Weyl-Titchmarsh functions of vector-valued Sturm-Liouville operators on $[0,1]$

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What I am NOT going to discuss. Scattering on $\mathbb{R}_+$:

SCALAR **vs** *MATRIX* potentials [Agranovich-Marchenko '50s–60s]

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where $V = V^* : [0,+\infty) \to \mathbb{C}^{N\times N}, \quad \int_0^{+\infty} x|V(x)|dx < +\infty.$

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Spectral data: *scattering matrix* $S(z)$;
finite number of *negative eigenvalues* $\lambda_j = -k_j^2$, $j = 1, .., m$, and *normalizing matrices* $M_j^* = M_j \geq 0$ $(\mathrm{rank} M_j = \text{multiplicity of} \lambda_j)$.

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Defined via $U(x,z)$ which is a ("properly normalized") solution of $\mathcal{H}U = z^2U$, $U(0,z) = 0$, such that, as $x \to +\infty$:

$$U(x,z) = e^{izx} - S(-z)e^{-izx} + o(1), \quad z > 0;$$
$$U(x,-ik_j) = e^{-|k_j|x}[M_k + o(1)], \quad j = 1,..,m.$$

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Necessary and sufficient conditions. Scalar case:

- **(I)** $S(z) = \overline{S(-z)} = [S(-z)]^{-1}$ is continuous on $\mathbb{R}$,

$$F_s(x) = F_s^*(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (1 - S(z)) e^{izx} dz, \quad x \in \mathbb{R},$$

$$F_s \in L^1 + (L^2 \cap L^\infty), \quad \int_0^{+\infty} x |F_s'(x)| dx < +\infty;$$

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- **(II)**

$$m + \frac{1 - S(0)}{4} = \frac{\log S(+0) - \log S(+\infty)}{2\pi i}$$

[m is the number of negative eigenvalues].

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Necessary and sufficient conditions. Matrix case:

- **(II)** $F_s(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(1-S(z))e^{izx}dz$
  - the equation $-x(t) + \int_{-\infty}^{0} x(\xi)F_s(t+\xi)d\xi = 0$, $-\infty < t \leq 0$, has no non-trivial solution;
  - the equation $x(t) + \int_{0}^{+\infty} x(\xi)F(t+\xi)d\xi = 0$, $0 \leq t < +\infty$, has no non-trivial solution, $F(t) = \sum_{j=1}^{m} M_j^2 e^{-|k_j|t} + F_s(t)$;
  - the number of linear independent solutions of the equation $x(t) + \int_{0}^{+\infty} x(\xi)F_s(t+\xi)d\xi = 0$, $0 \leq t < +\infty$, is equal to the sum of the ranks of the normalizing matrices $M_1, ..., M_m$.

Sturm-Liouville operators on $[0, 1]$:

$$\mathcal{L}\psi = -\psi'' + V\psi \quad [\text{ acting in } L^2([0,1]; \mathbb{C}^N) \ ]$$

Dirichlet boundary conditions:

$$\psi(0) = \psi(1) = 0$$

Self-adjoint MATRIX potentials:

$$V(x) = [V(x)]^*, \quad V \in L^2([0,1]; \mathbb{C}^{N \times N})$$

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Self-adjoint MATRIX potentials:

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- *$\mathcal{L}$ has purely discrete spectrum $\lambda_1 < \lambda_2 < \lambda_3 < \dots$;*
- *possible multiplicities are $1, 2, .., N$.*

Sturm-Liouville operators on $[0,1]$: $\mathcal{L}\psi = -\psi'' + V\psi$

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Weyl-Titchmarsh function:

Let φ, χ be the solutions of $\mathcal{L}\psi = \lambda\psi$ such that

$$\begin{cases} \varphi(0) = 0, \ \varphi'(0) = I_N, \\ \chi(1) = 0, \ \chi'(1) = -I_N. \end{cases}$$

$$M(\lambda) = M(\lambda, V) := [\chi'\chi^{-1}](0,\lambda,V).$$

If $V = V^*$, then $M(\lambda) = [M(\overline{\lambda})]^*$ and $\mathrm{Im} M(\lambda) \geq 0$ for $\lambda \geq 0$.

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- *Eigenvalues of $\mathcal{L}$ coincide with singularities of M.*

Scalar case. Characterization.

