Explicit Formulas for Arnold's Generic Curve Invariants

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Abstract

We review the explicit formulas for Arnold's generic curve invariants due to Viro, Shumakovich and Polyak and add some remarks concerning the invariants of spherical curves and curves immersed into arbitrary orientable surfaces.

1. Statement of the problem

A generic curve is a smooth immersion of the circle into the plane whose only singularities are transversal double points. Up to a diffeomorphism of the plane, how many generic closed curves are there?

One can immediately invent a host of invariants for generic plane curves, for example, the total number of double points, the Whitney index or the (unordered) sequence of edge numbers of the regions into which the curve divides the plane.

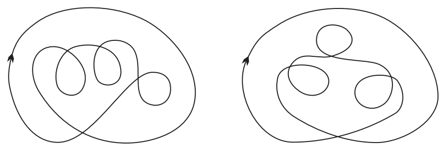

Figure 1: Two different curves with an equal number of double points $(= 4)$, equal Whitney numbers $(= -3)$ and equal sequences of edge numbers $(6, 6, 1, 1, 1, 1)$.

The problem we will discuss is the study of invariants and the classification of plane curves and also curves immersed into orientable surfaces.

Although this problem has a venerable history (it was first studied by Gauss), the most basic invariants of plane curves, J^+, J^- and St, were discovered only in 1993 by V. I. Arnold who used V. Vassiliev's approach to topological invariants through discriminants and singularity theory.

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The discriminant is the set of all non-generic curves viewed as a singular hypersurface in the space of all immersed curves. It divides the whole space into a number of connected components. Two generic curves are equivalent if and only if they belong to the same connected component, i. e. there exists a path in the space of generic curves connecting one curve with another. From this viewpoint, the role of the basic invariants should be played by functions that are constant on each connected component and have a prescribed behaviour when the path intersects the discriminant. In general position such a path meets the discriminant only in a finite number of its generic points. A generic point of the discriminant corresponds to a curve with only one generic singular point belonging to one of the three categories: a direct self-tangency, an inverse self-tangency or a triple point.

Arnold's seminal work [A1] gives an axiomatic description of the three invariants J^+, J^-, St that correspond to these types of singularities and proves their existence from the general properties of Serre fibrations. In 1994, two sets of explicit combinatorial formulas for the Arnold invariants have been found by O. Viro [V], A. Shumakovich [Sh] and M. Polyak [Po].

In the present text, we will give an overview of the work by Viro, Shumakovich and Polyak and supply it with several observations on our own part that might shed extra light on the meaning of these formulas and emphasize their relation with other viewpoints on the invariant theory of generic curves.

Since we are interested in plane curves as well as in the curves immersed into surfaces, we will treat only closed curves — observing that, in the plane, a parallel and in many respects simpler theory exists for *long curves* ([Tab]).

Now we state the problem in a more precise manner.

Let $\gamma : S^1 \to \mathbb{R}^2$ be a smooth immersion of the circle into the plane whose image is a smooth submanifold of $\mathbb{R}^2$ everywhere except for a finite number of double points with a transversal intersection. The problem is to study the invariants and find a classification of such immersions ("generic curves") with respect to an arbitrary diffeomorphism of the circle S^1 and an arbitrary diffeomorphism of the plane $\mathbb{R}^2$.

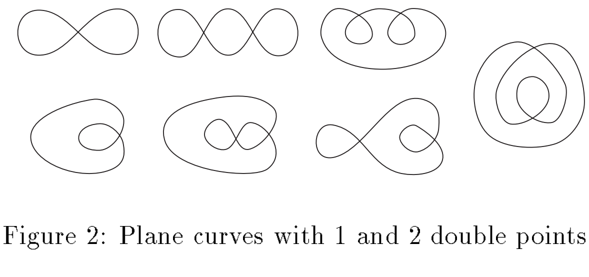


Figure 2: Plane curves with 1 and 2 double points.

Example 1. There are 2 non-equivalent plane curves with 1 double point and 5 non-equivalent plane curves with 2 double points (Fig. 2).

There are three non-equivalent variations of this problem: one can consider only orientation-preserving diffeomorphisms of the circle, or only orientation-preserving diffeomorphisms of the plane, or both (see [A1]). In this paper, we will only consider the totally non-oriented case.

