Vassiliev knot invariants II. Intersection graph conjecture for trees

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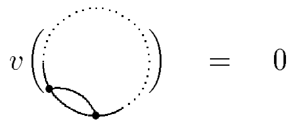
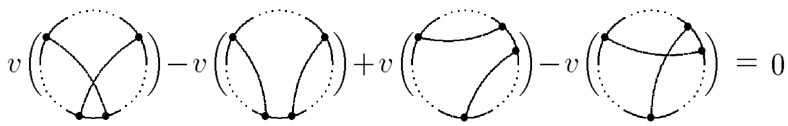
The space of Vassiliev knot invariants has a natural filtration by order [V]. Kontsevich's theorem [K] gives a purely combinatorial description of the corresponding graded space as a space of functions v ("invariants") on the set of chord diagrams satisfying certain linear equations:
one-term relations



and four-term relations



All such "invariants" form a Hopf algebra.

The intersection graph of a chord diagram is a more rough combinatorical object than the chord diagram itself, but it turns out that a lot of invariants

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depend only on the intersection graph. Here we prove that the one-term and four-term relations imply the coincidence of values of all invariants on two chord diagram which have the same intersection graph under the assumption that the intersection graph is a tree. A refined argument shows that this follows from the four-term relations alone.

Throughout this paper we use notations and definitions of the previous article [CDL1] which can be considered as an introduction to the present text.

The Hopf algebra of invariants is dual to the Hopf algebra of chord diagrams (see sec. 2.2 of [CDL1]), so the properties of either of them can be easily translated into the properties of the other. Below, we will be concerned with the Hopf algebra of chord diagrams. In particular, an equality $D_1 = D_2$ will mean equality of the two diagrams in the Hopf algebra $\mathcal{N}$ (see [CDL1]), i. e. their equality modulo one- and four-term relations.

Definition. *The intersection graph* $\Gamma(D)$ of a chord diagram D is a graph whose vertices correspond to the chords of D and two vertices are connected by an edge iff the corresponding chords intersect. (Two chords, a and b, are said to intersect if their endpoints a_1, a_2 and b_1, b_2 appear in the interchanging order a, b, a, b along the circle.)

Definition. A chord diagram D such that $\Gamma(D)$ is a tree will be called a *tree diagram*.

1 Main result

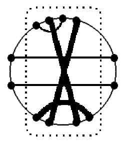
Theorem. *If* $\Gamma(D_1) = \Gamma(D_2)$ *is a tree then* $D_1 = D_2$.

To prove the Theorem we will describe the set of all chord diagrams with the same intersection tree graph. We will introduce elementary transformations on chord diagrams with the property that any two diagrams with the same intersection tree graph can be connected by a sequence of these transformations. Then we will prove that any two diagrams D_1 and D_2 differing by an elementary transformation are equal in the algebra of diagrams $\mathcal{N}$.

2 Key notions and statements

Definition. A *share* of a chord diagram is a collection of its chords such that there exist four points x_1, x_2, x_3, x_4 on the circle different from the endpoints of all chords and satisfying the two conditions:
— no chord connects two adjacent arcs x_1x_2, x_2x_3, x_3x_4 or x_4x_1;
— both ends of any chord of the collection belong to the union of the two opposite arcs x_1x_2 and x_3x_4.

Example. The collection of four chords inside the dotted oval



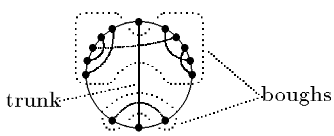
is a share. But the collection of three fat chords is not a share because there exists a fourth chord that separates their ends.

The collection formed by a single chord is always a share. The complement to a share is a share.

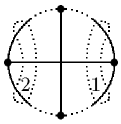
Here and below, we mark the shares by dotted ovals.

Definition. Pick a chord t in a tree diagram D, and call t the *trunk* of D. Consider the vertex $v(t)$ in $\Gamma(D)$ that corresponds to t. Then $\Gamma(D) \setminus v(t)$ is a forest. Each component of $\Gamma(D) \setminus v(t)$ is a tree. The collection of all chords D which correspond to all vertices of one component of $\Gamma(D) \setminus v(t)$ is a share. This share will be called a *bough* of the tree diagram with respect to the choosen trunk t.

Example.



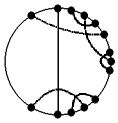
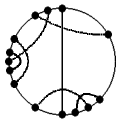
It is easy to see that each bough has the form of a "dumbbell" with two shares 1 and 2.

