AN UPPER BOUND FOR THE NUMBER OF VASSILIEV KNOT INVARIANTS

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Abstract

We prove that the number of independent Vassiliev knot invariants of order n is less than $(n-1)!$ — thus strengthening the *a priori* bound $(2n-1)!!$

1 Introduction to Vassiliev's theory

The notion of finite order knot invariants has appeared as a byproduct of V. Vassiliev's investigations on the topology of discriminants in functional spaces. Subsequent authors, notably J. Birman, X.-S. Lin, D. Bar-Natan, V. I. Arnold and A. B. Sossinsky, have much contributed to the simplification of notions, notations and techniques of the inventor's original work [V]. In the introduction that follows, we have tried to go still further and make our exposition of Vassiliev's theory as elementary as possible, sometimes deviating from the historical traditions established in [A], [B-L], [BN], and [S].

1.1 Invariants of singular knots

Unless otherwise specified, by a *knot* we will always understand a *singular knot*, i. e. a smooth immersion $f : S^1 \to \mathbb{R}^3$ which may have a finite number of simple double points. We will assume all knots to be *oriented* and use smooth ambient isotopy as the equivalence relation between knots. Note that protagonists of V. Vassiliev's original works were knots with a fixed initial point, i. e. mappings $f : \mathbb{R} \to \mathbb{R}^3$ having an asymptotic straight line; however, this difference does not influence the contents of the Vassiliev invariant theory.

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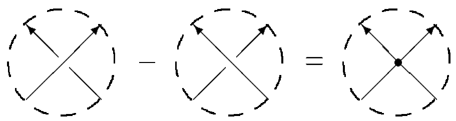
1.1.1. Definition. A *Vassiliev invariant* is a numerical function of a singular knot which is completely determined by its isotopy class and satisfies the following fundamental relation:



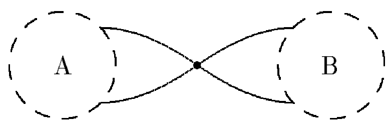
By these three figures we understand three knots which are identical outside a small ball (and can contain further double points there) and are as shown in the picture inside the ball. In particular, the knot in the right-hand side contains one double point more than either knot on the left.

Here and below, all the equations involving knot figures signify the numerical equality of values of one and the same invariant on these knots.

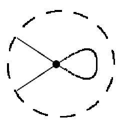
The adjective *numerical* implies a certain Abelian group over which the invariants are considered. Although the bulk of the results mentioned or proved below are valid over an arbitrary group, we will suppose that this group is a field of characteristic zero.

Remark. The above definition is more general than the commonly adopted one. Other authors require that Vassiliev invariants be necessarily of finite type (see subsection 1.2.1 below).

1.1.2. Corollary of definition. *The value of any Vassiliev invariant on a knot like*



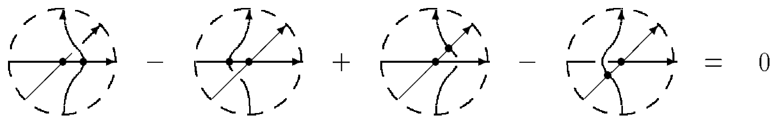
is equal to zero. In particular, any Vassiliev invariant vanishes on any knot that contains fragment



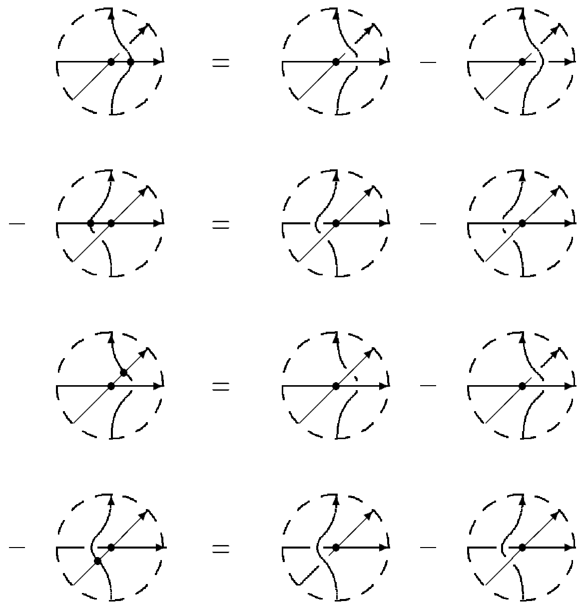
PROOF. It is easy to see that the two knots in the left hand side of fundamental relation 1.1.1 are equivalent in this case. □

