The algebra of 3-graphs*

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Abstract

We introduce and study the structure of an algebra in the linear space spanned by all regular 3-valent graphs with a prescribed order of edges at every vertex, modulo certain relations. The role of this object in various areas of low dimensional topology is discussed.

0 Introduction

Regular graphs of degree 3, i. e. graphs in which every vertex is incident with exactly three edges, often occur in mathematics. Apart from graph theory proper, where such graphs are referred to as 'cubic', they appear in a natural way in the topology of 3-manifolds, in the Vassiliev knot invariant theory and in connection with the four colour theorem. It turns out that in all these applications 3-valent graphs are endowed with a natural structure that consists in fixing, at every vertex of the graph, one of the two possible cyclic orders in the set of three edges issuing from this vertex.

In the theory of Vassiliev invariants 3-valent graphs can be viewed as elements of the primitive subspace $\mathcal{P}$ of the graded Hopf algebra $\mathcal{F}$ associated with the filtered algebra of finite order invariants. Among the elements that generate the algebra $\mathcal{F}$ (*connected Feynman diagrams* or *Chinese character diagrams* in the terminology of D. Bar-Natan [BN1]) there are two types of relations ('antisymmetry' and 'IHX', or, in another coordinate system, 1- and 4-term relations). These relations, restricted to the space spanned by the 3-valent graphs, lead to the 'space of 3-graphs'. In the space of 3-graphs there is a commutative multiplication defined by the connected sum of graphs. Although similar structures — algebras of Feynman diagrams and various modifications — have been studied in the literature (see [Va, BN1, Kon, Vai, CV, Vo]), nobody, to the best of our knowledge, has ever distinguished the algebra of 3-graphs as a self-contained object. We give the exact definition of this algebra in section 1.

In section 2 we give the tables of the 3-graph algebra in small dimensions and in section 3 we describe the computer calculations performed to obtain these tables.

In section 4 we discuss the construction (going back to Penrose [Pen] and Kontsevich [Kon]) of the functions on the algebra of 3-graphs associated with a Lie algebra with a fixed ad-invariant non-degenerate symmetric bilinear form. We also give the tables of the values of some of these invariants on the generators of small degree.

Section 5 is influenced by the brilliant paper by Bar-Natan [BN2] and dedicated to the famous four-colour theorem. The theorem, conjectured by F. Guthrie in 1852, was proved by

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K. Appel and W. Haken in 1976. Their proof heavily relies on an extensive computer search and makes up a book of 741 pages [AH] (see also [SK]). Recently, a rise of interest towards the four colour theorem is witnessed ([Kai, Se]). Various algebraic reformulations have appeared ([Pen, Ka1, Ka2, KaS, BN2]). There is an attempt to simplify the Appel–Haken proof [RSST], not unsuccessful, but based on the same, rather straightforward, reduction of the problem to the computer search through a large number of particular configurations. Our aim is to show that the algebra of 3-graphs is the right object lending conceptual clarity to investigations around the four colour theorem.

In section 6 we discuss the role of 3-graphs in the theory of knot invariants. The most interesting phenomenon here is that graph multiplication allows to introduce the structure of a non-commutative algebra in the space of primitive elements of the graded algebra $\mathcal{F}$ associated with the space of Vassiliev invariants. This means that the space $\mathcal{P}$ freely generating the algebra of Vassiliev invariants is itself endowed with an inner multiplication.

Finally, in section 7 we speak about the invariants of three dimensional manifolds (homology spheres) where the algebra of 3-graphs — only as a linear space, not as an algebra — has recently emerged in the works of T. Ohtsuki [Oh1] and T. Le [Le]

The reader may note that we have not found any extraordinary applications of the 3-graph algebra — such as would lead to new theorems. But the structure itself and the variety of its contacts with various mathematical subjects are so interesting that we decided to publish the present text as it is. In the row of combinatorial structures of low-dimensional topology including the algebras of chord diagrams [Va] and Feynman diagrams [Kon, Vai], spaces of Chinese [BN1] and Japanese [CV] characters, Vogel's Λ-modules [Vo], the algebra of 3-graphs is the simplest and, from the general point of view, the most natural object and therefore deserves to be studied.

1 The definition of the algebra

1.1 The space 3-graphs

We begin by giving an exact definition of the main object we will work with.

Definition. A *3-graph* is a regular 3-valent graph[1] with a fixed *rotation*. The *rotation* is the choice of a cyclic order of edges at every vertex, i. e. one of the two cyclic permutations in the set of three edges adjacent to this vertex.

In this definition *graphs* are allowed to have multiple edges and loops. In the case of graphs with loops the notion of rotation requires the following refinement: the cyclic order is introduced not in the set of *edges* incident to a given vertex, but in the corresponding set of *half-edges*.

A topology free combinatorial definition of the set of half-edges can be given as follows. Let E, V be the sets of edges and vertices, respectively, of the graph under study. Then the set of half-edges is a 'double fibering' $E \overset{\alpha}{\leftarrow} H \overset{\beta}{\rightarrow} V$, that is a set H supplied with two projections $\alpha : H \to E$ and $\beta : H \to V$ such that (a) the inverse image of every edge $\alpha^{-1}(e)$ consists of two elements, (b) the cardinality of each set $\beta^{-1}(v)$ equals the valency of the vertex

[1]Note that the number of vertices of such a graph is always even, and the number of edges a multiple of 3. Half the number of the vertices will be referred to as the *degree* of a 3-graph.

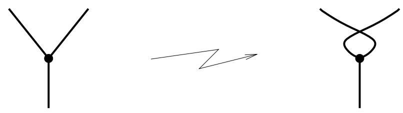
v, (c) for any point $h \in H$ the vertex $\beta(h)$ is one of the two endpoints of the edge $\alpha(h)$. Then the rotation is a cyclic permutation in the inverse image of each vertex $\beta^{-1}(v)$.

One can forget about edges and vertices altogether and adopt the following clear-cut, although less pictorial, definition of a 3-graph: this is *a set of cardinality* $6n$ *equipped with two permutations, one with the cyclic structure* $(3)(3)\ldots(3)$*, another with cyclic structure* $(2)(2)\ldots(2)$. The relation to the previous definition consists in that H is the set of half-edges, the first permutation corresponds to the rotation in the vertices and the second, to the transition from one endpoint of an edge to another.

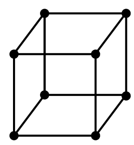
Definition. Two 3-graphs are said to be *isomorphic*, if there is a one-to-one correspondence between their sets of half-edges which preserves the rotation and induces a usual graph isomorphism — or, in other words, which preserves both structural permutations.

Remarks. **1.** It is convenient to include the circle into the set of 3-graphs viewing it as a graph with a zero number of vertices.

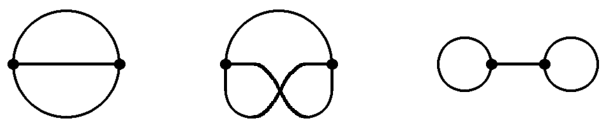
2. Although 3-graphs are abstract graphs, we shall draw them in the plane. Since not every graph is planar, the edges in the pictures will sometimes intersect. We will ignore such intersections and not consider them as vertices of the graph. Furthermore, we will suppose that the half-edges at every vertex are ordered counterclockwise. The change of rotation in a vertex can be depicted as follows:



Examples. **1.** Here is a typical example of a 3-graph (having 8 vertices and 12 edges, i. e. of degree 4 in our sense):

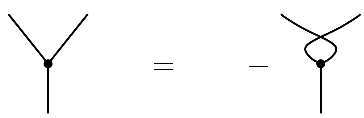


2. Up to an isomorphism, there are three different 3-graphs of degree 1:

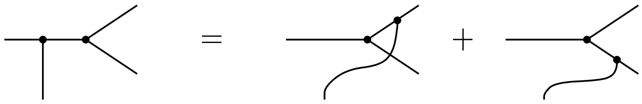


Definition. The space Γ_n is the quotient space of the linear space over $\mathbb{Q}$ generated by connected 3-graphs of degree n (i. e. having $2n$ vertices), modulo the following relations.

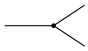
Antisymmetry relation (AS):

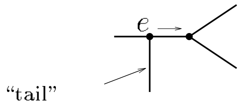


Kirchhof relation (IHX):



Remarks. **1.** Here and below unfinished fragments in the pictures denote graphs that are identical (and arbitrary) everywhere but in this explicitly shown fragment.

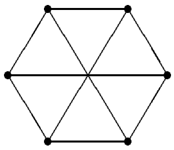
2. We use the title 'Kirchhof relation' because of the following analogy. Let us view the portion  of a given graph as a piece of an electrical circuit, where the variable vertex represents an 'electron' e with a 'tail' whose endpoint is fixed. Suppose that the electron moves towards a node of the circuit:



Then the IHX relation expresses the well-known Kirchhof rule: *the sum of currents entering a node is equal to the sum of currents going out of it.* This electrotechnical analogy is very useful, e. g. in the proof of the correctness of graph multiplication (see below).

3. By definition, Γ_0 is one-dimensional and generated by the circle.

Exercise. Check that the following graph is equal to zero as an element of the space Γ_3:



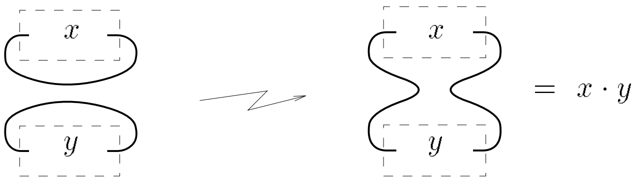
1.2 Multiplication

In the graded space

$$\Gamma = \Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \dots \ .$$

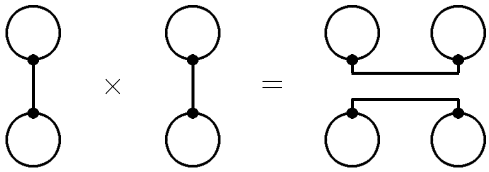
there is a natural structure of a commutative algebra.

Definition. The *product* of two 3-graphs x and y is the 3-graph obtained in the following way. Choose arbitrarily an edge in x and an edge in y. Cut each of these two edges in the middle and attach the endpoints that appear on one edge, to the endpoints appearing on the other.



