset \Omega \subset B(u,R) \quad [0<r<R \text{ are fixed}]$$

  is compact in the Carathéodory topology

  [so, the set of all (Ω, a, b) is compact too]

- $\omega(u, ba, \Omega)$ is continuous in the Carathéodory topology (Ω, a, b)

- Still,

  $$(\Omega^\delta, a^\delta, b^\delta) \xrightarrow{\text{Cara}} (\Omega, a, b)$$

  $$\Rightarrow \omega^\delta(u, b^\delta a^\delta, \Omega^\delta) \to \omega(u, ba, \Omega)$$

THUS, WE ARRIVE AT

UNIFORM ESTIMATE

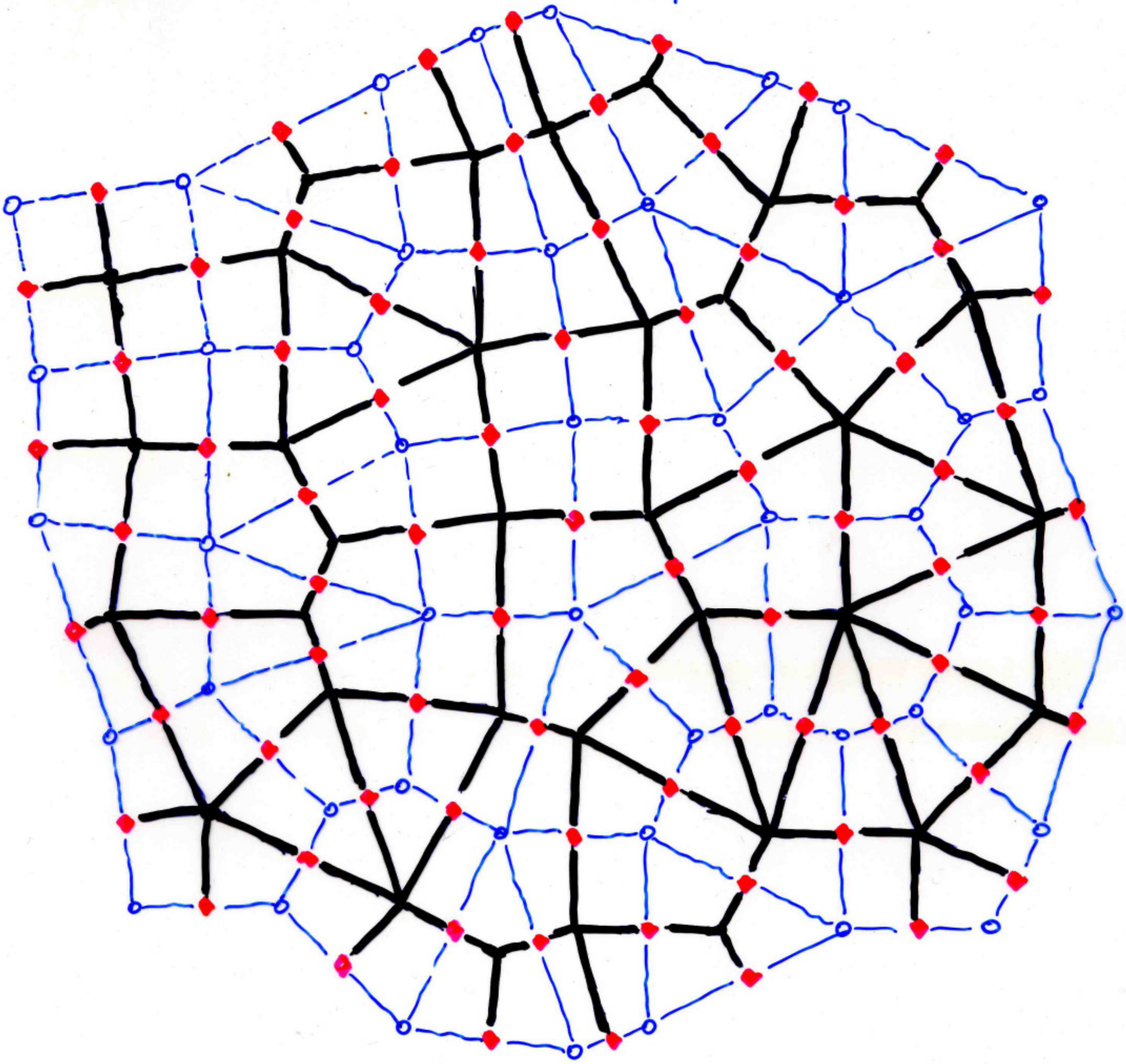
$$\boxed{\left|\omega^\delta(u, b^\delta a^\delta, \Omega^\delta) - \omega(u, b^\delta a^\delta, \Omega^\delta)\right| \leqslant \varepsilon(\delta) \xrightarrow[\delta \downarrow 0]{} 0}$$