[Marchenko-Ostrovski '75]

The Weyl-Titchmarsh function $m(\lambda, v)$ is a meromorphic function having simple poles at Dirichlet eigenvalues $\lambda_n(v)$ and

$$\operatorname*{res}_{\lambda=\lambda_n(v)} m(\lambda, v) = -[g_n(v)]^{-1} = -\left[\int_0^1 |\varphi(x, \lambda_n, v)|^2 dx\right]^{-1}$$

The sharp characterization of all *scalar* Weyl-Titchmarsh functions (equivalently, spectral data $(\lambda_n(v), g_n(v))_{n=1}^{+\infty}$) that correspond to potentials $v \in \mathcal{L}^2(0,1)$ (or other reasonable spaces) is available.

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Namely, the necessary and sufficient conditions are

$$\lambda_1 < \lambda_2 < \lambda_3 < ..., \qquad (\lambda_n - \pi^2 n^2 - v_0)_{n=1}^{+\infty} \in \ell^2, \quad v_0 \in \mathbb{R}$$

$$\text{and} \qquad (\pi n \cdot (2\pi^2 n^2 \cdot g_n - 1))_{n=1}^{+\infty} \in \ell^2.$$

Actually, $v_0 = \int_0^1 v(x)dx$.

Matrix case. Spectral data.

- *eigenvalues* $\lambda_1 < \lambda_2 < .. < \lambda_\alpha < ...$ (and *multiplicities* k_α);
- *residues of M:* $-\operatorname{res}_{\lambda=\lambda_\alpha} M(\lambda) = B_\alpha = B_\alpha^* \geq 0$, $\operatorname{rank} B_\alpha = k_\alpha$

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$$B_\alpha = P_\alpha g_\alpha^{-1} P_\alpha,$$

  where

  - $P_\alpha : \mathbb{C}^N \to \mathcal{E}_\alpha \subset \mathbb{C}^N$ is an *orthogonal projector* $(\operatorname{rank} P_\alpha = \dim \mathcal{E}_\alpha = k_\alpha)$
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Equivalent definition:

$$\mathcal{E}_\alpha = \operatorname{Ker} \varphi(1, \lambda_\alpha, V) = \left\{ h \in \mathbb{C}^N : \psi_{\alpha;h} = \varphi(\cdot, \lambda_\alpha, V)h \in \operatorname{Ker}(\mathcal{L} - \lambda_\alpha) \right\},$$

$$\langle \psi_{\alpha;h_1}, \psi_{\alpha;h_2} \rangle_{L^2([0,1];\mathbb{C}^N)} = \langle h_1, g_\alpha h_2 \rangle_{\mathcal{E}_\alpha}, \qquad g_\alpha = p_\alpha \left[\int_0^1 [\varphi^* \varphi](x, \lambda_\alpha, V) dx \right] p_\alpha^*$$

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Uniqueness Theorem: [M. M. Malamud '05, V. A. Yurko '06]

The matrix-valued function $M(\lambda)$ (or, equivalently, the collection of spectral data $(\lambda_\alpha, P_\alpha, g_\alpha)_{\alpha=1}^{+\infty}$) determines the potential uniquely.

Isospectral Flows.

[D.Ch., E.K.: Parametrization of the isospectral set for the vector-valued Sturm-Liouville problem.

J. Funct. Anal. 241(1), 359–373 (2006). arXiv:math.SP/0607810]

Fix some admissible spectrum $\{\lambda_\alpha\}_{\alpha\geq 1}$ (and multiplicities k_α) and all the residues $B_\alpha = P_\alpha g_\alpha^{-1} P_\alpha$, $\alpha \neq \beta$, *except one*. Then:

- g_β can be changed *arbitrarily* [M. Jr. Jodeit; B. M. Levitan '98]
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  There exists the *"forbidden subspace"* $\mathcal{F}_\beta$, $\dim\mathcal{F}_\beta = N - k_\beta$, which is uniquely determined by the spectrum and $(\mathcal{E}_\alpha)_{\alpha\neq\beta}$ such that all "deformations" $P_\beta \mapsto \widetilde{P}_\beta$: $\mathcal{F}_\beta \cap \mathrm{Ran}\widetilde{P}_\beta = \{0\}$ are permitted (the new potential is constructed *explicitly*).

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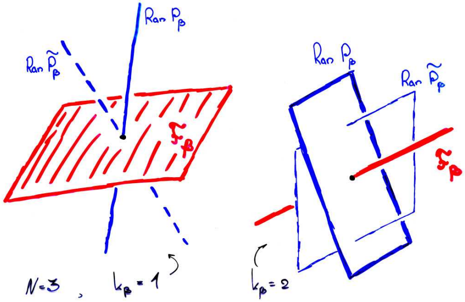

Toy example (discrete version). Block Jacobi matrices.