The product of 3-graphs can also be understood as the result of insertion of one graph, say x, into an edge of y.

Remark. The product of two connected graphs may yield a disconnected graph, for example:



This happens, however, only in the case when each of the two graphs becomes disconnected after cutting the chosen edge, and in this case both graphs are 0 modulo relations (see lemma 2 in Sec. 2.1).

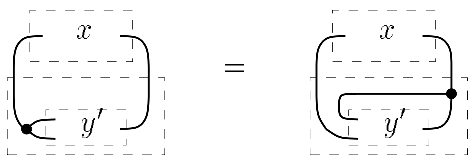
Theorem. *The product of 3-graphs, viewed as an element of the space Γ, is correctly defined.*

Note that, as soon as the correctness of multiplication is proved, one immediately sees that it is commutative.

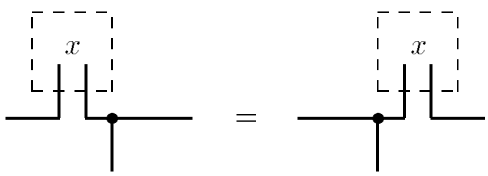
The claim that the product is correctly defined consists of two parts. First, that modulo the AS and IHX relations the product does not depend on the *choice* of the two edges x and y which are cut and pasted. Second, that it does not depend on the *way* they are put together (clearly, the two loose ends of one graph can be glued to the two loose ends of another graph in two different ways).

We establish these two facts in the following two lemmas. In the statement of Lemma 1, by a 'subgraph' is meant a part of the 3-graph (i. e. a subset of its set of half-edges) attached to the remaining part by two edges — like a factor in the product.

Lemma 1. (a) *A 'tail' can be dragged through a subgraph:*

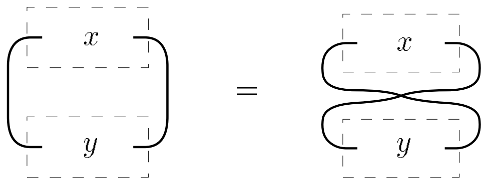


(b) *A subgraph can be carried through a vertex:*



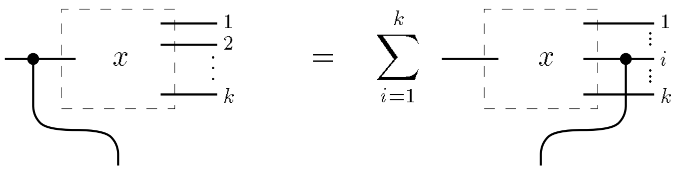
Taking a closer look at these pictures, the reader will understand that both formulations of Lemma 1 have exactly the same meaning, to wit: *the results of insertion of a 3-graph x into the adjacent edges of the 3-graph y, are equal.* Since y is connected, this implies that the product does not depend on the choice of the edge in y.

Lemma 2. *Two different ways to glue together two graphs with edges cut give the same result in the space* Γ:



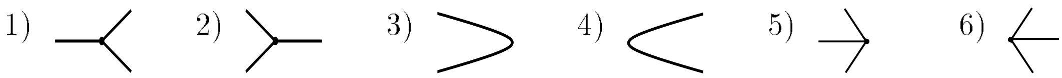
We now pass to the proof of Lemmas 1 and 2. Lemma 1 is a particular case ($k = 1$) of the following more general assertion.

Lemma 3. *In the space* Γ *the general Kirchhof law is obeyed:*



Proof of Lemma 3. Fix a horizontal line in the plane and consider an immersion of the given graph into the plane, generic with respect to the projection onto this line. More exactly, we assume that (1) the projections of all vertices onto the horizontal line are distinct, (2) when restricted to an arbitrary edge, the projection has only Morse critical points, and (3) the images of all critical points are distinct and different from the images of vertices.

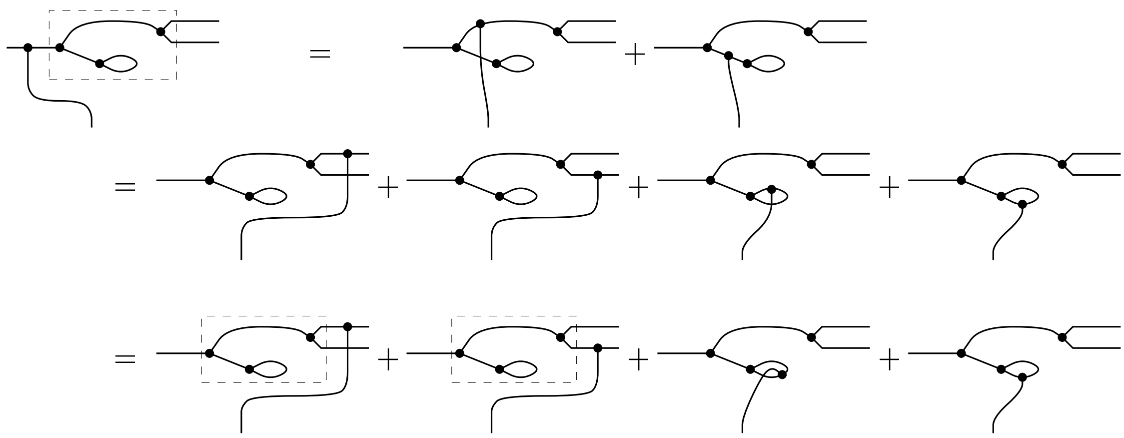
Bifurcation points are the images of vertices and critical points of the projection. Our proof of Lemma 3 proceeds by induction on the number of bifurcation points. If there is only one bifurcation points, it must be the image of a vertex, in which case the equality to be proved coincides with the IHX relation. Suppose that the lemma is established for s bifurcation points. The $(s + 1)$-th bifurcation point falls under one of the six categories:



In the first two cases the lemma follows from the IHX relation, in cases 3 and 4 — from the AS relation. Cases 5 and 6 by a deformation of the immersion are reduced to a combination of the previous cases.

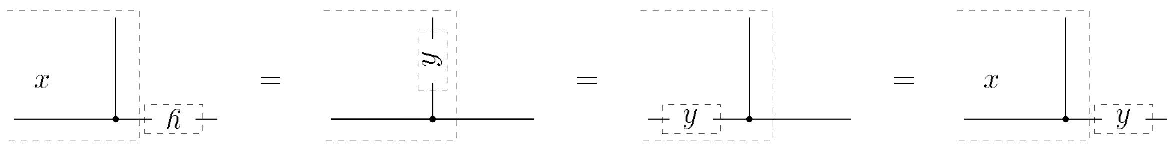
This completes the proof of Lemma 3.

Example.

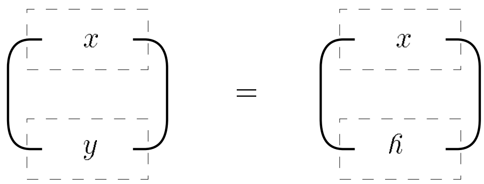


The two last summands cancel out because of the antisymmetry relation.

Proof of Lemma 2. If the 3-graph x is a circle, there is nothing to prove. We can therefore assume that x contains at least one vertex of degree 3. Choosing the vertex of x which is nearest to the right exit of y, one can, by Lemma 1, effectuate the following manoeuvres:



Therefore,



Now, rotating y through 180° in the ambient plane, we get the right-hand side of the required equality. Lemma 2, and thus the correctness of multiplication of 3-graphs, are established.

The product of 3-graphs is extended to the product of arbitrary elements of the space Γ in the usual way, by linearity.

Corollary. *The product of arbitrary elements of* Γ *is correctly defined.*

This follows from the fact that the product of 3-graphs is correct and a linear combination of either AS or IHX type, when multiplied by an arbitrary graph, preserves its type.

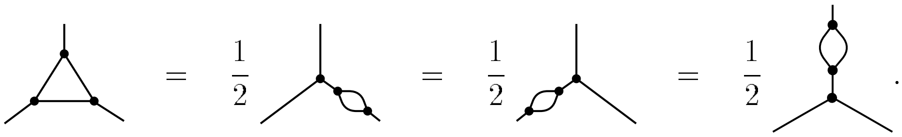
2 The Zoo

2.1 Some identities

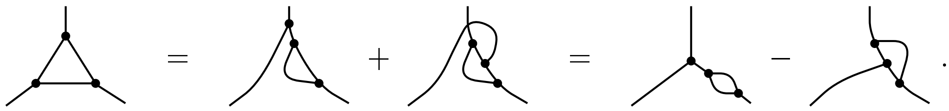
We will prove several identities elucidating the structure of the 3-graph algebra. They are easy consequences of the defining relations.

The first identity (Lemma 1) links the two operations defined in the set of 3-graphs: *the insertion of a triangle into a vertex* and *the insertion of a bubble into an edge*. To insert a bubble into an edge of a graph is the same thing as to multiply this graph by the element $b = \ominus \in \Gamma_1$. The correctness of multiplication in Γ implies that the result of the bubble insertion does not depend on the edge chosen, then it follows from Lemma 1 that the result of the triangle insertion does not depend on the choice of the vertex.

Lemma 1. *A triangle is equal to one half of a bubble:*

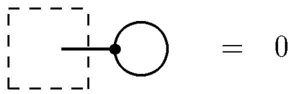


Proof of Lemma 1.

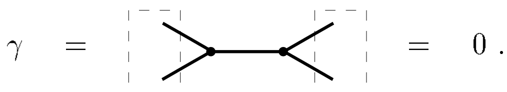


The second lemma describes two classes of 3-graphs which are equal to 0 in the algebra Γ, i. e. modulo the AS and IHX relations.

Lemma 2. (a) *A graph with a loop is* 0 *in* Γ*:*

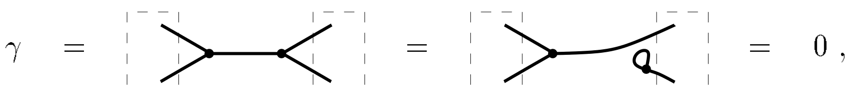


(b) *If the edge connectivity of the graph* γ *is* 1*, i. e. it becomes disconnected after the removal of an edge, then* $\gamma = 0$:



Proof of Lemma 2. (a) A graph with a loop is zero because of antisymmetry. Indeed, changing the rotation in the vertex of the loop yields a graph which is, on one hand, isomorphic to the initial one, and on the other hand, differs from it by a sign.

