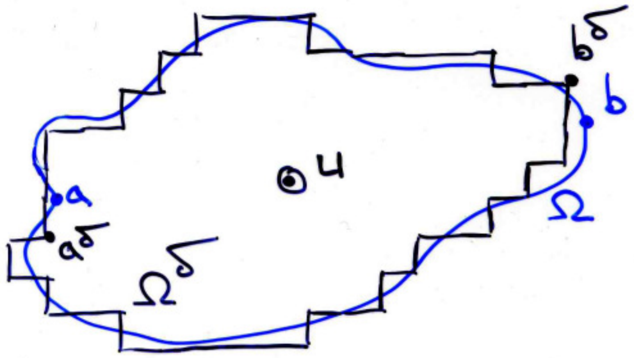
Start with some "nice" (Hausdorf) convergence $(\Omega^\delta, a^\delta, b^\delta) \to (\Omega, a, b)$

Let

$h^\delta(\cdot) := \omega^\delta(\cdot, b^\delta a^\delta, \Omega^\delta)$



① HARNACK'S ESTIMATE

$$\left|\frac{H(u_k) - H(u)}{\delta}\right| \leq \frac{\text{const}}{\text{dist}(u, \partial\Omega^\delta)} \cdot \underbrace{\max_{\Omega^\delta} |H|}_{\leq 1}$$

$\Rightarrow \{h^\delta\}$ are uniform Lip's inside

$\Rightarrow$ PRECOMPACTNESS IN $C(K)$, $K \subset \Omega$

$\Rightarrow$ $\exists$ subsequential limit

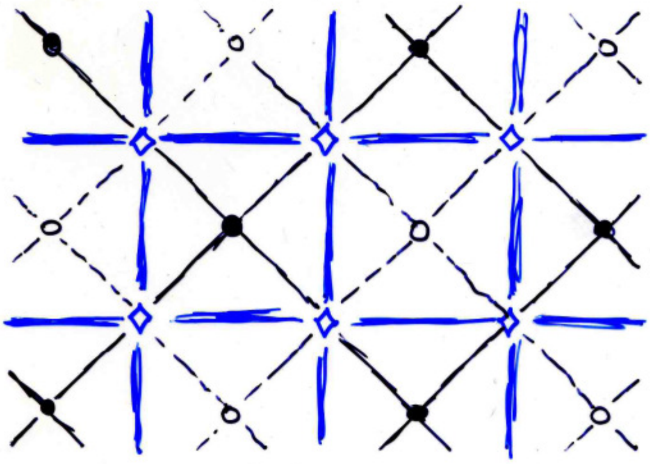
$$h^{\delta_k} \overset{K}{\rightrightarrows} h \quad , \quad h : \Omega \to [0,1]$$

② APPROXIMATION PROPERTY

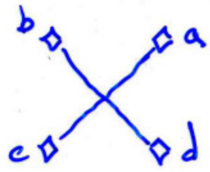
$\forall \varphi : \mathbb{R}^2 \to \mathbb{R}$ - smooth

$$\Delta^\delta(\varphi|_{\Gamma^\delta}) = (\Delta\varphi)|_{\Gamma^\delta} + O(\delta)$$

"STRONG - HOLOMORPHIC" FUNCTIONS



DISCRETE CAUCHY-RIEM[ANN] EQUATIO[NS]



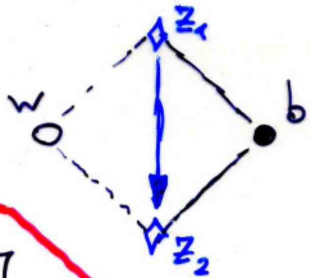
$F(b) - F(d) = i\,(F(a) - F(c$ …

Remark: 1 complex equati[on] per • or ∘

DEFINITION:

We call F S-HOLOMORPHIC,

if $\forall\, z_1 \sim z_2$:

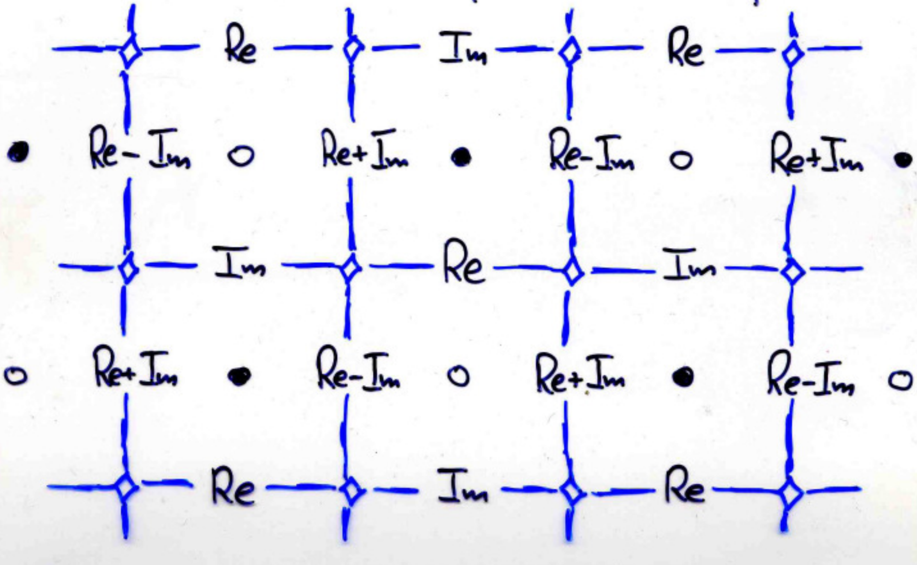


$\parallel e^{+i\frac{\pi}{4}}$ on the picture

$$\boxed{\Pr\left[F(z_1)\,;\ 1/\sqrt{i(w-b)}\right] = \Pr\left[F(z_2)\,;\ 1/\sqrt{i(w-b)}\right]}$$

Remark:

1 real equation per ◇—◇ :



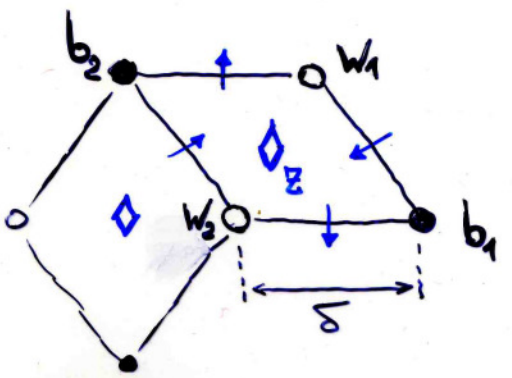
EXERCISES:

La Pietra

① F - δ-HOL $\Rightarrow$ F - HOL

② F - δ-HOL $\Rightarrow$ $H := \mathrm{Im} \int^{\delta} (F(z))^2 \, d^{\delta} z$ is well-defined

separately on $\bullet$'s and $\circ$'s:



$H(b_2) - H(b_1) := \mathrm{Im}\,[(F(z))^2 \cdot (b_2 - b_1$

$H(w_2) - H(w_1) := \mathrm{Im}\,[(F(z))^2 \cdot (w_2 - w_1$

③ H can be defined simultaneously on $\bullet$'s and $\circ$'s:

for each pair $w \sim b$

$$H(b) - H(w) := 2\delta \, \Big| \mathrm{Pr}\Big[F(z); \frac{1}{\sqrt{i(w-b)}}\Big] \Big|^2$$

ⓐ well-defined:

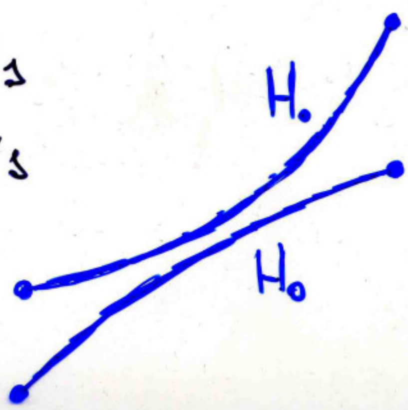
$$(H(b_2) - H(w_2)) + (H(b_1) - H(w_1)) = (H(b_2) - H(w_1)) + (H(b_1) - H(w_2$$

Hint: "$2\delta |F(z)|^2$"