[A.I.Aptekarev, E.M.Nikishin '83: The scattering problem for a discrete Sturm-Liouville operator;

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Let $b_p^* = b_p$, $a_p = a_p^* > 0$ be $N \times N$ matrices and

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots & 0 \\ a_1^* & b_2 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{n-2}^* & b_{n-1} & a_{n-1} \\ 0 & \dots & 0 & 0 & a_{n-1}^* & b_n \end{pmatrix}.$$

- $\sigma(\mathcal{J})$: $\lambda_1 < \lambda_2 < \dots < \lambda_m, \qquad k_1 + k_2 + \dots + k_m = Nn;$

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- $\sigma(\mathcal{J})$: $\lambda_1 < \lambda_2 < \cdots < \lambda_m, \qquad k_1 + k_2 + \cdots + k_m = Nn$;
- residues of the (rational) M-function:

  $\chi_{n+1} := 0,\ \chi_n := I,\ a_{p-1}^*\chi_{p-1} + b_p\chi_p + a_p\chi_{p+1} = \lambda \cdot \chi_p,$

  $$B_s = P_s g_s^{-1} P_s := -\operatorname*{res}_{\lambda=\lambda_s} M(\lambda), \quad M(\lambda) := -[\chi_1\chi_0^{-1}](\lambda).$$

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- $\sigma(\mathcal{J})$: $\lambda_1 < \lambda_2 < \dots < \lambda_m$, $\qquad k_1 + k_2 + \dots + k_m = Nn$;
- $(\lambda_s, B_s)_{s=1}^m$, $B_s = P_s g_s^{-1} P_s$, $\mathrm{rank} P_s = k_s$, should be such that

  there exists no (nontrivial) vector-valued polynomial
  $F : \mathbb{C} \to \mathbb{C}^N$, $\deg F \le n-1$: $\quad P_s F(\lambda_s) = 0$, $s = 1, \dots, m$.

Main result: [D.Ch., E.K. '08]

For all $v_1^0 < v_2^0 < .. < v_n^0$ the mapping $V \mapsto (\lambda_\alpha, P_\alpha, g_\alpha)_{\alpha=1}^{+\infty}$ is a **bijection** between the set of potentials

$$V = V^* \in L^2([0,1]; \mathbb{C}^{N\times N}) : \qquad \int_0^1 V(x)dx = \mathrm{diag}\{v_1^0, v_2^0, .., v_N^0\}$$

and the class of spectral data satisfying (A)–(C):

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and the class of spectral data satisfying (A)–(C):

(A) The spectrum is *asymptotically simple*, i.e., $\exists \alpha^\diamond \ge 0, n^\diamond \ge 1$:

$$k_1^\diamond + k_2^\diamond + .. + k_{\alpha^\diamond}^\diamond = N(n^\diamond - 1) \quad \text{and} \quad k_\alpha^\diamond = 1 \ \text{ for all } \alpha \ge \alpha^\diamond + 1.$$

It allows us to define the double-indexing (n,j), $n \ge n^\diamond$, $j = 1, .., N$, instead of the single-indexing $\alpha > \alpha^\diamond$. Namely, we set

$$\lambda_{n,j} = \lambda_{\alpha^\diamond + N(n-n^\diamond)+j}, \ g_{n,j} = g_{\alpha^\diamond + N(n-n^\diamond)+j} \ \text{ etc. } \ \text{for } n \ge n^\diamond.$$

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and the class of spectral data satisfying (A)–(C):

(A) The spectrum is *asymptotically simple*.

(B) The *asymptotics* of spectral data *in ℓ^2–sense* hold true:

$$(\lambda_{n,j} - \pi^2 n^2 - v_j^0)_{n=n^\diamond}^{+\infty} \in \ell^2, \qquad (\pi n \cdot (2\pi^2 n^2 g_{n,j} - 1))_{n=n^\diamond}^{+\infty} \in \ell^2,$$

$$(|P_{n,j} - P_j^0|)_{n=n^\diamond}^{+\infty} \in \ell^2 \quad \text{and} \quad (\pi n \cdot |\textstyle\sum_{j=1}^N P_{n,j} - I_N|)_{n=n^\diamond}^{+\infty} \in \ell^2.$$

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and the class of spectral data satisfying (A)–(C):

(A) The spectrum is *asymptotically simple*.