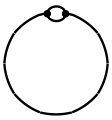
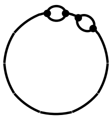
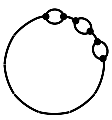
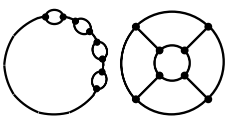
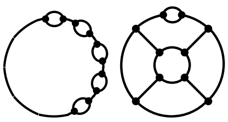
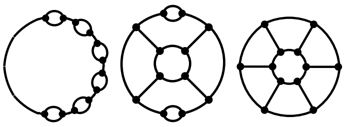
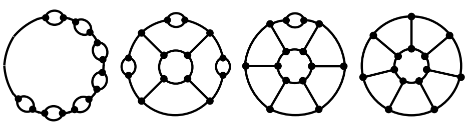
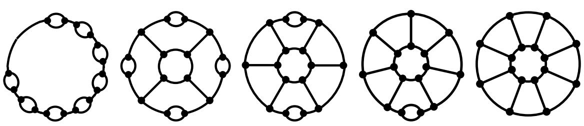
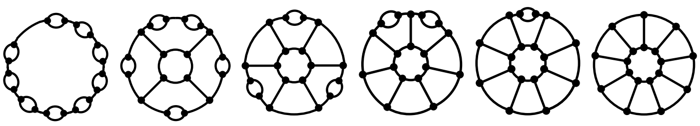
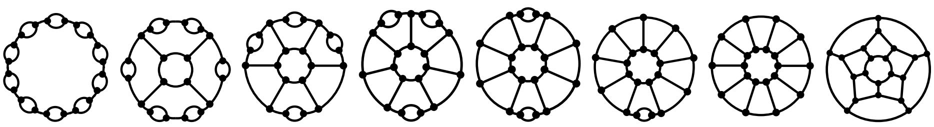
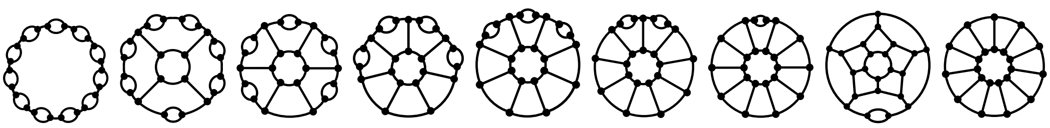
(b) By Lemma 1, γ is equal to a graph with a loop:



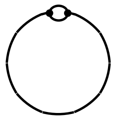
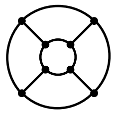
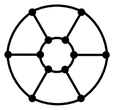
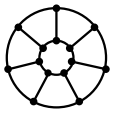
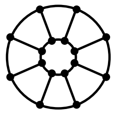
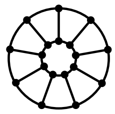
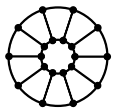
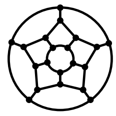
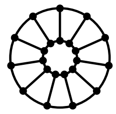
and the assertion follows from (a).

2.2 The Youngsters

The following table shows the bases of the linear spaces Γ_n for $n \leq 11$.

n	$\dim \Gamma_n$	additive generators
1	1	
2	1	
3	1	
4	2	
5	2	
6	3	
7	4	
8	5	
9	6	
10	8	
11	9	

Therefore, the following elements can be chosen as multiplicative generators of the algebra Γ up to degree 11 (notations in the lower line mean: b — 'bubble', w_i — 'wheels', d — 'dodecahedron'):

n	1	4	6	7	8	9	10		11
									
	b	w_4	w_6	w_7	w_8	w_9	w_{10}	d	w_{11}

Conjecture. The algebra Γ is generated by planar graphs.

2.3 Relations

The reader might have noted that the previous table of multiplicative generators does not contain the elements w_4^2 of degree 8 and $w_4 w_6$ of degree 10. This is not accidental. It turns out that the following relations hold in the algebra Γ:

$$\begin{aligned} w_4^2 &= \frac{5}{384}b^8 - \frac{5}{12}b^4 w_4 + \frac{5}{2}b^2 w_6 - \frac{3}{2}b w_7, \\ w_4 w_6 &= \frac{305}{27648}b^{10} - \frac{293}{864}b^6 w_4 + \frac{145}{72}b^4 w_6 - \frac{31}{12}b^3 w_7 + 2b^2 w_8 - \frac{3}{4}b w_9. \end{aligned}$$

O. Dasbach [Das] has proved the general fact that *the product of an arbitrary pair of homogeneous elements of Γ of positive degree belongs to the ideal generated by b.*

Since there are non-trivial relations between the generators, the algebra of 3-graphs, in contrast to the algebra of Feynman diagrams (see Sec. 6), is not free and hence does not possess the structure of a Hopf algebra.

Conjecture 1. The mapping $\Gamma \to \Gamma$ of multiplication by the bubble ($x \mapsto bx$) has no kernel.

This is an old conjecture: in the context of Vassiliev invariants, it was stated several times by various mathematicians, beginning from 1992, and stood several unsuccessful attempts of proof. We generalize it as follows.

Conjecture 2. The algebra Γ does not have zero divisors.

3 Computations

The experimental data reported in the previous section were obtained by computer calculations. The C/C++ programs, auxiliary shell-scripts and data files are available by anonymous ftp from math.botik.ru (see [Ftp]).

The main object dealt with by our programs is a regular three-valent graph without loops nor multiple edges, given by its incidence matrix of 0's and 1's. The matrix representation presupposes a certain numbering in the set of vertices. If $a < b < c$ are the numbers of vertices adjacent to a given one, then we assume that the rotation in this vertex is defined by the cycle (abc). If the list of all graphs with a fixed number of vertices x_1, x_2, ..., x_N is found, then any 3-graph y of the same order will be equal, in algebra Γ, to $\pm x_i$ (to determine the number i and the sign $\pm$ one has to find an isomorphism of abstract graphs y and x_i and then to count the number of vertices where this isomorphism violates the cyclic order of edges).

The computation of generators and relations in the algebra Γ up to degree 10 (graphs with 20 vertices) was carried out in two steps:

1. Preparation of material. At this stage we compile the list of all regular 3-valent graphs of a given order.

Since every graph with multiple edges, i. e. containing a bubble, as well as every graph with a triangle, can be represented as a product of graphs of smaller degree, it is sufficient to have the list of all connected graphs without loops nor multiple edges.

The list of 3-graphs is created by the program `gr3g` which consists of two main modules: the generator of the 'raw' stream of graphs based on a backtracking algorithm, and the filter which throws away the graphs isomorphic to those already chosen. We are obliged to I. A. Nesterov who in the summer of 1995 wrote a prototype version of this program and obtained the lists of 3-graphs up to 16 vertices.

2. Setting up and solving the equations. When making the equations for the graphs of a given order we assume that, for all smaller orders, we already know the set of generators and polynomial expressions of all graphs through the generators.

The program `relgr` finds the Kirchhof relations among the graphs of a given order. As soon as a graph with a triangle appears while writing out the equations, it is expressed through the graphs of smaller order, according to Lemma 1 of Sec. 2.1. Before running the program, one has the option of choosing the supposed new generators. If no assumptions about the new generators are made, the program will pick up any appropriate elements as such.

After the system of linear equations is set up, the program begins solving it. At this stage, two objectives are pursued: (1) find the expression of all graphs through the generators of degrees smaller than or equal to the current degree, (2) find, if any, the relations between the generators.

In the most difficult case — for graphs with 20 vertices — the computations were organized as follows:

1. The list of all 3-graphs was compiled in June 1996 on a network of 40 workstations in the University of Aizu, Japan. One of the computers (the fastest we had) was generating the raw stream of graphs, all the others were busy filtering it with respect to isomorphism. The computations took about one week, which allowed us to use the *manual dynamic parallelizing* — a method unusual in multiprocessor practice.

2. Making and solving the system of equations was performed in May 1997 using the most powerful computer we got access to: a dual Pentium Pro 200 with 256 Mb of memory. We thank B. Segal (CERN, Geneva) for his help.

The lists of 3-graphs with ≤ 20 vertices are available at [Ftp] (see also [Cub]). For each degree we give the lists of adjacency matrices and the polynomials that express the graphs through the generators.

Apart from the software for finding the generators and relations, the same ftp-server also contains two programs `sl_inv` and `so_inv` which serve to find the invariants of a given graph associated with Lie algebras (see next section). Both programs are based on the simplest algorithm of checking all possible vertex markings (for sl) or edge markings (for so) and calculating the number of components of the corresponding plane curves.

The description of the multiplicative structure of the graph algebra in degree 11 was obtained indirectly. First of all, we leaned upon a recent result of Jan Kneissler [Kn] that $\dim \Gamma_{11} = 9$. Then, using the sl- and so-invariants, we have checked that a certain set of 9 monomials of degree 11 in the generators that we found is linearly independent.

4 The algebra of graphs and Lie algebras

Having at one's disposal a set of homomorphisms into the ground field is always helpful in the study of an algebra.

Below, we describe the construction, going back to R. Penrose and M. Kontsevich (see also [BN2]), of a function on the algebra of 3-graphs associated with a metrized Lie algebra. More exactly, suppose that $\mathfrak{g}$ is a Lie algebra and B a non-degenerate ad-invariant symmetric bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. Then we can define a $\mathbb{C}$-valued function on the set of 3-graphs which satisfies the AS and IHX relations and thereby descends to to a linear function $K_{\mathfrak{g}}^{B} : \Gamma \to \mathbb{C}$. Moreover, it turns out that for a simple Lie algebra $\mathfrak{g}$ (say, with the Killing form as B) this function, divided by the dimension of $\mathfrak{g}$, yields an algebra homomorphism $\Gamma \to \mathbb{C}$.