One can also study the similar problem for immersions of the circle into arbitrary 2-surfaces. For example, there exists one spherical curve with 1 double point, two spherical curves with 2 double points and one curve with 2 double points on the torus which is not spherical (see Fig. 3).

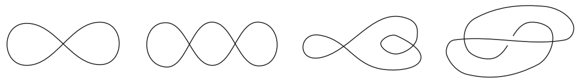


Figure 3: Spherical and toric curves with 1 and 2 double points.

Note that although in the case of spherical curves a complete invariant (Gauss diagram) is well-known, the study of the invariants which have a special behaviour with respect to the discriminant is interesting in itself. Of course, Arnold's basic invariants for spherical curves can be expressed through the Gauss diagram, but these expressions are quite non-trivial (Polyak's formulas modified for spherical case, see last section).

2. Gauss diagrams

In this section, we will describe the classical construction of Gauss diagram which gives an important combinatorial invariant of plane curves as well as of curves immersed into closed orientable surfaces.

Let $\gamma : S^1 \to \mathbb{R}^2$ be a generic curve with n double points. Denote the double points by n letters a_1, a_2, ..., a_n and mark the preimages of these points on the circle by the corresponding letters with exponents $+1$ or -1 according to the following rule.

The natural counterclockwise orientation of the circle S^1 induces a direction for travelling along the curve. A passage of the double point is said to be *positive* if the traveller sees another branch of the curve as going from left to right.

Definition 1 *A Gauss diagram of order n is a circle with n distinguished ordered pairs of distinct points, considered up to an arbitrary diffeomorphism of the circle and up to the operation of simultaneous reversing of order in each pair.*

Gauss diagrams are in one-to-one correspondence with *signed Gauss* *words*, i. e. elements of the quotient set

$$W(n)/S_n \times D_{2n} \times \mathbb{Z}_2,$$

where $W(n)$ is the set of all permutations of $2n$ letters $a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}$ in which the symmetric group S_n acts by permutations of subscripts, the dihedral group D_{2n} acts by cyclic shifts and reflections, and the non-trivial element of the group $\mathbb{Z}_2$ changes the exponents of all letters.

A Gauss diagram can be depicted as a chord diagram with oriented chords. The arrow on a chord goes from the positive to the negative passage of the double point. Simultaneous reversing of the orientations of all chords corresponds to an orientation-reversing diffeomorphism of the plane.

The curves of Figure 1 have different Gauss diagrams:

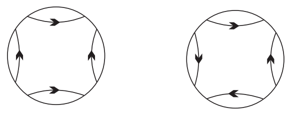


Figure 4: Gauss diagrams of the curves of Figure 1.

The construction of the Gauss diagram $G(\Gamma)$ is valid not only for a plane curve Γ, but also for a generic immersion of the circle into an arbitrary orientable surface.

There are three different Gauss diagrams of order 2 (see Fig. 5). Among them, two first diagrams correspond to plane curves, while the last one can only be realized by a curve in the torus (and hence in an arbitrary surface of genus higher than one).

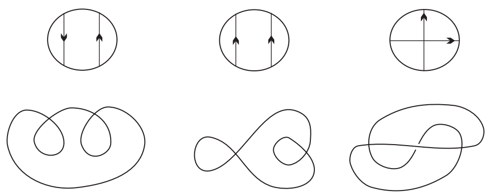


Figure 5: Curves corresponding to Gauss diagrams of order 2.

In general, any arbitrary Gauss diagram corresponds to a curve immersed into a surface of a sufficiently high genus. This was proved by J. Scott Carter [C] who closely followed L. Heffter's [H] classical construction of the closed oriented surface containing a given graph with a prescribed cyclical order of edges at every vertex.

Theorem 1 (J. Scott Carter [C]) *For any Gauss diagram G there is a minimal number g (the genus of G) such that G is the Gauss diagram of* *a certain generic curve in the closed orientable surface of genus g. This curve is defined uniquely up to a diffeomorphism of the surface. In other words, the Gauss diagram constitutes a complete invariant for those curves immersed into a surface that cannot be immersed into a surface of smaller genus. In particular, the Gauss diagram of a plane curve is the complete invariant of its spherical class. Moreover, if two immersions of S^1 into two different surfaces Q_1, Q_2 have the same Gauss diagram, then one can remove some handles from either or both of the surfaces Q_1, Q_2 so that the resulting immersions will be equivalent.*

The minimal surface of a Gauss diagram can be obtained by attaching disks to the curve viewed as a 1-skeleton in such a way that in the double points these pieces are patched together consistently with the cyclic order of the 4 branches of the curve. This can always be done in a unique way; an example is shown in Fig. 6.