ON ARBITRARY ISORADIAL GRAPHS

[δ = common radius of circles]

ISORADIAL GRAPH:

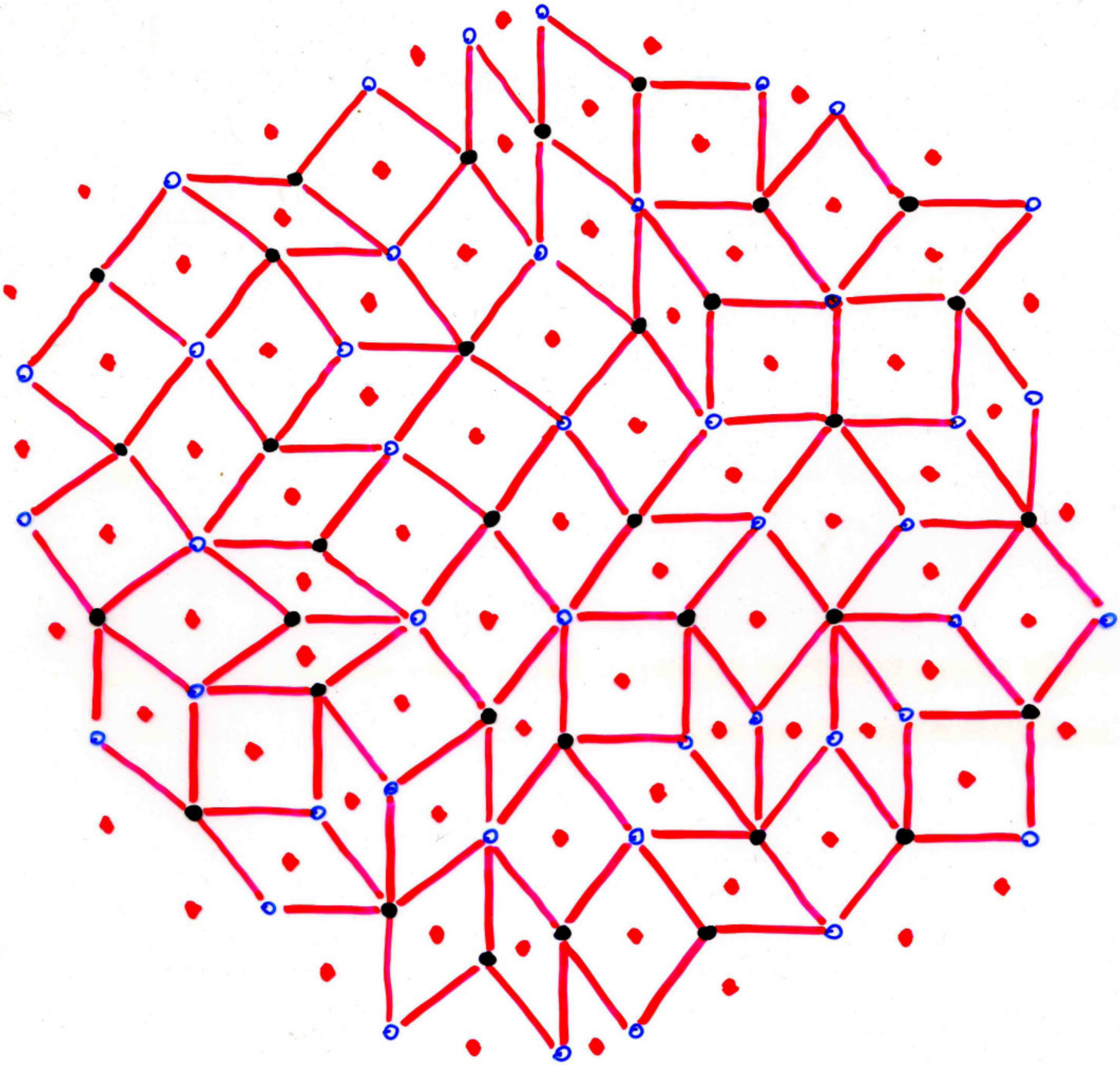
[dual vertices = centers of circles]