4.1 Construction of the invariant $K_{\mathfrak{g}}^{B}$

First, we define the tensor $J \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. The Lie bracket of the given Lie algebra $[\cdot, \cdot]$ can be regarded as an element of the space $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. The identification of $\mathfrak{g}^*$ and $\mathfrak{g}$ with the help of the fixed bilinear form B takes this tensor into an element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ which we denote by J. Due to the properties of the commutator $[\cdot, \cdot]$ and the form B, the tensor J is skew-symmetric with respect to the permutations of the three tensor factors in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. This fact will soon be made use of.

Recall that we can view a 3-graph as a set H of cardinality $6n$ ('the set of half-edges'), endowed with two partitions: into 3-element subsets, each with a fixed cyclic order of its elements, and into 2-element subsets. The 3-element subsets correspond to the vertices of the graph and form the *vertex partition*, while the 2-element subsets correspond to its edges and form the *edge partition*.

Consider three linear spaces:

- $\bigotimes_{H} \mathfrak{g}$ — the tensor product of $6n$ copies of the Lie algebra $\mathfrak{g}$, whose factors correspond to the half-edges of the graph,
- $\bigotimes_{V} (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})$ — the tensor product of $2n$ copies of the space $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$, whose factors correspond to the vertices of the graph,

- $\bigotimes\limits_E (\mathfrak{g}\otimes\mathfrak{g})$ — the tensor product of $3n$ copies of the space $\mathfrak{g}\otimes\mathfrak{g}$, whose factors correspond to the edges of the graph.

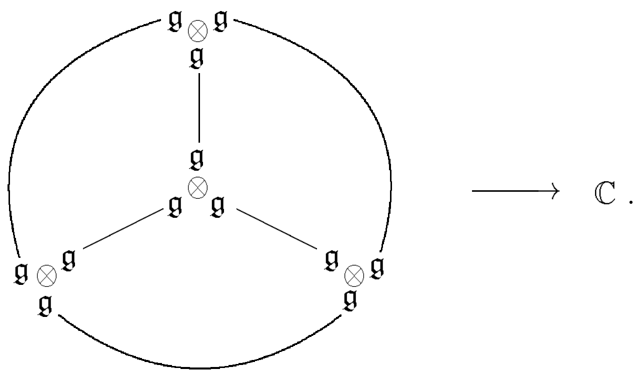
The vertex partition of the set H defines an isomorphism $\alpha : \bigotimes\limits_V (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}) \to \bigotimes\limits_H \mathfrak{g}$. The edge partition defines an isomorphism $\beta : \bigotimes\limits_H \mathfrak{g} \to \bigotimes\limits_E (\mathfrak{g}\otimes\mathfrak{g})$. A particular choice of the isomorphism α is fixed by choosing a specific order of the three half-edges around each vertex — we will suppose that it is arbitrary, but agreeing with the rotation in G. In the same way, a particular choice of the isomorphism β depends on the preferred ordering of the two half-edges that correspond to one and the same edge of the graph — for our purpose this order will prove irrelevant, and we fix it arbitrarily.

Now consider the composition of these two isomorphisms and the mapping to the ground field induced by the bilinear form $B : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}$:

$$\bigotimes_V (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}) \xrightarrow{\alpha} \bigotimes_H \mathfrak{g} \xrightarrow{\beta} \bigotimes_E (\mathfrak{g}\otimes\mathfrak{g}) \to \mathbb{C}$$

The value $K_{\mathfrak{g}}^B(G)$ of the function $K_{\mathfrak{g}}^B$ on a graph G, by definition, is equal to the value of the composition mapping on the element $J\otimes J\otimes\ldots\otimes J$. Despite the arbitrariness inherent in the construction of the isomorphisms α and β, the final result is uniquely defined due to the fact that the tensor J is invariant under even permutations, while the bilinear form B is symmetric.

Speaking pictorially, we suspend a copy of the tensor cube of $\mathfrak{g}$ with the distinguished element J at every vertex of the graph, then take the product over all vertices and compute the complete contraction of the resulting tensor of rank $6n$, applying the bilinear form B to every pair of factors that correspond to the endpoints of one edge:



Note that for every 3-graph of degree n we build one and the same element in the $6n$-th tensor power of the Lie algebra — but the mapping of this space into the ground field, defined by a specific pairing of the tensor factors, depends on the choice of a particular graph.

Verification of relations.

- The AS relation follows the fact that the tensor J changes sign under odd permutations of the three factors in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$.

- The IHX relation is a corollary of the Jacobi identity in $\mathfrak{g}$.

Changing the bilinear form. The bilinear form B preserves all the required properties when multiplied by a constant. A direct tracing of the above described construction shows that the corresponding function $K_{\mathfrak{g}}^{B}$ changes in the following way:

$$K_{\mathfrak{g}}^{\lambda B}(G) = \lambda^{-n} K_{\mathfrak{g}}^{B}(G)$$

for $G \in \Gamma_n$.

Multiplicativity. For a simple r-dimensional Lie algebra $\mathfrak{g}$ and any choice of the ad-invariant non-degenerate symmetric bilinear form B the function

$$\frac{1}{r} K_{\mathfrak{g}}^{B} : \Gamma \to \mathbb{C}$$

is multiplicative.

This fact follows from the properties of the Casimir operator of a simple Lie algebra.

To prove it, we need some preliminary remarks. First of all, note that the construction of the function $K_{\mathfrak{g}}^{B}$ is valid not only for 3-graphs, but also for graphs which, apart from 3-valent, also have univalent vertices. In this case we contract only those pairs of factors in the big tensor product that correspond to the edges connecting the 3-valent vertices — and as a result, instead of a number, we get an element of the space $\bigotimes_{U} \mathfrak{g}$, where U is the set of univalent vertices. The element thus obtained is always ad-invariant. In particular, if the algebra is simple and if the graph has only two univalent vertices, then the resulting tensor must be proportional to the quadratic Casimir element $c \in \mathfrak{g} \otimes \mathfrak{g}$. We recall that the Casimir element for a given form B is the element of $\mathfrak{g} \otimes \mathfrak{g}$ which corresponds to $B \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ and to $\mathrm{id} \in \mathfrak{g}^* \otimes \mathfrak{g}$ under isomorphisms induced by the form B. One usually considers the Casimir element for the Killing form, but for us this is not important. What matters is that the value of the bilinear form $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ on the corresponding Casimir element is always equal to the trace of the identity operator, i. e. to the dimension of the Lie algebra.

We pass to the proof of the assertion. Cut an arbitrary edge of the graph G_1 and consider the tensor that corresponds to the obtained graph with two univalent vertices. This tensor is proportional to the Casimir element $c \in \mathfrak{g} \otimes \mathfrak{g}$ with coefficient $\frac{1}{r} K_{\mathfrak{g}}^{B}(G_1)$. Similarly, for the second graph G_2 we get the tensor $\frac{1}{r} K_{\mathfrak{g}}^{B}(G_2) \cdot c$. Now, if we put together one pair of univalent vertices of the graphs G_1 and G_2 thus cut, then the partial contraction of the element $c \otimes c \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ will give an element proportional to $c \in \mathfrak{g} \otimes \mathfrak{g}$, while the coefficients involved get multiplied. This proves the multiplicativity of the function $\frac{1}{r} K_{\mathfrak{g}}^{B}$.

4.2 How to compute sl_N- and so_N-polynomials

For the classical Lie algebras sl and so the invariant just constructed admits an elegant combinatorial interpretation, which we will now describe. The proof for sl_N (or, which is the same thing, for gl_N) can be found, e. g., in [BN2] and [CD2]. The proof for so_N is based on similar considerations. See also [Pen], [BN1].

Computing $K_{\mathfrak{g}}^B$ for $\mathfrak{g} = sl_N$ supplied with the bilinear form $B(x,y) = \mathrm{tr}(xy)$. A *vertex marking* of a 3-graph G is an assignment of numbers ± 1 to the vertices of G. If the number of vertices is $2n$, then the total number of vertex markings is 2^{2n}. For every marking consider a plane curve obtained from the graph immersion by *thickening* its edges, i. e. by replacing every edge by two parallel lines. In the vicinity of a vertex the behaviour of the line follows one of the two patterns, depending on the sign of this vertex (we recall that the rotation agrees with the orientation of the plane):

Let $|\mu|$ be the number of negative vertices in the marking μ and $b(\mu)$ the number of connected components of the curve obtained. Then

$$K_{sl_N}(G) = \sum_{\mu \in MV} (-1)^{|\mu|} N^{b(\mu)}$$

where MV is the set of all vertex markings. The sum consists of 2^{2n} terms.

If the parameter N is variable, then $K_{sl_N}(G)$ is a polynomial in N. We will call it *sl-polynomial* and denote by $sl(G)$.

The polynomial $sl(G)$ is divisible by the dimension of the Lie algebra $N^2 - 1$ and the quotient is a multiplicative function on the algebra of 3-graphs, which we call the *reduced sl-polynomial* and denote by $\widetilde{sl}(G)$.

Computing $K_{\mathfrak{g}}^B$ for $\mathfrak{g} = so_N$ supplied with the bilinear form $B(x,y) = \mathrm{tr}(xy)$. To compute the value of the so-polynomial on a given 3-graph, one has to perform the similar actions with edges instead of vertices. Here is the precise description.

An *edge marking* of a 3-graph is an assignment of numbers ± 1 to the edges of the graph. For every edge marking consider the plane curve which is obtained from the graph immersion by thickening the edges, with the edges marked by -1 twisted in the way shown below:

In the neighbourhood of the vertices we take the direct connection of the lines, like in the positive vertices for sl_N. Denoting by $|\mu|$ the number of negative edges in the edge marking μ and by $b(\mu)$ the number of connected components of the curve obtained, we can write the formula:

$$K_{so_N}(G) = \sum_{\mu \in ME} (-1)^{|\mu|} N^{b(\mu)}$$

where ME is the set of all edge markings. The sum consists of 2^{3n} summands.

Dividing the so-polynomial by the dimension of the algebra $N(N-1)/2$, we obtain the *reduced so-polynomial* $\widetilde{so}(G)$, which is an algebra homomorphism $\Gamma \to \mathbb{Z}[N]$.