(B) The *asymptotics* of spectral data *in ℓ^2–sense* hold true.

(C) The collection $(\lambda_\alpha\,; P_\alpha)_{\alpha=1}^{+\infty}$ satisfies the following property:

Let $\xi : \mathbb{C} \to \mathbb{C}^N$ be an entire vector-valued function such that $\xi(\lambda) = O(e^{|\mathrm{Im}\sqrt{\lambda}|})$ as $|\lambda| \to \infty$ and $\xi \in L^2(\mathbb{R}_+)$.

If $P_\alpha \xi(\lambda_\alpha) = 0$ for all $\alpha \geq 1$, then $\xi(\lambda) \equiv 0$.

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Remarks [concerning (C)]:

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- *Nontrivial* in the vector-valued case; describes all "forbidden subspaces" $\mathcal{F}_\beta$ (restrictions $\mathcal{E}_\beta \cap \mathcal{F}_\beta = \{0\}$) simultaneously.
- Equivalent to the following (if all $\lambda_\alpha > 0$):

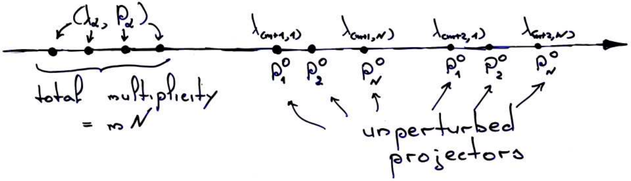
  Let $P_\alpha = h_\alpha h_\alpha^*$, *where* $h_\alpha = (h_\alpha^{(1)}; \ldots; h_\alpha^{(k_\alpha)})$ *and* $h_\alpha^{(j)} \in \mathbb{C}^N$ *are orthonormal. Then the vector-valued functions* $e^{\pm i\sqrt{\lambda_\alpha}t} h_\alpha^{(j)}$, $j = 1, \ldots, k_\alpha$, $\alpha \geq 1$, *together with the constant vectors* $e_1^0, \ldots, e_N^0$ *span* $L^2([-1, 1]; \mathbb{C}^N)$.

(C) $\xi : \mathbb{C} \to \mathbb{C}^N$, $\xi(\lambda) = O(e^{|\mathrm{Im}\sqrt{\lambda}|})$ *as* $|\lambda| \to \infty$ *and* $\xi \in L^2(\mathbb{R}_+)$.

$$\textit{If } P_\alpha \xi(\lambda_\alpha) = 0 \textit{ for all } \alpha \geq 1, \quad \textit{then } \xi(\lambda) \equiv 0.$$

Remarks [concerning (C)]:

- (C) can be rewritten *explicitly*, if $P_{n,j} = P_j^0$ for all $n \geq m+1$:



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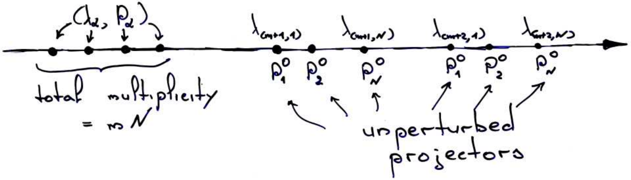

$$\mathcal{T} = \begin{pmatrix} T_0 & T_1 & \dots & T_{m-1} \\ T_1 & T_2 & \dots & T_m \\ \dots & \dots & \dots & \dots \\ T_{m-1} & T_m & \dots & T_{2m-2} \end{pmatrix} = \mathcal{T}^* > 0,$$

where

$$T_k = \sum\nolimits_{\lambda_\alpha < \lambda_{m+1,1}} F(\lambda_\alpha) P_\alpha F(\lambda_\alpha) \cdot \lambda_\alpha^k = T_k^*,$$

$$F(\lambda) \equiv \mathrm{diag}_{j=1,..,N} \left\{ \prod\nolimits_{n=m+1}^{+\infty} \left(1 - \frac{\lambda}{\lambda_{n,j}}\right) \right\}.$$

Proof. Asymptotics.