There is another very important corollary that can be deduced directly from the definition.

1.1.3. Four term relation for knots. *Any Vassiliev invariant satisfies*



PROOF. By the fundamental relation



In the right-hand parts of these equalities, every term appears twice with the opposite signs, so that everything cancels. □

1.2 Finite order invariants

1.2.1. Definition. A Vassiliev invariant *is of order no greater than* n if it vanishes on any knot with more than n double points.

The set of all Vassiliev invariants of order no greater than n is a vector space. Denoting this space by V_n, we obtain a filtration

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \cdots$$

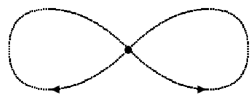
in the entire space of Vassiliev invariants.

1.2.2. Example. $\dim V_0 = 1$.

Indeed, any $v \in V_0$ takes zero values on knots with at least one singular point. Fundamental relation 1.1.1 implies that we can change the over- and under-crossings without changing the value of v. Hence v is constant. □

1.2.3. Example. $V_1 = V_0$.

Fundamental relation implies that any invariant of order no greater than 1 takes equal values on any two knots with one double point. Since its value on the knot



is zero, we have $v \in V_0$. $\square$

1.2.4. Example. $\dim V_2 = 2$.

Up to an additive constant, the only second order invariant $v \in V_2 \setminus V_0$ is the coefficient of x^2 in the Conway polynomial (see [V] or [B-L]).

1.3 Symbols of Vassiliev invariants

We will say that an invariant v has *order n* if $v \in V_n$ and $v \notin V_{n-1}$.

1.3.1. Definition. The *symbol of an n-th order invariant* is the restriction of the invariant to the set of knots having exactly n double points.

Due to the fundamental relation, considering the values of an n-th order invariant on a knot with n double points, we are free to make *flips*, i. e. change an overcrossing by an undercrossing or vice versa. The values of a symbol do not depend on the specific 'knottedness' being completely defined by the order of the double points along the knot.

This observation can be stated more formally in terms of *chord diagrams*.

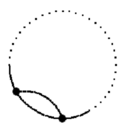
1.3.2. Definition. A *chord diagram* of order n is an oriented circle with a distinguished set of n disjoint pairs of distinct points, up to an equivalence defined by orientation preserving diffeomorphism of the circle.

A chord diagram can be depicted as a counterclockwise oriented circle with a set of n chords all of whose endpoints are distinct. Note that the geometry of the chords is irrelevant; what matters is the way they bind their endpoints into pairs.

The *chord diagram of a knot* $f : S^1 \to \mathbb{R}^3$ is the circle S^1 with the preimages of every double point connected by a chord.

The value of a symbol on a given knot depends only on its chord diagram. In the relations that follow, we will simply picture the chord diagrams instead of the correspomding values of symbols.

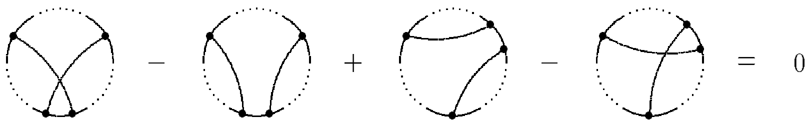
1.3.3. Isolated chord relation. *The value of any symbol on a chord diagram with an isolated chord, i.e. on a diagram like*



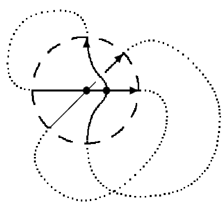
is equal to zero. Here and below, the dotted arcs suggest that there might be further chords attached to their points, while the solid portions of the circle honestly display everything they have.