G (black) is the original (spin) lattic

G^* (white) is the dual lattice

$\Lambda = G \cup G^*$



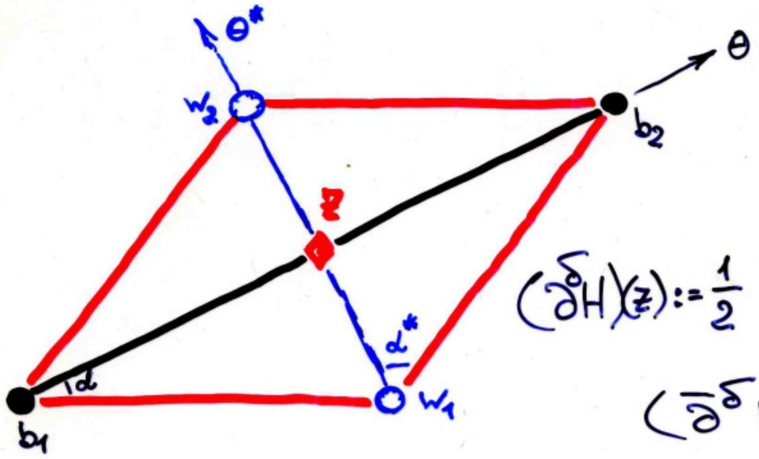
◊ = set of • = "diamond" lattice

ISORADIAL GRAPHS

La Pietr

Diff. operators, weights:

(1)



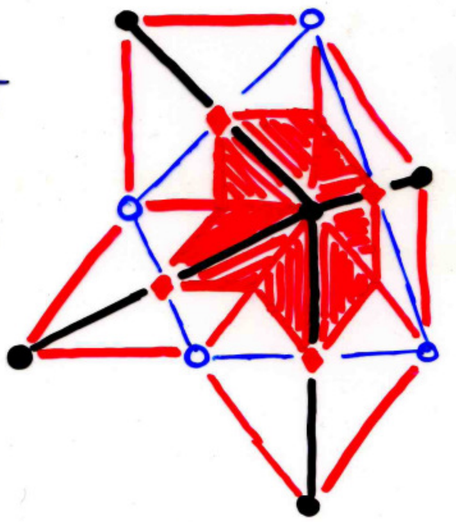
$\partial^\delta, \bar\partial^\delta : \mathcal{F}(\Lambda) \to \mathcal{F}$

$$(\partial^\delta H)(z) := \frac{1}{2}\Big[\frac{H(b_2)-H(b_1)}{b_2-b_1} + \frac{H(w_2)-H(w_1)}{w_2-w_1}$$

$(\bar\partial^\delta H)(z)$ similarly

(2) $m^\delta_\diamond(z) = \delta^2 \sin 2\alpha = \delta^2 \sin 2\alpha^* =$ area of rhombus

$m^\delta_\Lambda(u) =$ area of



$= \frac{\delta^2}{4} \sum \sin 2\alpha$

(3) $\partial^\delta, \bar\partial^\delta : \mathcal{F}(\diamond) \to \mathcal{F}(\Lambda)$

$\partial^\delta = -(\bar\partial^\delta)^*$, $\quad \bar\partial^\delta = -(\partial^\delta)^*$