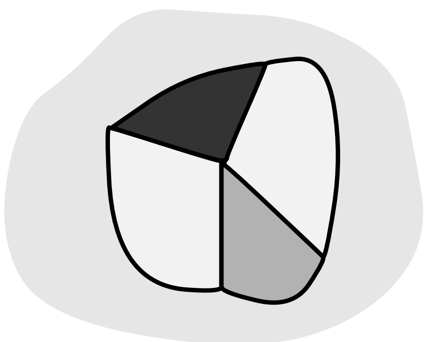
4.3 Values on generators of small degree

In the following table we quote the values of sl- and so-polynomials on the generators of Γ described in Sec. 2. This data was found by computer (see Sec. 3). For the algebra so the polynomials are expressed through the variable $M = N - 2$.

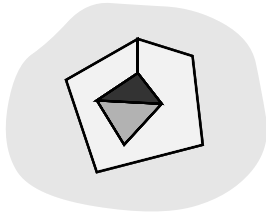
degree	element	$\widetilde{sl}$-polynomial	$\widetilde{so}$-polynomial
1	b	$2N$	$2M$
4	w_4	$2N^4 + 24N^2$	$2M^4 - 6M^3 + 60M^2 - 48M$
6	w_6	$2N^6 + 64N^4 + 96N^2$	$2M^6 - 10M^5 + 160M^4 - 368M^3$ $+816M^2 - 576M$
7	w_7	$2N^7 + 128N^5 + 128N^3$	$2M^7 - 12M^6 + 308M^5 - 816M^4$ $+1328M^3 - 768M$
8	w_8	$2N^8 + 256N^6 + 256N^4 + 384N^2$	$2M^8 - 14M^7 + 588M^6 - 1688M^5$ $+3216M^4 - 4256M^3$ $+9152M^2 - 6912M$
9	w_9	$2N^9 + 512N^7 + 512N^5 + 512N^3$	$2M^9 - 16M^8 + 1128M^7$ $-3376M^6 + 7104M^5 - 11200M^4$ $+12672M^3 + 12288M^2 - 18432M$
10	w_{10}	$2N^{10} + 1024N^8 + 1024N^6$ $+1024N^4 + 1536N^2$	$2M^{10} - 18M^9 + 2184M^8 - 6656M^7$ $+14880M^6 - 26432M^5 + 36096M^4$ $-35840M^3 + 111360M^2 - 95232M$
10	d	$2N^{10} + 22N^8 + 228N^6 - 232N^4$	$2M^{10} - 18M^9 + 88M^8 - 188M^7$ $+1254M^6 + 1038M^5 - 4948M^4$ $-21832M^3 + 60144M^2 - 35520M$
11	w_{11}	$2N^{11} + 2048N^9 + 2048N^7$ $+2048N^5 + 2048N^3$	$2M^{11} - 20M^{10} + 4268M^9$ $-13072M^8 + 30240M^7 - 58240M^6$ $+91008M^5 - 110080M^4 + 97536M^3$ $+290816M^2 - 331776M$

5 Graph algebra and the four colour theorem

The four colour theorem asserts that every planar map can be painted with four colours so that regions that have a common piece of boundary are coloured differently. For example:



Of course, it is supposed that a country cannot be adjacent to itself, as in the following picture:



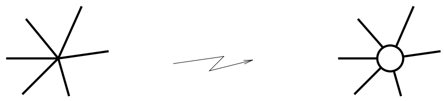
where the light country is its own neighbour.

The goal of this section is to show that the algebra of 3-graphs has a close relation to the four colour and helps to understand its algebraic meaning.

5.1 Preliminary results

Lemma 1 (A. Cayley [Ca], A. Kempe [Ke], 1879). *It is enough to prove the four colour theorem for maps with the property: there is no point where more than three vertices meet.*

Proof. Suppose that we know how to colour any map satisfying the assumptions of the lemma. Let us be given a map where at some point more that three countries meet. Adding small auxiliary countries around every such point



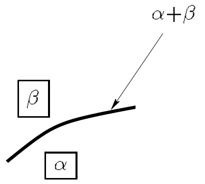
produces a map which satisfies the assumptions of the lemma. A colouring of this map evidently induces a required colouring of the initial map. The lemma is proved.

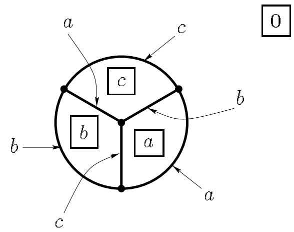
Let G be the boundary graph of the map, i. e. the graph whose edges are the boundaries of the countries, while the vertices are the points where the boundaries meet. According to the previous lemma, G can be assumed to be a regular three-valent graph. Therefore, the Cayley–Kempe lemma elucidates the reason why the 3-graphs arise in connection with the four colour theorem.

Definition. An *edge 3-colouring* of a three-valent graph is a colouring of the set of its edges with three colours, such that the three edges meeting at any vertex have different colours.

Lemma 2 (P. Tait [Ta], 1880) *The colourings of a plane map with four colours and a fixed colour of one country are in one-to-one correspondence with the edge 3-colourings of the boundary graph.*

Proof. Suppose we have a four coloured map. Identifying the four colours with the four elements 0, a, b, c of the Klein group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, we are able to construct an edge 3-colouring of the boundary graph G using the three colours a, b and c according to the rule: *if two countries have colours α and β, then the edge between them is coloured with $\alpha + \beta$.*



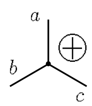


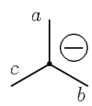
The inverse problem — construct a map colouring from a given edge 3-colouring of the boundary graph — is solved by *travelling with a magic brush.* We take a brush of the fixed colour and start at the chosen country. The algorithm is: *paint the current country with the current colour of the brush, then cross an arbitrary border and add the colour of the border to the current colour of the brush.* During the travel, the borders may be chosen arbitrarily — with the only ultimate purpose of visiting all countries.

The fact that this construction is correct and gives a mapping inverse to the one previously defined is ensured by the properties of group addition in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This proves the lemma.

An important moment in the history of the four colour theorem was in the year 1898, when P. Heawood [He] introduced the *sign of a vertex* for an edge 3-colouring of a plane graph.

Definition. A vertex of a plane three-valent graph is assigned the sign '*plus*', if going around this vertex counterclockwise the three colours are encountered in one fixed order, say a, b, c. If the colours appear in the opposite order — a, c, b, then the vertex is assigned a '*minus*'.





The following theorem was discovered in the 1950's by R. Penrose who stated it in [Pen] with the remark that his proof of this simple fact was too complicated to be published. Below, we give an elegant proof due to L. Kauffman [Ka1] (see also [Ka2, KaS]).[2]

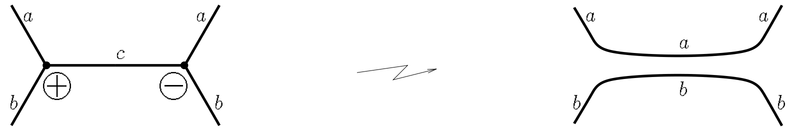
Theorem. *The product of the signs of all vertices does not depend neither on the particular edge 3-colouring, nor on the choice of the graph with $2n$ vertices and equals $(-1)^n$.*

Proof. Let V be the set of vertices of the graph G, $\#(V) = 2n$, and let ε_v stand for the sign of the vertex v. Then $\prod\limits_{v \in V} \varepsilon_v = \prod\limits_{(v_1, v_2) \in c} \varepsilon_{v_1} \varepsilon_{v_2}$, where the last product is taken over all

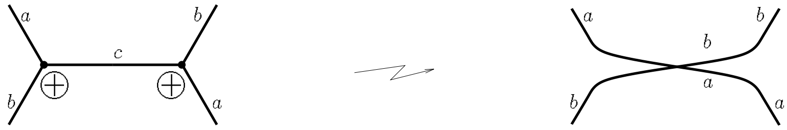
[2]We have proposed this theorem as a problem at the Russian mathematical olympiad for high school students in 1995 [Kv]. The students found at least two solutions, different from Kauffman's proof (and from each other).

c-edges (v_1, v_2), that is all the edges dyed with colour c. The total number of c-edges is equal to $\#(c) = \#(V)/2 = n$. They fall into two classes: $c^=$ and $c^\times$ — according to whether the signs at both endpoints are equal or different.

$c^=$:



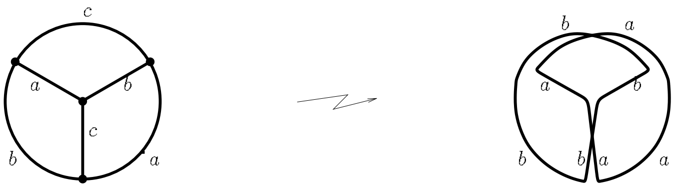
$c^\times$:



Therefore, the product of the signs of all vertices is

$$(-1)^{\#(c^=)} = (-1)^{2n}(-1)^{\#(c^\times)} = (-1)^n(-1)^{\#(c^\times)}.$$

It remains to show that the number $\#(c^\times)$ is even. This can be done by the following trick due to Kauffman. Resolving all the c-edges as shown in the picture above, we obtain a set of circles coloured with colours a and b. Circles of the same colour do not intersect. The number of edges of type $c^\times$ is equal to the number of intersections between the a-circles and b-circles. By Jordan's theorem every circle divides the plane into two connected components, hence this number is even. Here is an example:



The theorem is proved.

5.2 The number of colourings of a 3-graph

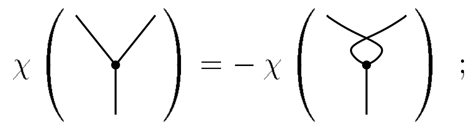
The four colour theorem can be stated as a property of the following function defined on the set of plane graphs:

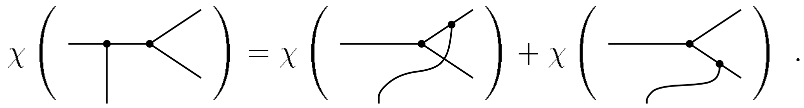
$$\chi(G) = \sum_{\substack{\text{edge} \\ \text{3-colourings } G}} \prod_{v \in V} \varepsilon_v \,.$$

Indeed, according to the previous theorem, all the terms of this sum are the same and equal to $(-1)^n$. Therefore, up to a sign, $\chi(G)$ is equal to the number of edge 3-colourings of G. The four colour theorem is thus equivalent to the fact that $\chi(G) \neq 0$ for any plane graph G.