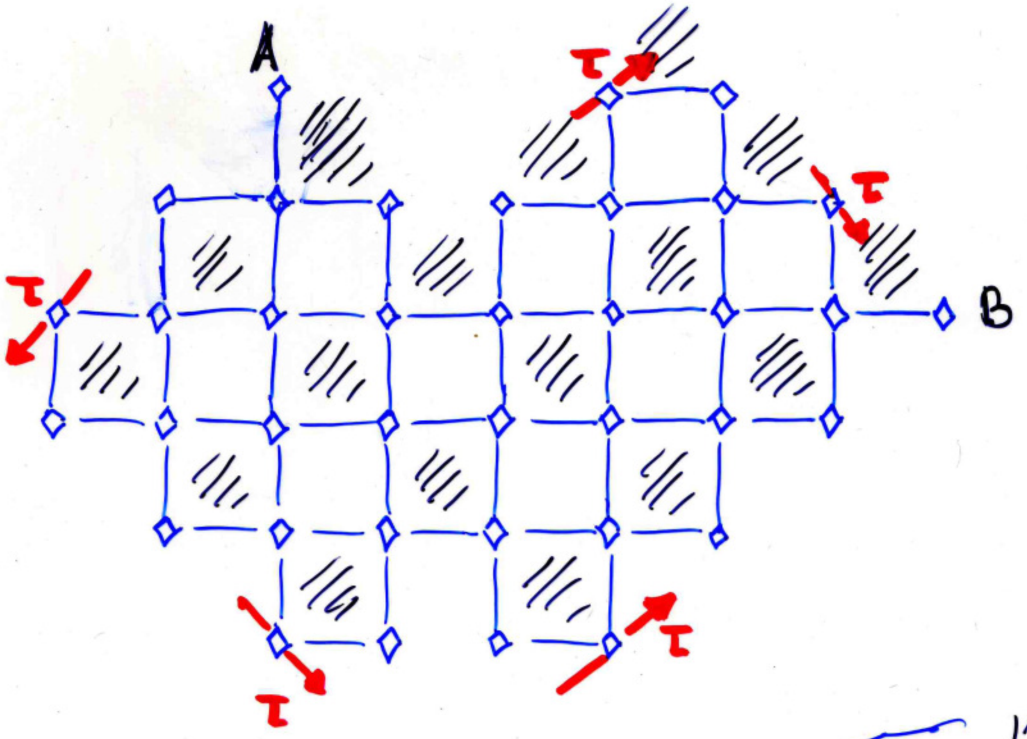
ⓑ compatible with standard definition

④ H is subharmonic on $\bullet$'s and superharmonic on $\circ$'s

[i.e., $\Delta^{\delta} H_{\bullet} \geq 0$
$\Delta^{\delta} H_{\circ} \leq 0$]



LaPietra



BOUNDARY CONDITIONS

$F(z) \uparrow\uparrow 1/\sqrt{I}$

$\Downarrow$

$H_\circ \equiv$ const on AB

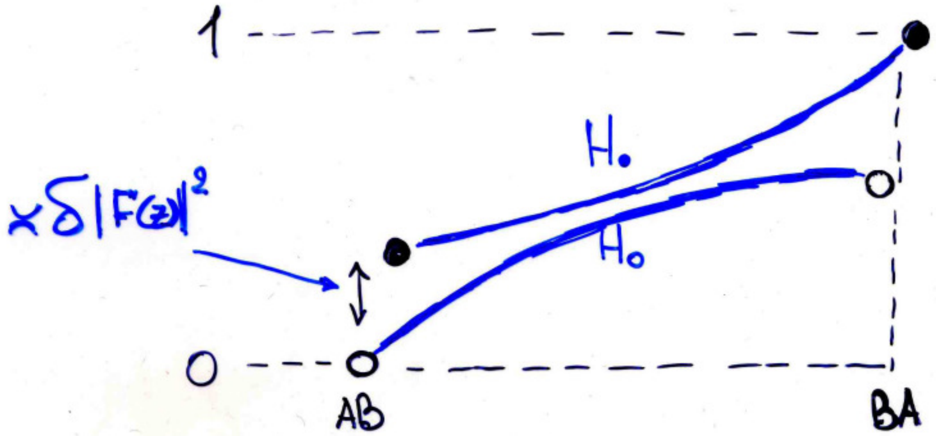
$H_\bullet \equiv$ const on BA

NORMALIZATION

$|F(A)| = F(B) = 1/\sqrt{2\delta}$

$\Rightarrow$

$$\boxed{\begin{array}{ll} H_\circ \equiv 0 & \text{on } AB \\ H_\bullet \equiv 1 & \text{on } BA \end{array}}$$