(4) $\Delta^\delta : \mathcal{F}(\Lambda) \to \mathcal{F}(\Lambda)$

[in fact, $\mathcal{F}(\Gamma) \to \mathcal{F}(\Gamma)$]

$$\Delta^\delta = 4\partial^\delta\bar\partial^\delta = 4\bar\partial^\delta\partial^\delta = \frac{1}{2m^\delta_\Gamma(u)} \sum \tan\alpha_s \cdot (H(u_s) - H(u$$

relative probabilities for RW on the isoradial graph

Remark:

$\partial^\delta\bar\partial^\delta \neq \bar\partial^\delta\partial^\delta : \mathcal{F}(\diamond) \to \mathcal{F}(\diamond)$

NO NATURAL DEFINITION OF HARMONIC FUNCTIONS ON $\diamond$

[square lattice is simpler]

SIMILARLY,

WE PROVE THE UNIFORM ESTIMATE

$$\left| P^\delta(v, a^\delta, \Omega^\delta) - P(v, a^\delta, \Omega^\delta) \right| \leq \varepsilon(\delta) \xrightarrow[\delta \downarrow 0]{} 0$$

DISCRETE POISSON KERNEL — CONTINUOUS POISSON KERNEL

(BOUNDARY VALUES: 0 on $\partial\Omega^\delta \setminus a^\delta$
>0 at a^δ

NORMALIZATION: $P^\delta(u, a^\delta, \Omega^\delta) = 1$
FOR SOME FIXED $u \in \Omega^\delta$)

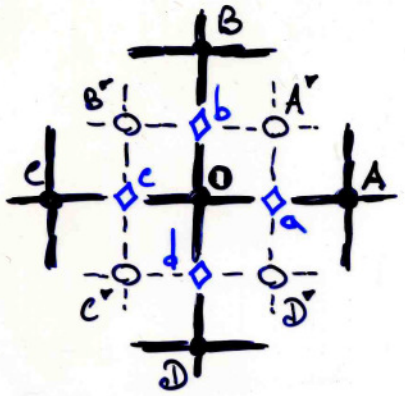
IT GIVES UNIVERSALITY OF LERW ON ISORADIAL GRAPHS

[see "core argument" in Lawler - Schramm - Werner]

ALSO, UST and HE are OK.

La Pietra

DISCRETE HOLOMORPHIC FUNCTIONS (square lattice for simplicity)



① H is defined on $\bullet$

$$F := \partial^\delta H = \frac{1}{2}(\partial_x - i\partial_y)^\delta H$$

$$F(a) = \frac{1}{2\delta}(H(A) - H(O)) \in \mathbb{R}$$

$$F(c) = \frac{1}{2\delta}(H(O) - H(C)) \in \mathbb{R}$$

$$F(b) = -\frac{i}{2\delta}(H(B) - H(O)) \in i\mathbb{R}$$

$$F(d) = -\frac{i}{2\delta}(H(O) - H(D)) \in i\mathbb{R}$$

Exercise:

CHECK H is harmonic at O (i.e. $(\Delta^\delta H)(O) = 0$)

if and only if

$$\boxed{F(b) - F(d) = i\,(F(a) - F(c))}$$

DISCRETE CAUCHY-RIEMAN[N] EQUATION[S]

② $\widetilde{H}$ is defined on $\circ$

$$F(a) = -\frac{i}{2\delta}(\widetilde{H}(A') - \widetilde{H}(D')) \in i\mathbb{R}$$

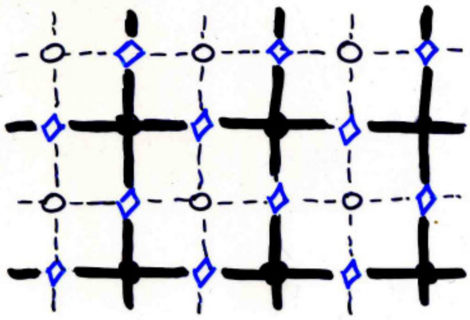
$$F(c) = -\frac{i}{2\delta}(\widetilde{H}(B') - \widetilde{H}(C')) \in i\mathbb{R}$$

$$F(b) = \frac{1}{2\delta}(\widetilde{H}(A') - \widetilde{H}(B')) \in \mathbb{R}$$

$$F(d) = \frac{1}{2\delta}(\widetilde{H}(D') - \widetilde{H}(C')) \in \mathbb{R}$$