Note that the Heawood sign of the vertex ε_v is well-defined not only for plane graphs, but also for all 3-graphs, i. e. graphs with rotation. In this case the sign ε_v is $+1$, if and only if the cyclic order of colours at the vertex v agrees with the rotation. We can, therefore, extend the domain of the function $\chi(G)$ to the set of 3-graphs.

Proposition 1. *The function χ satisfies the relations*

(AS): 

(IHX): 

and hence descends to the algebra Γ.

Remark. It is natural to set the value of χ on the circle (the unit of the algebra of 3-graphs) equal to 3.

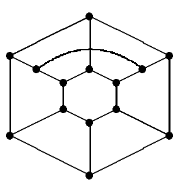
Proof of proposition 1. The validity of the AS relation is obvious. In fact, the change of rotation in one vertex of a 3-graph G leads to the change of sign of one vertex in every edge 3-colouring and hence to the change of sign in every term in the sum that defines $\chi(G)$.

Let us prove the IHX relation. Without loss of generality we can assume that the middle edge in the left hand side of the relation has colour c. Then the configuration formed by the remaining four edges belongs either to the class $c^{=}$ or to the class $c^{\times}$ (see subsection 5.1). In the first case the colouring of the graph on the left induces a colouring of the first summand on the right, in the second case, a colouring of the second summand. Every term of the sum that defines $\chi(G)$ is thus represented in either of the two summands on the right. There are additional terms in the right hand side — those that do not correspond to anything on the left — but they appear in mutually cancelling pairs:

This proves our assertion.

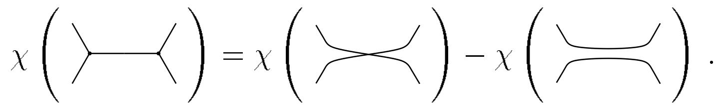
Corollary. The number of colourings $\chi(G)$ of a 3-graph G which represents a zero in the algebra Γ, is equal to 0.[3]

Remarks. 1. Here's an example of a graph G which is different from 0 in Γ but having $\chi(G) = 0$:



[3]This implies the following a priori non-evident fact: *a plane graph cannot be equal to zero in* Γ. It would be interesting to find a proof of this fact without resort to the four colour theorem.

2. The function $\chi : \Gamma \to \mathbb{Z}$ satisfies a relation which is stronger than IHX, although not homogeneous in the sense of the graduation in Γ:

(L) : 

It can be proved similarly to the IHX relation.

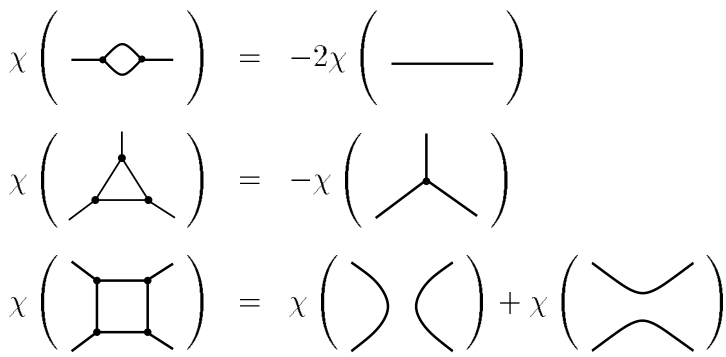
This relation was known to Lagrange (and probably to Euler) as the identity $a \times (b \times c) = b(a,c) - c(a,b)$ for the vector product of vectors in real 3-space[4]. It was rediscovered by R. Penrose [Pen] in the context of graph theory and Lie algebras. Then it was again rediscovered in [CV] in the context of Vassiliev invariants and in [Gar] in connection with the invariants of 3-manifolds.

In general, the relation (L) takes you outside of the class of connected graphs. We set the value of $\chi(G)$ on a disconnected graph G to be equal to the product of its values on the connected components.

With these relations at hand, one is tempted to try an approach to the four colour theorem based on induction on the number of vertices. Suppose that the theorem is true for all plane graphs with less than $2n$ vertices and let G be a plane graph with $2n$ vertices. Using relations (L), (AS) and (IHX), we can represent $\chi(G)$ as a linear combination of several terms of the form $\chi(G_i)$, where G_i's are plane graphs with fewer than $2n$ vertices. If we manage to find such a decomposition where

- either all graphs have the same number of vertices modulo 4 and all the coefficients have the same sign (positive or negative),
- or the signs of some coefficients differ from the others, but then corresponding graphs have a different number of vertices modulo 4

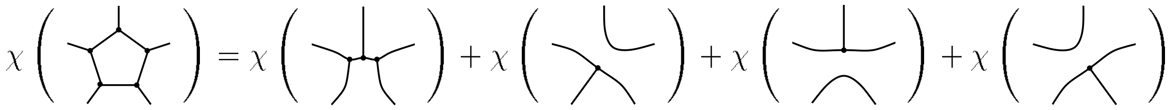
— then we can infer by induction that $\chi(G) \neq 0$. And, indeed, the required decompositions can be found, if the graph G contains a biangle, a triangle or a quadrangle [Pen]:



An easy argument involving Euler characteristic shows that any plane graph contains a cycle with the number of sides less than or equal to 5. Hence, to prove the four colour

[4]To understand the connection between the two formulation, one has to view both sides of Lagrange's identity with variable a, b, c as elements of the fourth tensor power of the Lie algebra so_3.

theorem, it remains to find a relation for graphs with a pentagon. The best formula that we could obtain (and later found in [Pen]) reads:



and does not lead to the proof of the four colour theorem, because the graphs in the right side have numbers of vertices, different modulo 4, and hence the corresponding values of χ have different signs.

5.3 3-colourings and the Lie algebra so_3

.

Three-space $\mathbb{R}^3$ endowed with the operation of vector product can be viewed as the Lie algebra $so_3(\mathbb{R})$.[5] For the ad-invariant non-degenerate form $B(x,y) = -\frac{1}{2}\operatorname{tr}(xy)$ the function $K^B_{so_3}$, constructed in Section 4, coincides with the number of colourings χ. Indeed, the following three matrices constitute a basis of so_3, orthonormal with respect to the bilinear form B:

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \;;\quad b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \;;\quad c = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

The commutation relations $[a,b] = c$, $[b,c] = a$, $[c,a] = b$ imply that the tensor J, assigned to each vertex of the graph, can be written as

$$J = a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b - b \otimes a \otimes c - c \otimes b \otimes a - a \otimes c \otimes b .$$

The contraction of the $3n$-th power of this tensor in accordance with the rule of Sec. 4.1 is equivalent to counting the edge 3-colourings of a graph (signs at both endpoints of an edge must coincide; every vertex gives a factor of $+1$ or -1 depending on the cyclic order of the three 'colours' a, b, c). Therefore, the colouring counting function χ is a particular case of the general construction of a function on the graph algebra associated with by a metrized Lie algebra. The general multiplicativity property proved in Sec. 4.1 now implies the following assertion.

Proposition. *The function $\chi(\cdot)/3$ is multiplicative, i. e. represents an algebra homomorphism $\Gamma \to \mathbb{R}$.*

In the terminology introduced in this section we can state the four colour theorem like this: *the value of the so_3-invariant on any plane 3-graph is different from 0.*

6 Graph algebra and Vassiliev invariants

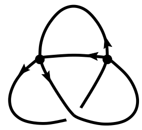
We will explain the connection of the graph algebra with finite order knot invariants, directly starting from Vassiliev's filtration in the linear space generated by the isotopy classes of knots. A more traditional introduction to the theory of Vassiliev invariants can be found, e. g. in [BN1, CD1, CDL]. The reader may wish to consult these references for the details and proofs which are here omitted.

[5]Instead of so_3 one can use the $\mathbb{C}$-isomorphic Lie algebra sl_2.

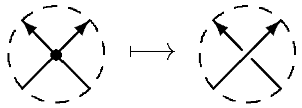
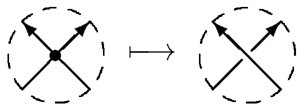
6.1 Singular knots and Vassiliev's filtration

Definition. A *singular knot* is a smooth embedding $K : S^1 \to \mathbb{R}^3$, which is an immersion everywhere but in a finite number of simple double points (i. e. points of transversal self-intersection of two branches of the curve). We denote by $\mathcal{K}_n$ the set of all singular knots with n double points.

Here is an example of a knot with two double points:



Every double point of a singular knot can be *resolved* in two ways:

- positively  and
- negatively  .

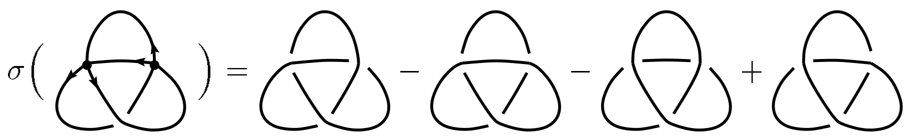
Note that both operations are correctly defined on isotopic classes of knots and do not depend on the orientation chosen on the knot.

The *complete resolution* of a singular knot $K \in \mathcal{K}_n$ is the linear combination of 2^n non-singular knots

$$\sigma(K) = \sum_{\varepsilon_1,\ldots,\varepsilon_n = \pm 1} (-1)^{|\varepsilon|} K_{\varepsilon_1,\ldots,\varepsilon_n},$$

where $|\varepsilon|$ stands for the number of minus ones in the sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, and the smooth knot $K_{\varepsilon_1,\ldots,\varepsilon_n}$ is obtained from K by the resolution of every double point p_i — positively or negatively depending on the sign ε_i.

Example.