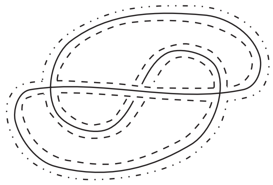


Figure 6: The Carter's surface of this curve is a torus, which can be obtained by attaching disks along each of the two dotted lines.

If n is the order of the Gauss diagram (the number of double points) and c is the number of faces attached, then the Euler characteristic of Carter's curface is $2 - 2g = n - 2n + c$, hence $g = (2 + n - c)/2 \leq (n + 1)/2$.

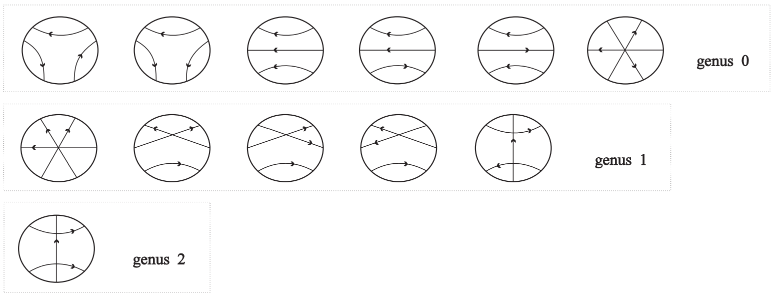


Figure 7: Gauss diagrams of order 3. Out of the total number of 12 different Gauss diagrams of order three, 6 diagrams have genus zero, 5 diagrams are of genus one and 1 diagram of genus two.

3. Arnold's basic invariants

Let X be the space of all smooth immersions of the circle S^1 into the plane $\mathbb{R}^2$. Elements of X will be referred to as *curves*.

According to a classical theorem due to H. Whitney [Wh], the space X is made up of a countable number of connected components each containing all curves with the same value i of *rotation number*. The rotation number, or Whitney index, of a curve is the degree of the mapping $S^1 \to S^1$ that sends every point into the direction of the tangent vector at this point, i. e. the total number of revolutions made by the tangent vector when the point traverses the curve once in the positive direction. Since we identify the two immersions that differ by a mirror symmetry of the plane, we will suppose the Whitney index to be non-negative.

Choose a standard curve K_i for every non-negative value of i:

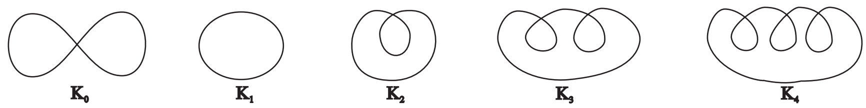


Figure 8: Standard curves.

Denote by Σ^+ the set of all curves that have a point of direct self-tangency, i. e. a point which is visited at least twice with tangent vectors having the same direction (Fig. 9a). In a similar way, let Σ^- be the set of all curves that have a point of inverse self-tangency (Fig. 9b) and Σ^{St}, the set of all curves having at least one triple point.

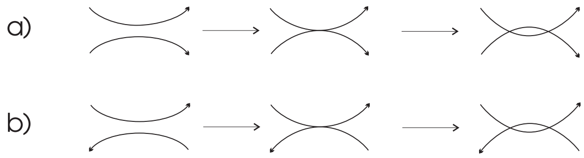


Figure 9: Direct and inverse self-tangency.

The discriminant is the union of the three varieties Σ^+, Σ^- and Σ^{St}. Set

$$G = X \setminus (\Sigma^+ \cup \Sigma^- \cup \Sigma^{St}).$$

An invariant of generic curves is a function on the space G which is constant on connected components, i. e. whose value may only change when we move through the discriminant hypersurface. Therefore the question initially raised by V. Arnold in [A1] is quite natural: are there any invariants of generic curves which have *constant* jumps on the discriminant? To state the main theorem of [A1], we need to introduce a coorientation of each of the three components, Σ^+, Σ^- and Σ^{St}, of the discriminant.