$\Rightarrow$ DISCRETE C-R AT $\bullet$ IS ALWAYS TRU[E]

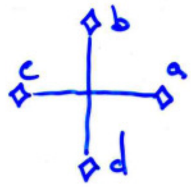
Definition:



Function $F: \diamond \to \mathbb{C}$

is DISCRETE HOLOMORPHIC

if it satisfies



$$F(b) - F(d) = i\,(F(a) - F(c))$$

everywhere.

REMARKS:

- F is hol. $\Rightarrow$ both $\mathcal{R}F := \begin{cases} \mathrm{Re}\,F & \text{on } \text{—}\diamond\text{—} \\ \mathrm{Im}\,F & \text{on } \text{|} \end{cases}$

  $\mathcal{I}F := \begin{cases} \mathrm{Im}\,F & \text{on } \text{—}\diamond\text{—} \\ \mathrm{Re}\,F & \text{on } \text{|} \end{cases}$

  are discrete holomorphic

- Similar (but more complicated) ON ISO-GRAPH
- F, G are discrete hol. $\not\Rightarrow$ FG is discrete ho

EXERCISE:

Check that F is discrete holomorphic if and only if

DISCRETE INTEGRAL

$\int^{\delta} F(z)\, d^{\delta} z$ IS WELL-DEFINED

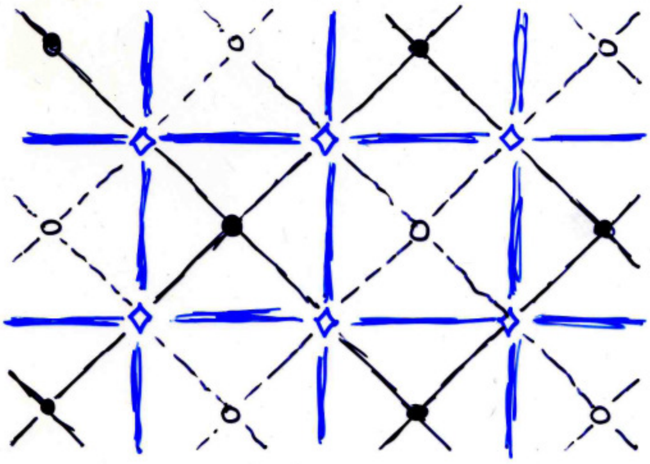
[i.e. doesn't depend on the path of integration

[SEPARATELY ON Γ AND Γ^*]

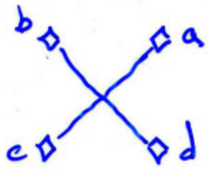
original lattice $\bullet$ — dual lattice $\circ$

Cauchy-Riemann on Γ^* — Cauchy-Riemann on Γ

"STRONG - HOLOMORPHIC" FUNCTIONS



DISCRETE CAUCHY-RIEM[ANN] EQUATIO[NS]



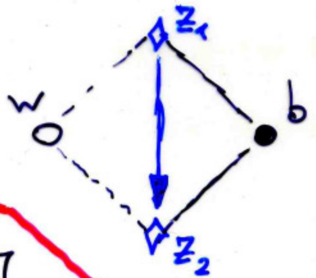
$F(b) - F(d) = i\,(F(a) - F(c$ ...

Remark: 1 complex equati[on] per • or ○

DEFINITION:

We call F S-HOLOMORPHIC,

if $\forall\, z_1 \sim z_2$:



$\parallel e^{+i\frac{\pi}{4}}$ on the picture

$$\boxed{\Pr\left[F(z_1);\ 1/\sqrt{i(w-b)}\right] = \Pr\left[F(z_2);\ 1/\sqrt{i(w-b)}\right]}$$

Remark:

1 real equation per ◇—◇ :