Let $\mathcal{X}$ be the linear space, say over $\mathbb{C}$, freely generated by the isotopy classes of (non-singular) knots in $\mathbb{R}^3$. In $\mathcal{X}$, we consider the decreasing filtration

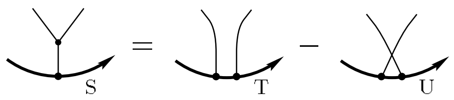
$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \mathcal{X}_3 \supseteq \ldots ,$$

where $\mathcal{X}_n$ is the subspace generated by complete resolutions of singular knots with n double points, i. e. the linear span of the set $\sigma(\mathcal{K}_n)$.

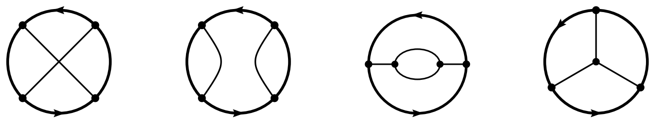
The quotient space $\mathcal{X}_n/\mathcal{X}_{n+1}$ allows a combinatorial description in terms of *Feynman diagrams*.

Definition. A *Feynman diagram* of degree n is a regular three-valent graph on $2n$ vertices with a distinguished oriented cycle, called *Wilson loop*, and a fixed *rotation* (cyclic order of edges) in all the vertices that do not belong to the Wilson loop.

Definition. The *space of Feynman diagrams* $\mathcal{F}_n$ is the linear space, generated by all Feynman diagrams of degree n modulo *STU relations*:



Example. Up to a sign, there exist 4 different non-zero Feynman diagrams of degree 2:



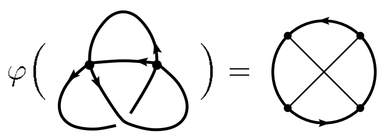
whereas the STU relations ensure that $\dim \mathcal{F}_2 = 2$.

Remark. STU relations imply AS and IHX relations for the inner vertices of a Feynman diagram ('inner' means 'not belonging to the Wilson loop') [BN1].

Feynman diagrams are related to singular knots by means of the following construction.

Definition. The *Feynman diagram* $\varphi(K)$ *of a singular knot* $K : S^1 \to \mathbb{R}^3$ is the circle S^1 endowed with the collection of n chords that connect the pairs of points having equal images under the mapping K.

Example. For the singular knot that appeared as an example above, we have:



In the direct sum of linear spaces

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n$$

there is a correctly defined multiplication: each of the two Wilson loops is cut at some inner point, then they are put together to form the Wilson loop of the product. Let $\widetilde{\mathcal{F}}$ be the quotient algebra over the ideal, generated by the Feynman diagram .

Theorem. *The assignment $\sigma(K) \mapsto [\varphi(K)] \in \widetilde{\mathcal{F}}_n$ extends to a correctly defined linear mapping $\mathcal{X}_n \to \widetilde{\mathcal{F}}_n$, whose kernel coincides with $\mathcal{X}_{n+1}$. This yields a linear space isomorphism $\mathcal{X}_n / \mathcal{X}_{n+1} \cong \mathcal{F}_n$.*

This fact is one of the possible reformulations of the famous Kontsevich's theorem (see [BN1, CD3] for a proof).

6.2 Connected Feynman diagrams and 3-graphs

In the Feynman algebra $\mathcal{F}$ there is a naturally defined comultiplication (both the multiplication and the comultiplication agree with the corresponding operations in the space of knots $\mathcal{K}$, see [BN1, CD1, CDL]). In fact, $\mathcal{F}$ is a commutative and cocommutative Hopf algebra.

It is well-known that a commutative and cocommutative Hopf algebra coincides with the symmetric algebra over its space of primitive elements, or, in other words, it can be viewed as the algebra of polynomials in the primitive elements. For a graded algebra the primitive space is also graded: $\mathcal{P} = \oplus_{n=1}^{\infty} \mathcal{P}_n$, where $\mathcal{P}_n = \mathcal{P} \cap \mathcal{F}_n$. In the case of the algebra $\mathcal{F}$ the subspace $\mathcal{P}_1$ is one-dimensional and generated by the element $\ominus$. The quotient algebra $\widetilde{\mathcal{F}} = \mathcal{F}/(\ominus)$ is also a Hopf algebra, whose primitive space is naturally identified with the direct sum $\oplus_{n=2}^{\infty} \mathcal{P}_n$.

By construction, the space of primitive elements $\mathcal{P}$, considered by itself, does not possess any structure of an algebra. Primitive elements only generate the algebra $\mathcal{F}$ in the same way as the variables $x_1, \ldots, x_n$ generate the polynomial algebra $\mathbb{R}[x_1, \ldots, x_n]$. However, the link between the space $\mathcal{P}$ and the algebra of 3-graphs Γ allows to introduce in $\mathcal{P}$ a natural (non-commutative) multiplication. We shall now describe how this is done.

It is known [BN1] that the primitive space of the Feynman algebra coincides with the subspace spanned by the *connected Feynman diagrams*.

A Feynman diagram is said to be *connected*, if the graph obtained by the removal of the Wilson loop is connected. Here are examples of a connected and a disconnected Feynman diagrams:

The relation between the linear spaces Γ and $\mathcal{P}$ is expressed by two facts:

1. There is a linear inclusion $i : \Gamma_n \to \mathcal{P}_{n+1}$, defined on the generators as follows: we cut an arbitrary edge of the graph and attach a Wilson loop to the two resulting endpoints. The orientation of the Wilson loop can be chosen arbitrarily.

2. There is a projection $\pi : \mathcal{P}_n \to \Gamma_n$, which consists in introducing the rotation in the vertices of the Wilson loop according to the rule 'forward–sideways–backwards' and then forgetting the fact that the Wilson loop was distinguished.

The composition homomorphism $\Gamma \to \Gamma$ in the sequence

$$\Gamma \xrightarrow{i} \mathcal{P} \xrightarrow{\pi} \Gamma$$

coincides with the multiplication by the bubble $g \mapsto bg$.

The multiplicative structure of the graph algebra implies that there is an action $\Gamma \times \mathcal{P} \to \mathcal{P}$ which gives rise to an operation $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$.

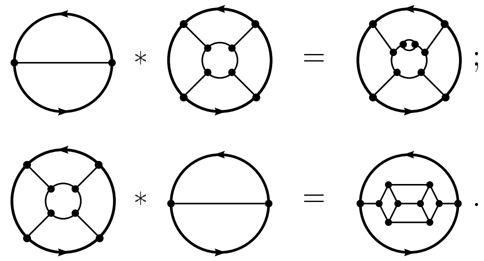
We first define the action $*' : \Gamma \times \mathcal{P} \to \mathcal{P}$. For two given elements $g \in \Gamma$ and $p \in \mathcal{P}$ we construct its product $g *' p$ using the same idea of the connected sum. More specifically, we cut an arbitrary edge of the graph g. Then we cut an edge in p not lying on the Wilson loop and attach the new endpoints of P to those of g, like we did for the algebra Γ itself. The graph that results is $g *' p$; it has one Wilson loop and is connected, hence it belongs to P.

Now define the multiplication $*$ of connected Feynman diagrams $p, q \in \mathcal{P}$ by the rule

$$p * q = \pi(p) *' q,$$

where $\pi : \mathcal{P} \to \Gamma$ is the homomorphism of forgetting the Wilson loop defined above.

Example. *The operation $*$ is, in general, non-commutative:*



These two elements of the space $\mathcal{P}$ are different; they can be distinguished, e. g., by the sl_2-invariant defined and studied in [CV].

The *correctness* of the definitions given can be verified by the arguments similar to those used in Sec. 1 for the algebra Γ.

Conjecture. The algebra of primitive elements $\mathcal{P}$ has no divisors of zero.

7 The algebra of 3-graphs and homology 3-spheres

The goal of this section is to show that the algebra of 3-graphs is closely related to the finite order invariants of integral homology 3-spheres, recently introduced by T. Ohtsuki [Oh1]. More precisely, in the infinite-dimensional linear space $\mathcal{M}$ freely generated by homology spheres, there is a filtration

$$\mathcal{M} \supseteq \mathcal{M}_0 \supseteq \mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \mathcal{M}_3 \supseteq \dots .$$

It turns out that $\mathcal{M}_{3n} = \mathcal{M}_{3n+1} = \mathcal{M}_{3n+2}$, while the quotient $\mathcal{M}_{3n}/\mathcal{M}_{3(n+1)}$ is isomorphic to the space spanned by all (possibly disconnected) 3-graphs with $2n$ vertices, viewed modulo AS and IHX relations. In this space there is a natural multiplication, defined by disjoint union of graphs, and comultiplication, together making up the structure of a Hopf algebra. The space of *connected* 3-graphs, i. e. our algebra Γ, serves as the primitive space of this Hopf algebra.

We do not know of any topological applications of the 3-graph algebra Γ in the context of three-manifolds, and in this section Γ will manifest itself only as a linear space.

7.1 Dehn surgery

Suppose that M is a closed oriented 3-dimensional manifold and $L = L_1 \cup \dots \cup L_n$ a link in M. Let $N(L_j)$ be a regular tubular neighbourhood of the component L_j. Choose a section f_j of the unit normal bundle $\partial N(L_j) \cong S^1 \times L_j \to L_j$. Its homotopy class is uniquely determined by an integer — the linking number between f_j and L_j. Since we are only interested in the homotopy classes of the sections, we will use the same letter f_j to denote both the section and its linking number with L_j. A pair $\mathcal{L} = (L, \mathbf{f})$, where $\mathbf{f} = (f_1, \dots, f_n)$, is referred to as a *framed link* in M.

Definition. *Dehn surgery* is a means of constructing a new manifold $M_{\mathcal{L}}$ starting from M a framed link $\mathcal{L}$ in M (see [MF, PS, Ro]). The construction of $M_{\mathcal{L}}$ proceeds as follows. Take a solid torus $T = S^1 \times D^2$ with a distinguished meridian $\mu = \{*\} \times \partial D^2$. Cut the neighbourhoods $N(L_j)$ of all components of the link L out of M and paste instead a copy of the solid torus T over a diffeomorphism of boundaries $h_j : \partial T \to \partial N(L_j)$, which takes a fixed meridian $\mu \subset \partial T$ into the section f_j:

$$M_{\mathcal{L}} = (M \setminus \cup N(L_j)) \bigcup_{h_j} T_j,$$

where T_j is the j-th copy of T.