PROOF. This diagram can be represented by a knot with a loop like the one in 1.1.2. □

1.3.4. Four term relation.

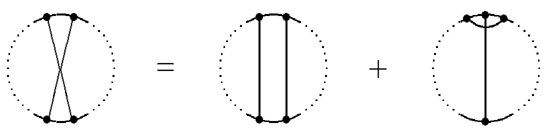


PROOF. The first diagram appearing in the above relation can be represented by the knot



which has a fragment shown above in the four term relation for knots 1.1.3. The three remaining knots of 1.1.3 will produce the other diagrams of the present relation. □

1.3.5. Corollary.



PROOF. This obviously follows from 1.3.3 and 1.3.4. □

1.3.6. Kontsevich's theorem. *All linear relations between values of the symbol of an arbitrary Vassiliev invariant, may be generated by the isolated chord relations 1.3.3 and four term relations 1.3.4.*

In other words, any function on the set of all chord diagrams of order n that satisfies relations 1.3.3 and 1.3.4 is the symbol of a certain n-th order Vassiliev invariant. This statement looks like a purely combinatorial one; however, up to now, only analytical proofs, valid over $\mathbb{C}$, of this fact are known[1] (see [K], [BN] for details).[2]

2 The main result

Right after his discovery of finite order knot invariants, V. Vassiliev proposed an algorithm to generate all the invariants of a given order. This algorithm, however explicit, is extremely complex, and the dimensions of the spaces V_n/V_{n-1} are known only for $n \leq 9$ (the value 44 for $n = 9$ was found by D. Bar-Natan on January 12, 1993, after his computer worked for several days with an extra 32 Mb of borrowed memory). See [BN] for a discussion of these values and the estimate for $\dim V_n/V_{n-1}$ from below.

The only upper bound for this dimension known so far was the evident *a priori* bound $(2n-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$ which gives the total number of ways to draw n chords in a circle (this number can be decreased by a factor of the order of $2n$ if the rotations of the circle are taken into consideration, but nobody has performed the exact computations to this end).

The main result of this note is the upper bound

$$\dim V_n/V_{n-1} \leq (n-1)!$$

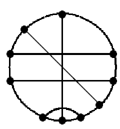
Denote by Δ_n the quotient space of the space generated by all chord diagrams of order n by the subspace generated by relations 1.3.3 and 1.3.4. There is a natural mapping $V_n \to \Delta_n^*$ which sends every Vassiliev invariant of order n to the corresponding linear function on Δ_n. The subspace V_{n-1} coincides with the kernel of this mapping, so that

$$\dim V_n/V_{n-1} \leq \dim \Delta_n .$$

Over a field of characteristic zero, by Kontsevich's theorem, this inequality actually becomes an equality. However, to estimate the number of Vassiliev invariants from above, it is sufficient to use the inequality and find an upper bound for $\dim \Delta_n$.

2.1. Definition. A *spine diagram* is a chord diagram such that one (or more) of its chords intersects all the other chords of the diagram.

For example,



is a spine diagram of order 5.

Remark. Both in this definition and below, we understand *intersection* of chords in a combinatorial sense: a chord joining a_1 and a_2 is said to *intersect* a chord joining

[1] Validity over $\mathbb{C}$ implies validity over $\mathbb{Q}$ and hence over any field of characteristic 0.

[2] *Note added in revision.* About the time this text was sent to the journal, there appeared two papers, [P] and [C], whose authors claim to have found a combinatorial proof of Kontsevich's theorem.

b_1 and b_2, if, and only if, when travelling around the circle, the points a_1 and a_2 interleave between the points b_1 and b_2.