If two curves are close to a generic point of Σ^+ and lie on either side of this hypersurface, then they are identical everywhere but in a small disk, where one of them has two additional double points. We say that this curve is on the *positive* side of Σ^+.

The same definition also holds for the coorientation of the component Σ^-.

The fact that the triple point component is also coorientable is not so evident, and the corresponding definition is more tricky. We explain this construction in Fig. 10.

It takes some imagination to understand that a triple point can be resolved in only two essentially different ways, i. e. that the neighbourhood of a curve with one triple point in the space of generic curves consists of two connected components. Either of these two components is given a sign + or − according to the following rule:

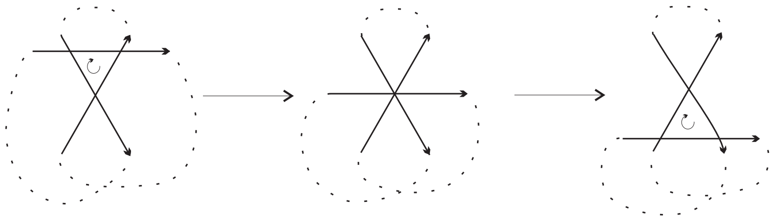


Figure 10: Coorientation of Σ^{St}.

Consider the new-born triangle that has replaced the triple point. Choose an arbitrary direction on the curve. It supplies each of the three sides of the triangle with a certain direction. There is also a cyclic order in the set of the three sides defined by the order in which the sides are passed when moving along the curve in the choosen direction. This cyclic order induces another direction on each of the three sides. The *sign* of the new-born triangle is equal to

$$(-1)^{\text{the number of sides on which the two directions are the same}}.$$

This sign does not depend on the choice of the direction on the curve.

Definition 2 *The strangeness St is an invariant which jumps by* 1 *only at a positive crossing of the cooriented stratum Σ^{St}.*

J^+ is an invariant which jumps by 2 *only at a positive crossing of the cooriented stratum Σ^+.*

J^- is an invariant which jumps by -2 only at a positive crossing of the cooriented stratum Σ^-.

Theorem 2 (V. Arnold [A1]) *For any sequence $\mathbf{k} = (k_0, k_1, k_2, \ldots)$ there are unique invariants $St_{\mathbf{k}}$, $J^+_{\mathbf{k}}$, $J^-_{\mathbf{k}}$ with normalization*

$$St_{\mathbf{k}}(K_i) = k_i, \qquad J^+_{\mathbf{k}}(K_i) = k_i, \qquad J^-_{\mathbf{k}}(K_i) = k_i.$$

Different choices of the normalizing sequence $\mathbf{k}$ may lead to additional remarkable properties of the invariants $St_{\mathbf{k}}$, $J^+_{\mathbf{k}}$, $J^-_{\mathbf{k}}$. In [A1] Arnold used the following normalization:

for St:	$k_0 = 0$,	$k_i = i - 1$	$(i = 1, 2, 3, \ldots)$;
for J^+:	$k_0 = 0$,	$k_i = -2(i - 1)$	$(i = 1, 2, 3, \ldots)$;
for J^-:	$k_0 = -1$,	$k_i = -3(i - 1)$	$(i = 1, 2, 3, \ldots)$

— and proved that it is the only normalization which leads to invariants, additive under the connected sum of curves. Below for the invariants with this normalization we will omit the subscript $\mathbf{k}$.

Another choice of normalization $\mathbf{k}$ gives invariants of spherical classes of curves (see Sec. 6).

To calculate the invariants of a curve by definition, we connect the given curve with one of standard curves K_i by a smooth path in the space of immersions that meets the discriminant only in its generic points. The value of the invariant on the standard curve is known. Accumulating all jumps of the invariant during deformation, we obtain the value of the invariant on the initial curve.

Example 2. As an example, consider the plane curve

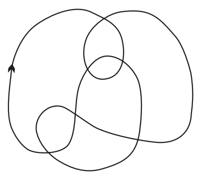
$\Gamma =$ 

Figure 11: An example curve.