The diffeomorphism class of the manifold $M_{\mathcal{L}}$ is defined by M and $\mathcal{L}$ and depends neither on the choice of the sections f_j within the homotopy classes specified nor on the choice of the particular diffeomorphisms h_j.

P. Lickorish [Li] (see also [Wa, PS, MF]) has proved that any closed oriented 3-dimensional manifold can be obtained from S^3 by a Dehn surgery along a suitable framed link in S^3. Framed links in S^3 that lead to one and the same manifold were described by R. Kirby [Kir].

7.2 Integral homology 3-spheres

Definition. An *integral homology 3-sphere* (*homology 3-sphere* for short) is a closed oriented three-manifold M, whose integral homology coincides with that of the sphere: $H_0(M) = \mathbb{Z}$, $H_1(M) = H_2(M) = 0$, $H_3(M) = \mathbb{Z}$.

The class of homology 3-spheres is closed with respect to the operation of Dehn surgery along *algebraically split 1-framed* links.

Definition. A framed link $\mathcal{L} = (L, \mathbf{f})$ inside a homology 3-sphere M is called *algebraically split 1-framed*, if all the framings f_j are ± 1 and all the pairwise linking indices of the components of L are zero.

Remark. Since we work with homology 3-spheres, the notion of the linking index is correctly defined.

The following two lemmas are important.

Lemma 1 [Oh1]. *Let M be a homology 3-sphere. A framed link $\mathcal{L}$ in M is an algebraically split 1-framed link, if and only if for any sublink $\mathcal{L}' \subseteq \mathcal{L}$ the result of the Dehn surgery $M_{\mathcal{L}'}$ is a homology 3-sphere.*

Lemma 2 [Oh1]. *Any homology 3-sphere M can be obtained from S^3 by the Dehn surgery along an algebraically split 1-framed link $\mathcal{L}$ in S^3: $M = S^3_{\mathcal{L}}$.*

7.3 Finite order invariants

Let $\mathcal{M}$ be the linear space (for simplicity, over $\mathbb{C}$) spanned by the diffeomorphism classes of homology 3-spheres. With a given homology 3-sphere M and an algebraically split 1-framed link $\mathcal{L}$ in M we associate a linear combination of homology 3-spheres

$$(M,\mathcal{L}) := \sum_{\mathcal{L}'\subseteq\mathcal{L}} (-1)^{\#\mathcal{L}'} M_{\mathcal{L}'} \in \mathcal{M},$$

where the summation ranges over all sublinks $\mathcal{L}'$ of the link $\mathcal{L}$, including the empty one; $\#\mathcal{L}'$ denotes the number of components of the sublink $\mathcal{L}'$.

Notation. By $\mathcal{M}_m$ we denote the subspace of $\mathcal{M}$, generated by all linear combinations $(M,\mathcal{L})$, where $\mathcal{L}$ is an arbitrary link with m components:

$$\mathcal{M}_m := \{(M,\mathcal{L}) | \#\mathcal{L} = m\}.$$

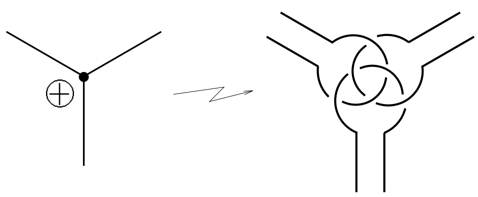
The subspaces $\mathcal{M}_m$ form a decreasing filtration

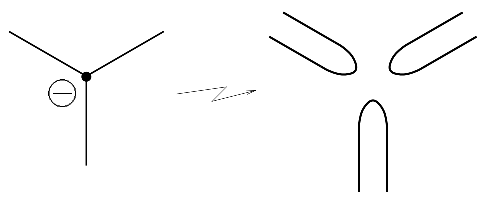
$$\mathcal{M} \supseteq \mathcal{M}_0 \supseteq \mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \mathcal{M}_3 \supseteq \dots$$

Definition. We say that a function $v : \mathcal{M} \to \mathbb{C}$ is an *invariant of order* m, if $v|_{\mathcal{M}_{m+1}} \equiv 0$ and $v|_{\mathcal{M}_m} \not\equiv 0$.

The linear spaces, generated by invariants of different orders, form an increasing filtration in the space of all invariants. The famous Casson invariant provides an example of a finite order invariant — its order is 3 (see [Oh1]). In [Oh2] Ohtsuki defined a series of invariants λ_n, generalizing the Casson invariant, equal to $\lambda_1/6$. A. Kricker and B. Spence [KrS] showed that λ_n has order $3n$. Thus, for every m divisible by 3 there is a non-trivial invariant of order m. On the other hand, it is proved in [GO] that $\mathcal{M}_{3n} = \mathcal{M}_{3n+1} = \mathcal{M}_{3n+2}$. Therefore, the order of any non-zero invariant of homology 3-spheres is a multiple of 3.

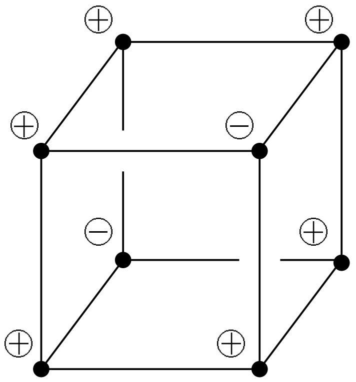
Definition. The *Le mapping* [Le] from the set of 3-graphs into the linear space $\mathcal{M}$ is defined as follows. Let G be a regular 3-valent (possibly disconnected) graph with $2n$ vertices, endowed with a rotation. We can embed G into 3-space and choose a generic projection of the space onto a plane, such that the rotation of the graph agrees with the orientation of the plane. Consider a vertex marking $(\varepsilon_1,\dots,\varepsilon_{2n})$ where each ε_i is either '+' or '−'. Starting with the marked graph $G_{\varepsilon_1,\dots,\varepsilon_{2n}}$, we construct a framed link $\mathcal{L}_{\varepsilon_1,\dots,\varepsilon_{2n}}(G) \subset S^3$, consisting of $3n$ components that correspond to the graph edges. Each component is unknotted and represents a thin oval stretching along the corresponding edge. In the vicinity of a vertex, the three ovals that meet behave as a Borromean link, if the mark is positive, and as a trivial link, if the mark is negative:



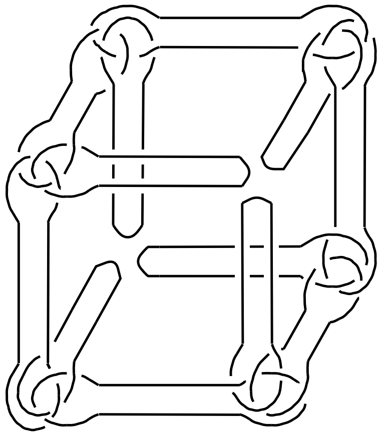


We take the framings of all components of the link thus constructed to be equal to 1. The framed link that results is denoted by $\mathcal{L}_{\varepsilon_1,\dots,\varepsilon_{2n}}(G)$. Here is an example:

$G_{\varepsilon_1,\dots,\varepsilon_{2n}} =$



$\mathcal{L}_{\varepsilon_1,\dots,\varepsilon_{2n}}(G) =$



It is easy to see that $\mathcal{L}_{\varepsilon_1,\dots,\varepsilon_{2n}}(G)$ is an algebraically split 1-framed link. We are finally in a position to write the formula that defines the Le mapping as an alternating sum of 2^{2n} homology spheres obtained by Dehn surgery along the constructed links:

$$\boxed{Le(G) = \sum_{\varepsilon_1,\dots,\varepsilon_{2n}} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{2n} \cdot S^3_{\mathcal{L}_{\varepsilon_1,\dots,\varepsilon_{2n}}(G)}}$$

In general, the Le mapping is not correctly defined — the element $Le(G)$ given by the previous formula may depend both on the immersion of G into space and on its projection onto the plane. However, the following theorem holds.

Theorem [Le, GO].

1) For any choice of the embedding and the projection the image $Le(G)$ of a graph G with $2n$ vertices belongs to $\mathcal{M}_{3n}$.

Let $\pi : \mathcal{M}_{3n} \to \mathcal{M}_{3n}/\mathcal{M}_{3(n+1)}$ be the natural projection of linear spaces. Then

2) The mapping $\pi \circ Le$ is defined correctly, i. e. the image $\pi \circ Le(G)$ does not depend on the specific embedding of the graph G into 3-space nor on its projection onto the plane.

3) The mapping $\pi \circ Le$ satisfies AS and IHX relations.

4) The mapping $\pi \circ Le$ descends to an isomorphism of the linear space generated by (disconnected) 3-graphs modulo AS and IHX relations, onto the quotient space $\mathcal{M}_{3n}/\mathcal{M}_{3(n+1)}$.

The invention of the Le mapping went through several steps. First Ohtsuki [Oh1] studied the mapping

$$G \mapsto (S^3, \mathcal{L}_{+,\dots,+}(G))$$

and proved its surjectivity. Although this mapping does not agree with AS and IHX relations, it provides a means to prove that the quotients $\mathcal{M}_m/\mathcal{M}_{m+1}$ are finite-dimensional. Then Garoufalidis and Ohtsuki [GO] introduced the mapping

$$G \mapsto \sum_{\varepsilon_1,\dots,\varepsilon_{2n}} \varepsilon_1 \cdot \ldots \cdot \varepsilon_{2n} \cdot (S^3, \mathcal{L}_{\varepsilon_1,\dots,\varepsilon_{2n}}(G)),$$

where $(S^3, \mathcal{L}_{\varepsilon_1,\dots,\varepsilon_{2n}}(G))$ is the above defined combination of 2^{3n} homology spheres (thus the whole sum consists of 2^{5n} terms), and proved that it satisfies both AS and IHX relations. Finally, Le [Le, Lemma 5.1] noticed that the formula of Garoufalidis and Ohtsuki can be simplified and rewritten in the form displayed in the frame above.

The most difficult part of Le's theorem — the injectivity of the mapping $\pi \circ Le$ — was proved in [Le] (see also [BGRT]).

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