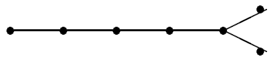


Definition. *An elementary transformation* of the tree diagram is a permutation of boughs with respect to some trunk.

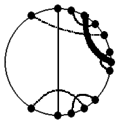
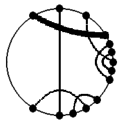
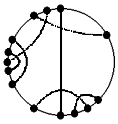
Proposition 1. *If D_1, D_2 are tree diagrams with $\Gamma(D_1) = \Gamma(D_2)$ then D_2 can be obtained from D_1 by a sequence of elementary transformations.*

Example. For diagrams

$D_1 =$  and $D_2 =$ 

$\Gamma(D_1) = \Gamma(D_2) =$ 

So D_2 can be obtained from D_1 by a sequence of elementary transformations (the corresponding trunks are shown in thick lines):

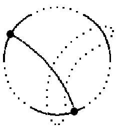
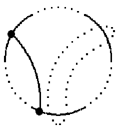
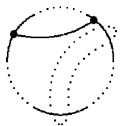
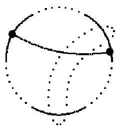
$D_1 =$  $\longrightarrow$  $\longrightarrow$  $= D_2$

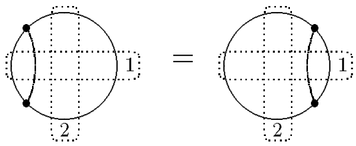
Proposition 2. *If D_1 and D_2 are two tree diagrams that differ by a permutation of boughs then $D_1 = D_2$.*

The Theorem is a consequence of Propositions 1 and 2.

In the proof of Proposition 2 we will use the following Statement and its Corollary.

Statement. (Generalized four-term relation) *For any share the following relation holds:*

 $-$  $+$  $-$  $= \quad 0$

Corollary. 

Note that in the last figure there are no dotted arcs, i. e. each diagram consists of the two marked shares and one more chord.

3 Proof of the Theorem

To prove the Theorem, it is sufficient to prove Propositions 1 and 2.

3.1 Proof of Proposition 1

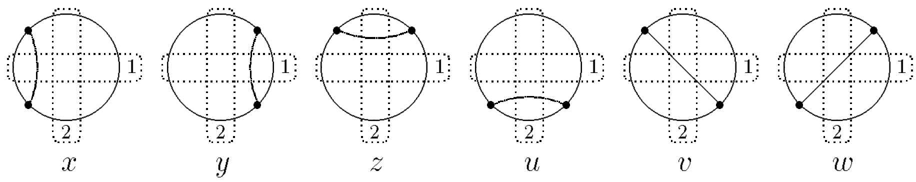
We can choose trunks $t_1 \in D_1$ and $t_2 \in D_2$ which correspond to one and the same vertex $v = v(t_1) = v(t_2)$ in the tree $\Gamma = \Gamma(D_1) = \Gamma(D_2)$. There is a one-to-one correspondence between the sets of boughs of D_1 and D_2. Rearranging the boughs of D_1, we can dispose them on the trunk in the order of the corresponding boughs in the diagram D_2. Now, each bough is a tree diagram with a naturaly distinguished trunk — the "handle" of the "dumbbell". So we can make a permutation of the new boughs with respect to a new trunk. Proceeding this way, we obtain a sequence of elementary transformations which convert D_1 into D_2.

3.2 Proof of the Generalized four-term relation

A usual four-term relation is determined by the choice of a chord and the choice of a point on the circle different from the endpoints of all chords. Given a share in a chord diagram and a fixed point p outside of it, take all the chords of the share one by one and write the usual four-term relations for these chords and point p. In the sum of all such equations, only four terms (those that give the generalized four-term relation) will survive, because every diagram with the variable endpoint of the additional chord inside the share occurs twice with opposite signs and thus cancels.

3.3 Proof of the Corollary.

Of the six chords diagrams



four different generalized four-term relations can be composed. They can be written as

$$\begin{cases} x - v + z &= 0 \\ y - w + z &= 0 \\ y - v + u &= 0 \\ x - w + u &= 0 \end{cases}$$

where in each equation a one-term relation is taken into account. These equations imply $x = y$.

3.4 Proof of Proposition 2

The proof proceeds by induction with respect to a specially chosen parameter which will be called *complexity*.

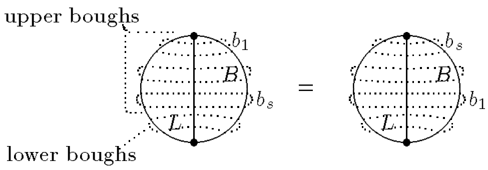
Let π be the permutation of boughs of a tree diagram D_1 which transforms it into D_2.