Figure 12 shows a deformation of this curve to K_1 and the corresponding jumps of the basic invariants.

It follows that $St_{\mathbf{k}}(\Gamma) = k_1 + 1$, $J^+_{\mathbf{k}}(\Gamma) = k_1 + 2$, $J^-_{\mathbf{k}}(\Gamma) = k_1 - 4$.

In particular, with the normalization we have adopted

$$St(\Gamma) = 1, \quad J^+(\Gamma) = 2, \quad J^-(\Gamma) = -4.$$

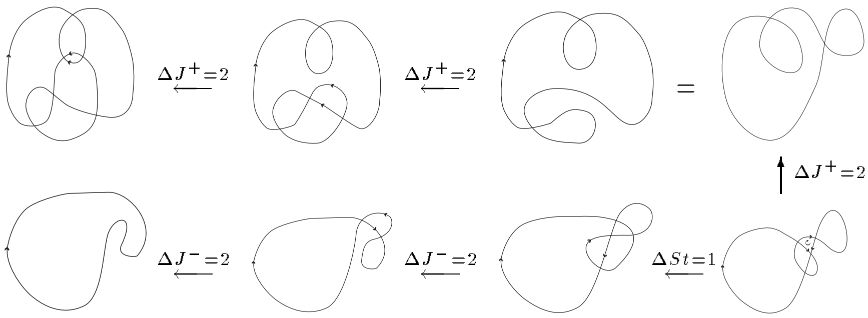


Figure 12: Calculation of the invariants.

4. Formulas of Viro and Shumakovich

Choose an arbitrary orientation of the curve $\Gamma \subset \mathbb{R}^2$. Pick a connected component C of $\mathbb{R}^2 \setminus \Gamma$ and a point $x \in C$.

Definition 3 *The index* $\mathrm{ind}_\Gamma(C)$ *of the region* C *is the algebraic number of turns made by the radius-vector from* x *to a point travelling along* Γ.

Evidently, $\mathrm{ind}_\Gamma(C)$ does not depend on the choice of $x \in C$ but does depend on the orientation of Γ.

The index of the exterior component is zero. If C_1 and C_2 are two adjacent components and C_1 lies on the left of Γ while C_2 lies on the right, then $\mathrm{ind}_\Gamma(C_2) = \mathrm{ind}_\Gamma(C_1) + 1$. We arrive at the *Alexander rule*[2] for the indices of four connected components in the vicinity of a double point of Γ (see Fig. 13).

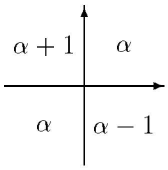


Figure 13: Alexander numbering.

[2] J. Alexander [Al] was the first to notice that any generic plane curve admits such a distribution of indices.

Now consider the smoothing of a double point determined by the orientation (Fig. 14).

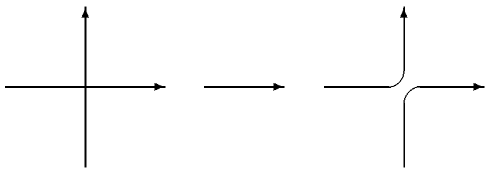


Figure 14: Smoothing.

Let $\widetilde{\Gamma}$ be obtained from Γ by smoothing all double points of Γ. It is a union of circles. Each smoothing joins two local components of the complement of Γ with the same index into one component of the complement of $\widetilde{\Gamma}$. Therefore a connected component $\widetilde{C}$ of $\mathbb{R}^2 \setminus \widetilde{\Gamma}$ inherits the index $\mathrm{ind}_{\tilde{\Gamma}}(\widetilde{C})$.

Theorem 3 (O. Viro [V])

$$\begin{aligned} J^+(\Gamma) &= 1 + n - \sum_{\tilde{C}} \mathrm{ind}^2_{\tilde{\Gamma}}(\widetilde{C})\chi(\widetilde{C}); \\ J^-(\Gamma) &= 1 - \sum_{\tilde{C}} \mathrm{ind}^2_{\tilde{\Gamma}}(\widetilde{C})\chi(\widetilde{C}), \end{aligned}$$

where n is the number of double points of Γ and χ is the Euler characteristic.