- (1) $(\lambda_{n,j} - \pi^2 n^2 - v_j^0)_{n=n^\diamond}^{+\infty} \in \ell^2$,
- (2) $(\pi n \cdot (2\pi^2 n^2 g_{n,j} - 1))_{n=n^\diamond}^{+\infty} \in \ell^2$,
- (3) $(|P_{n,j} - P_j^0|)_{n=n^\diamond}^{+\infty} \in \ell^2$,

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- (4) $(\pi n \cdot |\sum_{j=1}^N P_{n,j} - I_N|)_{n=n^\diamond}^{+\infty} \in \ell^2$

follows from the analysis of the sum $\sum_{\lambda_\alpha : |\lambda_\alpha - \pi^2 n^2| = O(1)} B_\alpha$ (which behaves better then the individual residues).

Proof. Asymptotics.

- (2) $(\pi n\cdot(2\pi^2 n^2 g_{n,j}-1))_{n=n^\diamond}^{+\infty}\in\ell^2$,
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Equivalent description (technical trick):

$$B_{n,j}=P_{n,j}g_{n,j}^{-1}P_{n,j},\quad P_{n,j}=\langle\cdot,h_{n,j}\rangle h_{n,j}:\ \langle h_{n,j},e_j^0\rangle>0.$$

Let $\quad H_n=\frac{1}{\sqrt{2}\pi n}(h_{n,1};\ldots;h_{n,N})=S_nU_n,\quad S_n=S_n^*,\ U_n^*=U_n^{-1}.$
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Note:

$$S_n^2 = H_n H_n^* = \frac{1}{2\pi^2 n^2} (B_{n,1} + ... + B_{n,N}).$$

Proof. Inverse problem. Main steps. *[Trubowitz's approach]*

Consider some $(\lambda_\alpha^\sharp, P_\alpha^\sharp, g_\alpha^\sharp)_{\alpha=1}^{+\infty}$ satisfying (A)–(C).

- Construct some (special) *diagonal* potential V^0 such that $\sigma(V^0) = \{\lambda_\alpha^\sharp\}_{\alpha=1}^{+\infty}$ (counting with multiplicities);

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$$\widetilde{P}_\alpha = P_\alpha^\sharp \quad \text{and} \quad \widetilde{g}_\alpha = g_\alpha^\sharp \quad \text{for all } \alpha > \alpha_*;$$

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$$\widetilde{P}_\alpha = P_\alpha^\sharp \ \text{ and } \ \widetilde{g}_\alpha = g_\alpha^\sharp \ \text{ for all } \alpha > \alpha_*;$$

- Use a *finite number* of *isospectral flows* to modify $(\widetilde{P}_\alpha, \widetilde{g}_\alpha)_{\alpha=1}^{\alpha_*}$.

Related questions:

- *Borg type results* (re-parametrization for this class of meromorphic functions). In the scalar case one can use zeros of $m(\lambda, v)$ (i.e., the spectrum of the mixed boundary value problem $\psi'(0) = 0$, $\psi(1) = 0$) instead of the normalizing constants g_n. *How many spectra does one need (in the vector-valued case) to determine the potential uniquely?*

Related questions:

- *Borg type results*
- *Geometry: splitting of eigenvalues* (topology of isospectral manifolds *essentially depends on* the multiplicities k_α).

Related questions:

- *Borg type results*
- *Geometry: splitting of eigenvalues*
- *Degenerate mean potential V^0: looking for "nice" parameters* (structure and asymptotics of the additional spectral data are simpler to describe, if the spectrum is asymptotically simple). If v^0 is a multiple eigenvalue of V^0, then the regularization

$$B_{n,(v^0)} := \sum_{\{\lambda_\alpha \text{ near } \pi^2 n^2 + v^0\}} B_\alpha;$$

$$D_{n,(v^0)} := \sum_{\{\lambda_\alpha \text{ near } \pi^2 n^2 + v^0\}} \lambda_\alpha B_\alpha;$$

seems promising.

Related questions:

- *Borg type results*
- *Geometry: splitting of eigenvalues*
- *Degenerate mean potential V^0: looking for "nice" parameters*
- *Other classes of potentials:*
  Recently, Ya.V.Mikityuk and N.S.Trush (Lviv) announced the result for the class W_2^{-1} [using M.G.Krein's approach].

Related questions:

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- *Geometry: splitting of eigenvalues*
- *Degenerate mean potential V^0: looking for “nice” parameters*
- *Other classes of potentials*
- *Other (separated but non-Dirichlet) boundary conditions*
  [smth. in S.Matveenko’s talk on Wednesday, Aug 5]

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- *[?] Some revision of the 1D inverse scattering problems with matrix potentials*

THANK YOU!