Definition. Numerate the boughs from top to bottom along the trunk: $b_1, \dots, b_k$. Let s be the minimum number such that π acts identically on $b_{s+1}, \dots, b_k$. The *complexity* $c(D_1, \pi)$ of the pair (D_1, π) is the total number of chords in the boughs $b_1, \dots, b_s$.

Induction base. If $c(D_1, \pi) = 2$ then $D_1 = D_2$

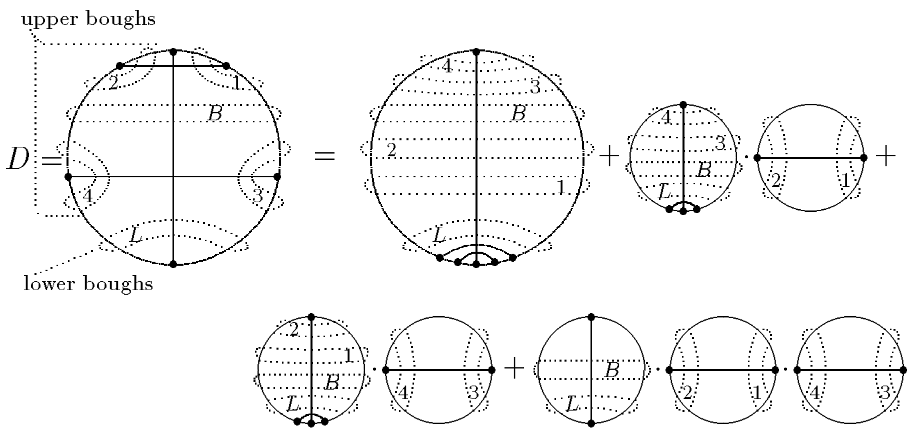
Induction hypothesis. Suppose that for $c(D_1, \pi) < m$ the statement of Proposition 2 holds.

Induction step. Let us prove the statement for $c(D_1,\pi)=m$. Since the permutation group is generated by transpositions of the first element with other elements of the underlying finite set, it is sufficient to prove the following fact:



where the total number of chords in the upper boughs is exactly m. Note that if the both shares of dumbbell b_1 are empty then this fact follows from the Corollary and induction hypothesis. In general case we need the following lemma.

Lemma 1. *Suppose that the statement of Proposition 2 holds for all pairs (D_1,π) with $c(D_1,\pi)<m$. Then*

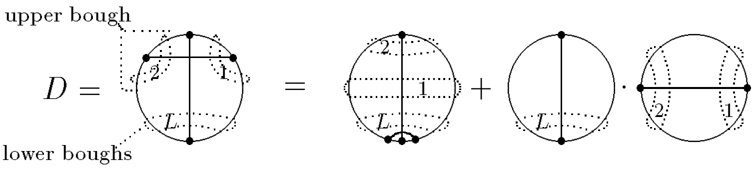


for a tree diagram D such that the total number of chords in the upper boughs is exactly m.

The number of chords in the upper boughs of the big diagram in the right hand side is $m-2$. By the induction hypothesis, the class of the diagram in $\mathcal{N}$ does not change under any permutation of its upper boughs. In particular, the right hand side stays invariant when the two pairs of shares, $(1,2)$ and $(3,4)$ are swapped. Proposition 2 thus follows from Lemma 1.

To prove Lemma 1, we need yet another lemma.

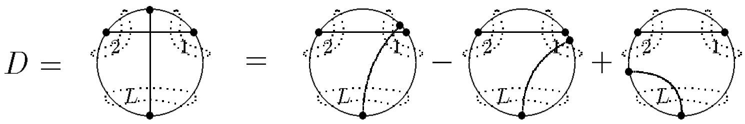
Lemma 2. *Suppose that the statement of Proposition 2 holds for all pairs* (D_1, π) *such that* $c(D_1, \pi) < m$. *Then*



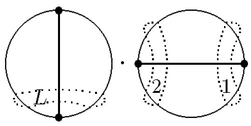
for any tree diagram D *with the number of chords in the upper boughs less than* m.