This theorem can be proved by checking that the right hand side expressions give correct values for the standard curves K_i and verifying that their jumps under the three elementary perestroikas conform to the definition of the invariants. The same remark also holds for the formula of Shumakovich (Theorem 4) and formula of Polyak (Theorem 5).

Example 3. Figure 15 shows the result of smoothing all the double points of our example curve.

Numbers in this picture indicate the indices $\mathrm{ind}_{\tilde{\Gamma}}(\widetilde{C})$ for all components $\widetilde{C}$. So,

$$\begin{aligned} J^+(\Gamma) &= 1 + 6 - (1 + 4) = 2, \\ J^-(\Gamma) &= 1 - (1 + 4) = -4. \end{aligned}$$

A. Shumakovich [Sh] gave a formula for the strangeness in similar terms.

Fix a base point $*$ on the (oriented) curve Γ. Then each double point p of Γ obtains a sign $s(p) = +1$ or -1 depending on whether the first *passage*

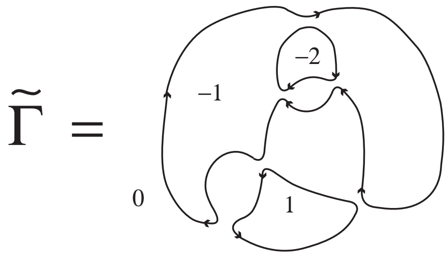


Figure 15: Smoothed curve.

of this point starting from the base point $*$ is positive or negative. (We have defined the sign of the passage earlier in Sec. 2). The signs of double points depend on the choice of the base point $*$.

Definition 4 *The index* $\mathrm{ind}_\Gamma(p)$ *of a double point* p *of* Γ *is the arithmetic mean of indices of the four components of the complement to* Γ *near* p. *(If one component approaches* p *twice from two opposite directions, then its index is counted twice.)*

Theorem 4 (A. Shumakovich [Sh]) *If the base point* $*$ *lies on the exterior contour of* Γ, *then*

$$St(\Gamma) = \sum_p s(p)\mathrm{ind}_\Gamma(p).$$

Example 4. We determine the signs and indices of all double points of our curve Γ:

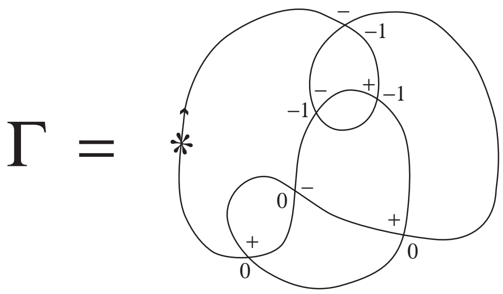


Figure 16: Indices and signs of double points of the example curve.

— and find that $St(\Gamma) = 1 - 1 + 1 = 1$.

5. Formulas of Polyak

M. Polyak in [Po] gave explicit formulas for Arnold's invariants in terms of *based Gauss diagrams*, i. e. Gauss diagrams with a fixed base point $*$, or more exactly, with a distinguished arc on the outer circle of the diagram.

The choice of the base point defines a sign $s(c)$ of every chord c according to the rule: $s(c) = s(p)$ where p is the double point that corresponds to the chord c and $s(p)$ is the sign introduced in the previous section. In other words, the chord is positive with respect to the chosen base point $*$, if, starting from $*$ and moving counterclockwise, its endpoint which is passed first corresponds to a positive passage of the double point.

Let A be a *based chord diagram.* This is an object that differs from a based Gauss diagram in that the chords of A are non-oriented.

A based chord diagram A defines a rational-valued function $G \mapsto \langle A, G\rangle$ on the set of based Gauss diagrams. In the definition that follows a *subdiagram* of type A means a based Gauss subdiagram of G which becomes A after forgetting the orientation of all its chords.

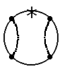
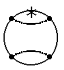
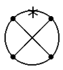
Definition 5

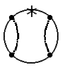
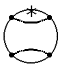
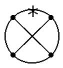
$$\langle A, G\rangle = \sum_{G_A} \prod_{c \in G_A} s(c),$$

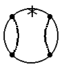
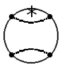
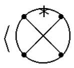
where the sum is taken over all subdiagrams G_A of type A and the product is taken over all chords of G_A.