REMARK:

ON THE SQUARE LATTICE magnetisation estimates are available

$\Downarrow$

estimate of $|F(z)|$ on the boundary $\Longrightarrow$ $H \to \omega(\cdot, BA)$

PROPOSITION: $\left\{ H = \operatorname{Im} \int F^2 dz \right\}$ | La Pietra

$$\left.\begin{array}{l} F - \text{J-HOL IN } \Omega \\ |H_\bullet|, |H_\circ| \leq M \quad \text{IN } \Omega \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{|F(z)|^2 \leq \frac{\text{const}}{\operatorname{dist}(z, \partial\Omega)} \cdot M}$$

Remark: The same as if F^2-hol.

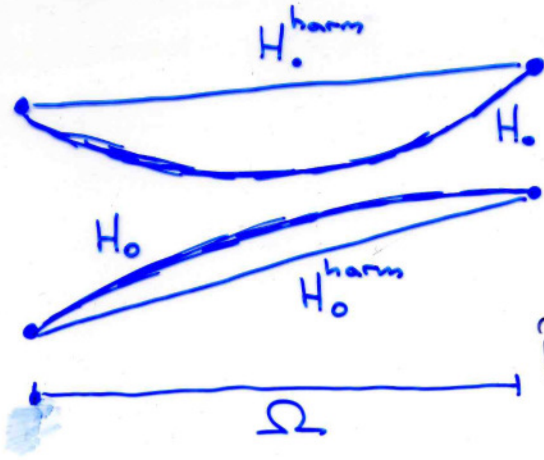
[In particular,

$$|H_\bullet - H_\circ| \leq \text{const} \cdot \frac{\delta}{d} M$$

$d \leftarrow$ dist to $\partial\Omega$]

SKETCH OF PROOF:

Let $\Omega = B(z, d)$ [some nice approximatio



$$H_\bullet = H_\bullet^{harm} - \widetilde{H}_\bullet$$

$$H_\circ = H_\circ^{harm} + \widetilde{H}_\circ$$

$\widetilde{H}_\circ, \widetilde{H}_\bullet$: superharmonic
$= 0$ on $\partial\Omega$
$\leq 2M$ inside

(1) $$\widetilde{H}_\bullet = \iint_\Omega^\delta G_\Omega(\cdot, u)(\Delta^\delta \widetilde{H}_\bullet)(u)\, dm^\delta(u)$$

$$\widetilde{H}_\bullet \leq 2M \Rightarrow \boxed{\|\Delta^\delta \widetilde{H}_\bullet\|_{L^1(\Omega')} \leq \text{const} \cdot M}$$

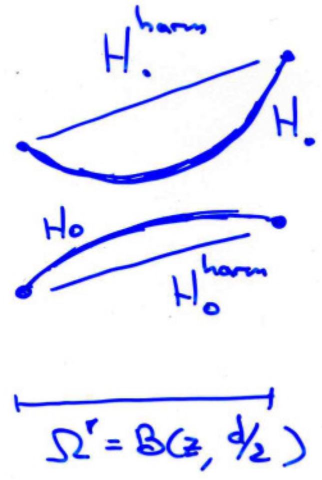
$\Omega' = B(z, d/2)$

"Proof":

$u \in B(z, d/2) \Rightarrow$

$$\Rightarrow \|G_\Omega(\cdot, u)\|_{L^1(\Omega)} \geq \text{const} \cdot d^2$$

② Consider Ω^r :



$$H_\bullet = H_\bullet^{harm} - \tilde{H}_\bullet$$
$$H_o = H_o^{harm} + \tilde{H}_o$$

$$\|\Delta^\delta \tilde{H}\|_{L^1(\Omega^r)} \leq const$$

[<u>Note:</u> $\Delta^\delta \tilde{H} = \Delta^\delta H$ doesn't depend on Ω]

$$\nabla \tilde{H} = \iint^\delta_{\Omega^r} \nabla G_{\Omega^r}(\cdot, u)(\Delta^\delta \tilde{H})(u)\, dm^\delta(u)$$

$$\Rightarrow \|\nabla \tilde{H}\|_{L^1(\Omega^{rr})} \leq \iint^\delta_{\Omega^r} \underbrace{\|\nabla G_{\Omega^r}(\cdot, u)\|_{L^1(\Omega^{rr})}}_{\leq\, const \cdot d} (\Delta^\delta \tilde{H})(u)\, dm^\delta(u)$$