3.5 Proof of Lemma 2

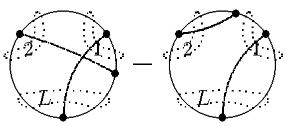
By the generalized four-term relation (see sec. 2)



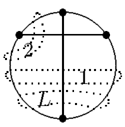
The last diagram is the product



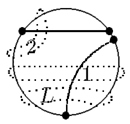
The difference of the first two diagrams on the right, by the generalized four-term relation, is equal to



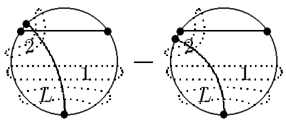
After its upper boughs are permuted, the first diagram becomes



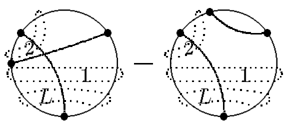
Since multiplication of diagrams in $\mathcal{N}$ is correctly defined, the second diagram is equal to



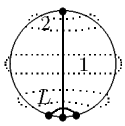
Therefore, by the usual four-term relation their difference equals



The generalized four-term relation applied to this difference yields



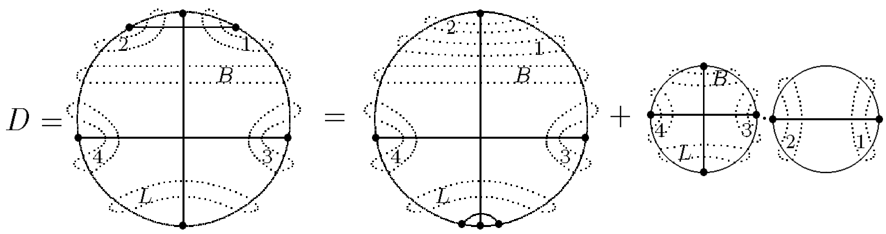
Here the last diagram vanishes by the one-term relation, while the first diagram is equal to



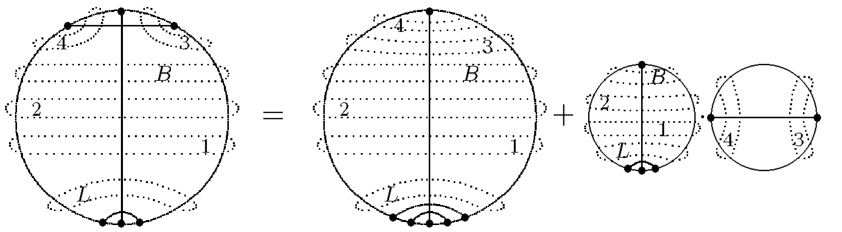
by Corollary of the generalized four-term relation (see sec. 2). This completes the proof of Lemma 2.

3.6 Proof of Lemma 1

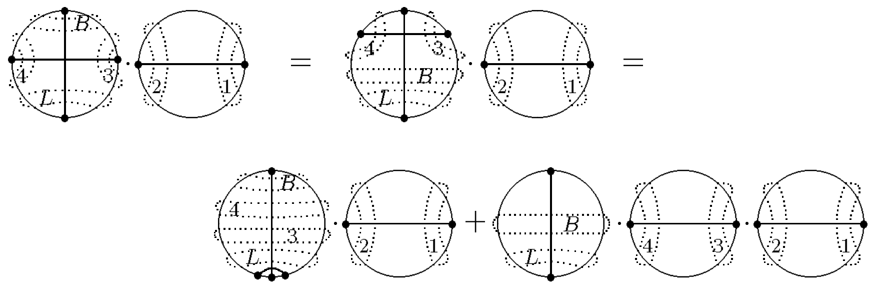
By Lemma 2



By the induction hypothesis and then by Lemma 2 the big diagram on the right is equal to

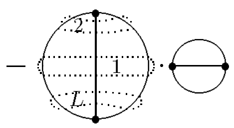


Applying the hypothesis and Lemma 2 once more, we obtain:



This completes the proof of Lemma 1, Proposition 2 and the Theorem.

Remark. Now that Proposition 2 is proved, another look at Lemmas 1 and 2 shows that their statements hold without any assumptions on the tree diagram. Moreover, it is easy to see that a more careful argument gives the analogues of the Theorem, Proposition 2 and so on in the algebra $\mathcal{M}$ (see [CDL1]) of diagrams considered modulo only four-term relations. In this case in the right hand part of the equality of Lemma 2 the following additional term appears:



Acknowledgements

In our first version of the proof of the Theorem we had only a topological proof of the Corollary from sec. 2, which did not use the four-term relation. The present proof of the Corollary (sec. 3.3) is due to Don Zagier who suggested it during the Jerusalem Combinatorics conference in May, 1993. We also thank A. Zvonkin, J. Birman and T. Stanford for useful discussions and interest in our work.

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