It turns out that, in the space of all generic curves with a fixed number of double points and a fixed Whitney index, Arnold's basic invariants J^+, J^-, St can be expressed in terms of the three functions $\langle A, \cdot\rangle$, where A is a based chord diagram of order two.

Theorem 5 (M. Polyak [Po]) *Let G be a based Gauss diagram of a generic curve Γ. Then*

$J^+(\Gamma) = \langle$  $, G\rangle - \langle$  $, G\rangle - 3\langle$  $, G\rangle - \dfrac{n + \mathrm{ind}^2\Gamma - 1}{2};$

$J^-(\Gamma) = \langle$  $, G\rangle - \langle$  $, G\rangle - 3\langle$  $, G\rangle - \dfrac{3n + \mathrm{ind}^2\Gamma - 1}{2};$

$St(\Gamma) = -\dfrac{1}{2}\langle$  $, G\rangle + \dfrac{1}{2}\langle$  $, G\rangle + \dfrac{1}{2}\langle$  $, G\rangle$ $+\dfrac{n + \mathrm{ind}^2\Gamma - 1}{4},$

where n is the number of double points of Γ and $\mathrm{ind}\Gamma$ is the rotation number of Γ.

Example 5. For our test curve

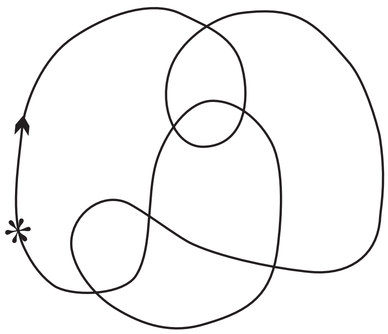


Figure 17: Example curve with a base point.

we have $\mathrm{ind}\Gamma = -1$, $n = 6$ and the based Gauss diagram looks as follows:

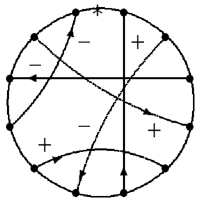
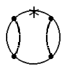
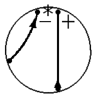
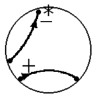
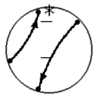
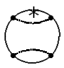
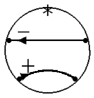
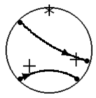
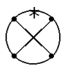
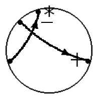
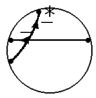
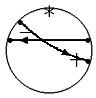
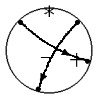
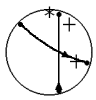
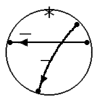
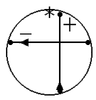
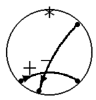
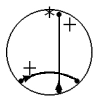
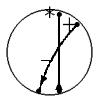


Figure 18: Based Gauss diagram.

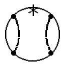
We have three subdiagrams of type :  ,  ,  ;

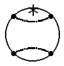
two of type :  ,  ;

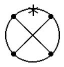
and ten of type :  ,  ,  ,  ,

 ,  ,  ,  ,  ,  .

Hence,

$\langle$  $, G\rangle \;=\; -1-1+1=-1,$

$\langle$  $, G\rangle \;=\; -1+1=0,$

$\langle$  $, G\rangle \;=\; -1+1-1-1+1+1-1-1+1-1=-2.$

and

$$\begin{aligned} J^+(\Gamma) &= -1+6-\frac{6+1-1}{2}=2; \\ J^-(\Gamma) &= -1+6-\frac{18+1-1}{2}=-4; \\ St(\Gamma) &= (1/2)-1+\frac{6+1-1}{4}=1, \end{aligned}$$

6. Arnold's inariants for spherical curves

After adding one point "at infinity" the plane $\mathbb{R}^2$ becomes the sphere S^2. Hence, a plane curve Γ can always be considered as a spherical curve. Since under this process different plane curves may become equivalent spherical curves, the natural question is: what plane curve invariants in fact represent invariants of spherical curves?