2.2. Main theorem. *Equivalence classes of spine diagrams modulo the one- and four-term relations generate Δ_n as a vector space.*

The theorem implies the upper bound

$$\dim V_n/V_{n-1} = \dim \Delta_n \leq (n-1)!$$

because, since every spine diagram is uniquely determined by the one-to-one correspondence of the points on the left with the points on the right of the 'spine', there cannot be more than $(n-1)!$ of them.

3 Proof of the main theorem

The theorem will be proved by induction. First we choose the induction parameter.

3.1. Definition. Let D be a chord diagram. The *degree* $\deg(c)$ of a *chord* $c \in D$ is the number of chords in D which intersect c. The *degree* $\deg(D)$ of a *chord diagram* is the maximum of degrees of its chords:

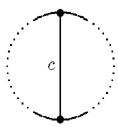
$$\deg(D) \;=\; \max_{c\in D} \deg(c).$$

3.2. Induction base. If $\deg(D) = n-1$ then D is a spine diagram.

3.3. Induction hypothesis. Suppose that every diagram D of degree greater than d is equal to a linear combination of spine diagrams modulo relations 1.3.3 and 1.3.4.

3.4. Induction step. We have to show that a diagram of degree d can be expressed through diagrams of bigger degrees.

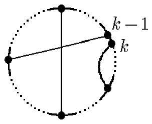
Let D be a diagram such that $\deg(D) = d$ and $c \in D$ a chord such that $\deg(c) = d$. We will draw this chord vertically:



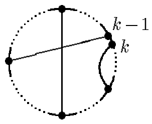
Since $d < n-1$ and we can rotate the whole picture through $180°$ if necessary, we may assume that there are two points in the right half of the circle that are connected by a chord.

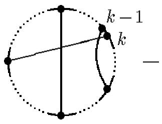
3.4.1 Let us enumerate the endpoints of the chords on the right from top to bottom. Let k be the number of the first point connected to a point in the right half-circle. We will reduce the problem to the case $k = 1$.

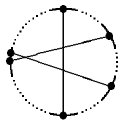
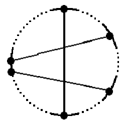
If $k \neq 1$, then D looks something like this:



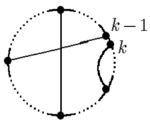
Applying the four term relation 1.3.4, we obtain:

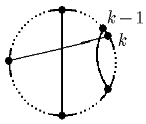
D = 

= 

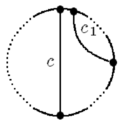
−  + 

The degrees of the last two diagrams in this relation are higher than d, hence they are equivalent to linear combinations of spine diagrams by the induction hypothesis. So

D = 

$\sim$ 

Proceeding in this way, we will arrive at a diagram with $k = 1$, so that without loss of generality we may assume that D has the form

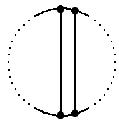


3.4.2. Now we are going to lower (= to increase the ordinal number of) the second endpoint of the chord c_1. We will do this by induction with respect to the following parameter.

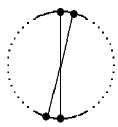
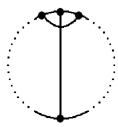
3.4.2.1. Definitions. The lower end of any chord which intersects c_1 and does not intersect c will be called *bound*. A point that lies on the lower arc between c and c_1 and is not bound, will be referred to as *loose*.

Let l be the number of loose points in a chord diagram, and b the number of bound points between the lower end of c_1 and the first loose point. We call the pair (l, b) the *index* of the chord diagram and use it as the induction parameter with respect to the lexicographical ordering: $(l_1, b_1) > (l_2, b_2)$ if and only if either $l_1 > l_2$ or $l_1 = l_2$ and $b_1 > b_2$.

3.4.2.2. Induction base I. If both components of the index are zero, $(l, b) = (0, 0)$, then our diagram D has the form

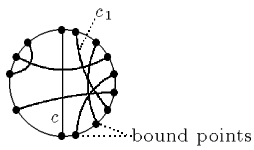
$D =$ 

By Corollary 1.3.5, this can be expressed as

$D =$  $-$ 

and the degrees of both diagrams on the right are bigger than d.