So,
$$\boxed{\|\nabla \tilde{H}\|_{L^1(\Omega^{rr})} \leq const \cdot d \cdot M}$$

[USE ASYMPTOTICS OF GREEN'S FUNCTION [KENYON]]

③ BY HARNACK'S ESTIMATE THE SAME HOLDS FOR H^{harm} :

$$\|\nabla H^{harm}\|_{L^1(\Omega^{rr})} \leq const \cdot d\, \|\nabla H^{harm}\|_{L^2(\Omega^{rr})}$$

$$\leq const\, \|H^{harm}\|_{L^2(\Omega)} \leq const \cdot d \cdot M$$

(4)

$$\|\nabla H\|_{L^1(\Omega'')} = \|F^2\|_{L^1(\Omega'')}$$

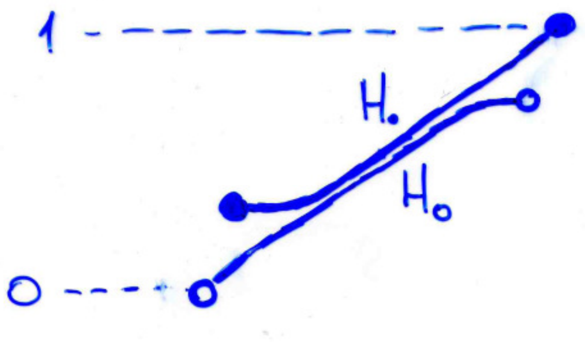
$$= \left(\|F\|_{L^2(\Omega'')}\right)^2 \leq \text{const} \cdot d \cdot M$$

MEAN VALUE PROPERTY
FOR HOLOMORPHIC FUNCTIONS:

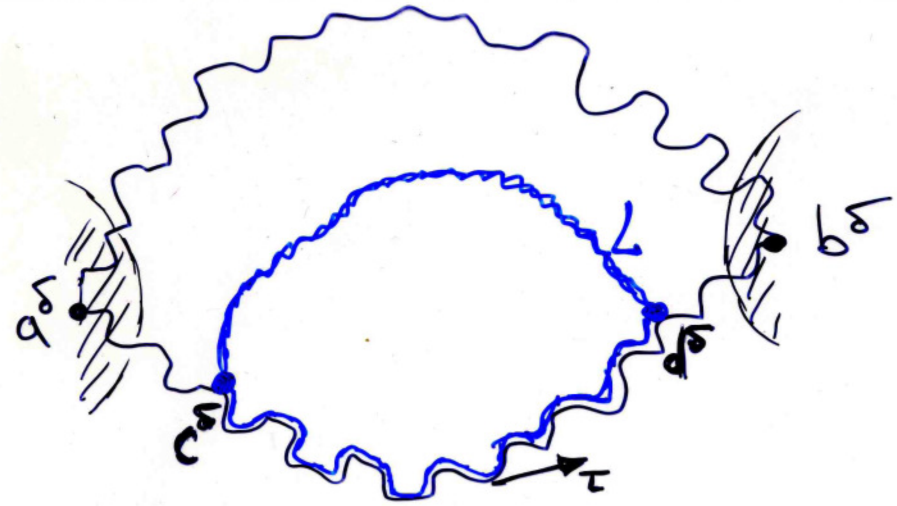
$$|F(z)|^2 \leq \frac{\text{const}}{d^2} \iint^{\delta} |F(z)|^2 \leq \frac{\text{const}}{d} \cdot M$$

$\boxtimes$

In particular, $H_\bullet$ and $H_\circ$ are uniformly close to each other inside Ω:



We still need to control $|F(z)|$ on the boundary.