In a recent paper [A3] Arnold noted that $SJ^+(\Gamma)=J^+(\Gamma)+\frac{1}{2}\,\mathrm{ind}^2\Gamma$, where $\mathrm{ind}\Gamma$ is the rotation number of Γ, is a spherical invariant. Similarly $SJ^-(\Gamma)=J^-(\Gamma)+\frac{1}{2}\,\mathrm{ind}^2\Gamma$ and $SSt(\Gamma)=St(\Gamma)-\frac{1}{4}\,\mathrm{ind}^2\Gamma$ are also spherical invariants. Since the rotation number $\mathrm{ind}\,\Gamma$ does not change under local perestroikas that consist in passing through Σ^+, Σ^-, Σ^{St}, the spherical invariants SJ^+, SJ^-, SSt can be considered as usual plane curve invariants $J^+_{\mathbf{k}}$, $J^-_{\mathbf{k}}$, $St_{\mathbf{k}}$ with a special normalization:

$$\begin{aligned} &\text{for } SSt: && k_0=0, && k_i=-\frac{(i-2)^2}{4} && (i=1,2,3,\dots); \\ &\text{for } SJ^+: && k_0=0, && k_i=\frac{(i-2)^2}{2} && (i=1,2,3,\dots); \\ &\text{for } SJ^-: && k_0=-1, && k_i=\frac{(i-3)^2-3}{2} && (i=1,2,3,\dots). \end{aligned}$$

Example 6. The spherical invariants of our curve Γ (Fig. 11) are $SSt(\Gamma)=3/4$; $SJ^+(\Gamma)=5/2$; $SJ^-(\Gamma)=-7/2$.

A natural complete invariant of spherical curves is its Gauss diagram. So each spherical invariant in principle can be expressed in terms of Gauss diagram. Explicit formulas for SJ^+, SJ^-, SSt can be obtained from Polyak's formulas by excluding the base point information. The following numbers take part in the answer.

Definition 6 *Let* $\otimes$ *be a Gauss subdiagram of a Gauss diagram* G*. The endpoints of its two chords divide the circle into four arcs. Let* a *be the number of endpoints of chords of* G *that belong to the upper arc, i. e. lie between the two arrowheads of the oriented chords, and* b*,* c*,* d *are the numbers of endpoints on the other three arcs in the counterclockwise order. Then the weight* $w_{\otimes}$ *of the Gauss subdiagram is the number* $w_{\otimes} = -a+b-c+d$*.*

Theorem 6

$$
\begin{aligned}
SJ^+(\Gamma) &= -\frac{n-1}{2} - \#\bigcirc_{\leftleftarrows} + \#\bigcirc_{\leftrightarrows} + \frac{3n}{2}\sum_{\otimes} w_{\otimes};\\
SJ^-(\Gamma) &= -\frac{3n-1}{2} - \#\bigcirc_{\leftleftarrows} + \#\bigcirc_{\leftrightarrows} + \frac{3n}{2}\sum_{\otimes} w_{\otimes};\\
SSt(\Gamma) &= \frac{n-1}{4} + \frac{1}{2}\#\bigcirc_{\leftleftarrows} - 1/2\#\bigcirc_{\leftrightarrows} - \frac{n}{4}\sum_{\otimes} w_{\otimes},
\end{aligned}
$$

where $\#\bigcirc_{\leftleftarrows}$ *and* $\#\bigcirc_{\leftrightarrows}$ *are the numbers of Gauss subdiagrams of* Γ *of the indicated form.*

Example 7. It is easy to see from the example of Sec. 5 that for our favourite curve Γ:

$$\#\bigcirc_{\leftleftarrows} = 3, \qquad \#\bigcirc_{\leftrightarrows} = 2$$

and weights of the ten Gauss subdiagrams of type $\otimes$ are equal respectively to

$$0,\quad 4,\quad 8,\quad 0,\quad 4,\quad -4,\quad 0,\quad 4,\quad 0,\quad 8.$$

So, $\sum_{\otimes} w_{\otimes} = 24$ and

$$SJ^+(\Gamma) \;=\; -\frac{5}{2} - 3 + 2 + \frac{3}{12}\cdot 24 = 5/2;$$

$$\begin{aligned} SJ^{-}(\Gamma) &= -\frac{17}{2} - 3 + 2 + \frac{3}{12} \cdot 24 = -7/2; \\ SSt(\Gamma) &= \frac{5}{4} + \frac{3}{2} - \frac{2}{2} - \frac{1}{24} \cdot 24 = 3/4. \end{aligned}$$

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