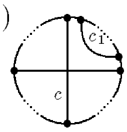
3.4.2.3. Induction base II. Suppose that $l = 0$ but $b \neq 0$. Then our diagram is like this (for example):

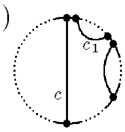


Here every chord that intersects c also intersects c_1, hence the degree of such diagram is greater than d.

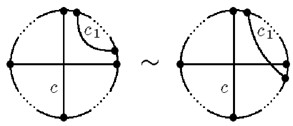
3.4.2.4. Induction hypothesis. Suppose that every diagram D whose index is smaller than (l, b) is equivalent to a linear combination of diagrams of degree greater than d.

3.4.2.5. Induction step I. Now let D be a diagram with index (l, b), where $b = 0$. The point adjacent to the lower endpoint of the chord c_1 is loose, and there are two possibilities for the corresponding chord:

1) 

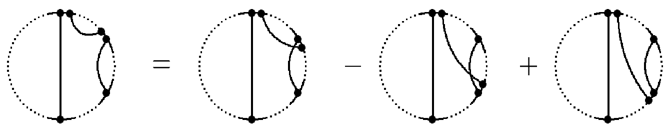
2) 

In the first case, by the four term relation 1.3.4 and the induction hypothesis

$D \quad =$ 

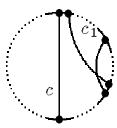
where the last diagram looks like D but the number of its loose points is $l-1$.

The second case is also easy, because relation 1.3.4 implies

$D \quad =$ 

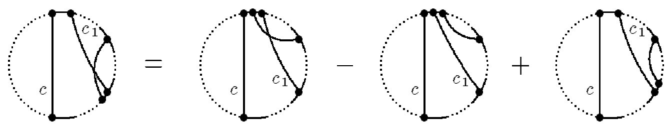
Each of the three diagrams in the right-hand side has less than l loose points.

3.4.2.6. Induction step II. Now suppose that the diagram D has index (l, b) with $b > 0$, so it looks typically like this:

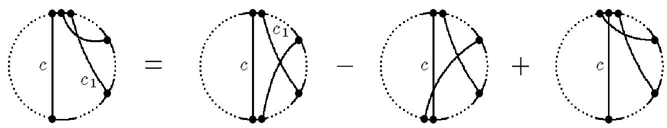


(the endpoint nearest to the lower endpoint of c_1 is bound). We have to express D, modulo four-term relations, through diagrams which either have greater degree or whose index is lexicographically smaller than (l, b).

Applying the four term relation, we get:

$D \quad =$ 

The indices of the last two diagrams in the right-hand side are equal to $(l, b-1)$. Denote the first diagram on the right by D'. We have:

$D' \quad =$ 

Here, the degrees of the last two diagrams on the right are greater than d. The first diagram of the right-hand side is similar to D, but the first bound point has moved further downwards, so that the value of b has decreased.

This completes the proof of the theorem. □

Acknowledgements

We are obliged to D. Bar-Natan and V. Vassiliev for encouraging discussions, to A. Sossinsky, S. Lando and the referee for useful remarks and to M. Kontsevich for sending us his numerous illuminating preprints 'which will never be finished'. To test the conjectures that were constantly appearing, we have extensively used Dror Bar-Natan's C++ program that enumerates the symbols of Vassiliev invariants in small dimensions (we thank A. Kaishev for compiling the source code under MS-DOS). We are especially grateful to V. Goryunov who pointed out an error in the nested induction argument of subsection 3.4.2.

The work described in this paper was initially inspired by Arnold's galvanizing lecture notes [A]. After reading that text, we spent several months crazily filling our wastepaper baskets with pictures of chord diagrams — in a quest to find the elusive law that governs the numeric sequence 1, 2, 3, 5, 8, 12, 18, ...

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