CONSIDER $\oint_L^\delta (F^\delta(z))^2 \, d^\delta z$

① Boundary conditions $F(z) \uparrow\uparrow 1/\sqrt{\tau}$

$$\Rightarrow \int_{c^\delta d^\delta}^{\delta} (F^\delta(z))^2 \, d^\delta z = \int_{c^\delta d^\delta}^{\delta} |F^\delta(z)|^2 \, |d^\delta z|$$

② $|F^\delta(z)|^2 \leqslant \text{const} \cdot \frac{1}{d}$, $d = \text{dist}(z, \partial\Omega^\delta)$

$$\Rightarrow \left| \int_{\substack{\text{inner} \\ \text{part of } L}}^{\delta} (F^\delta(z))^2 \, d^\delta z \right| \leqslant \text{const} \cdot |\log \delta|$$

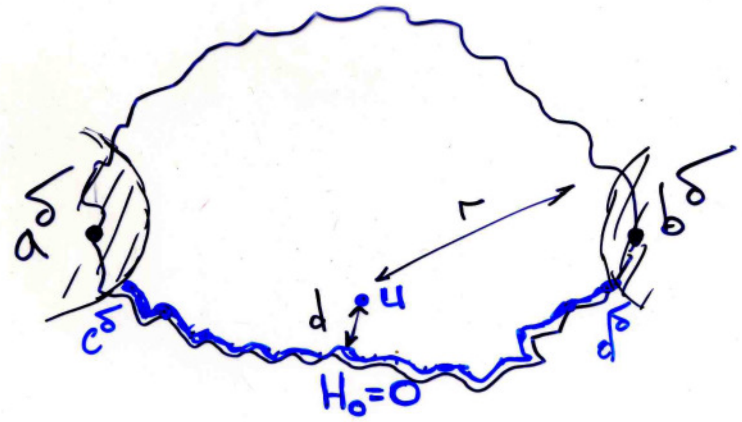
③ FORTUNATELY,

$$\int_L^\delta (F^\delta(z))^2 \, d^\delta z = - \sum (\Delta^\delta H_\bullet)(b) \, m^\delta(b) \leqslant 0$$

Σ over black vertices inside L

THUS:

$$\boxed{\int_{c^\delta d^\delta}^{\delta} |F^\delta(z)|^2 \, |d^\delta z| \leqslant \text{const} \cdot |\log \delta|}$$



very rough bound, but WE DIDN'T USE "EXTERNAL" INFORMATIC

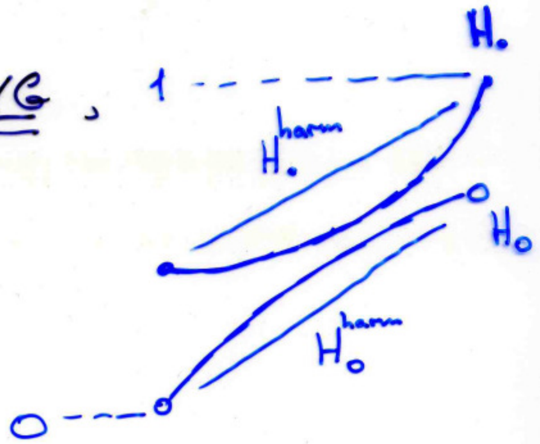
$$\underbrace{\int_{c^\delta d^\delta} |F^\delta(z)|^2 \, |d^\delta z|}_{\asymp} \le \text{const} \cdot |\log \delta|$$

$\vee$ ← !

$$\frac{1}{2\delta} \int_{c^\delta d^\delta} H_\bullet(z) \, |d^\delta z| \asymp \sum_{c^\delta d^\delta} H_\bullet$$

↑ closest to the boundary

APPLYING WEAK-BEUERLING,



$$H_\bullet^{harm}(u) \le \text{const} \cdot \underbrace{\frac{\delta^\beta |\log \delta|}{d^\beta}}_{\downarrow \delta \downarrow 0 \atop 0}$$

influence of $a^\delta b^\delta$

$$+ \ \text{const} \cdot \underbrace{\left(\frac{d}{r}\right)^\beta}_{\downarrow d \to 0 \atop 0}$$

influence of $b^\delta a^\delta$ and neighborhoods of a^δ, b^δ

COMPACTNESS ARGUEMENTS GIVE THE CONVERGENCE OF $F^\delta(z)$ TO $\sqrt{\varphi'(z)}$