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MATHEMATICS

Proof of Przytycki's Conjecture on Matched Diagrams

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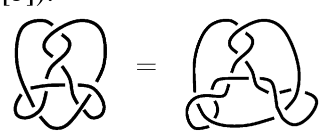
1. PROBLEM

A knot diagram is said to be matched if all crossings in this diagram can be divided into pairs of the form

Under any orientation of the knot, the branches in every such special pair have opposite directions. We say that the pair shown on the left in the figure is negative, and the pair on the right is positive.

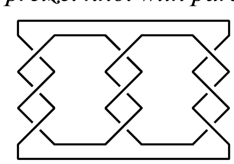
It is easy to understand that, e.g., all rational knots have such diagrams (it suffices to expand the corresponding rational number into a continued fraction with even denominators). Many other knots possess matched diagrams as well; they include the table knot 8_{15} (see [3]):



In 1987, the noted mathematician Jozef Przytycki [1] conjectured that there exist knots having no matched diagrams (this conjecture is also a part of Problem 1.60 in Kirby's well-known collection [2] of open problems in topology).

The conjecture remained open for 24 years in spite of many efforts of various mathematicians.

Theorem. *The pretzel knot with parameters* (3, 3, –3)



has no matched planar diagram.

The proof of this theorem is based on the construction of a special Seifert surface from a matched diagram and follows from two lemmas presented below.

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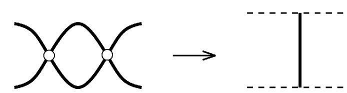
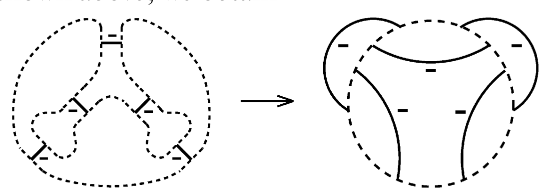
2. CONSTRUCTION OF THE SEIFERT SURFACE

Consider a matched diagram D of a knot K. Let us replace each pair of crossings by two parallel straight line segments having the same directions as the corresponding fragments of the knot and joined by a common perpendicular:



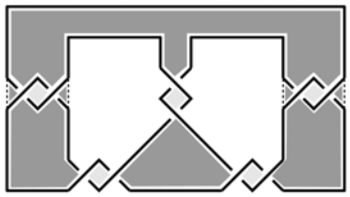
We connect the parallel segments by the remaining fragments of the diagram so as to obtain a simple closed curve. When this curve is developed into a circle, the perpendiculars mentioned above become disjoint chords located partly inside and partly outside the circle; thus, for the matched diagram of the knot 8_{15} shown above, we obtain



(The exterior and interior chords are interchanged, which is equivalent to the eversion of the planar diagram and does not affect the isotopic type of the knot.) The chords are endowed with signs corresponding to the signs of the corresponding special pairs.

It is easy to reconstruct the initial knot from such a signed diagram: it suffices to double the chords and build in a special pair of crossings of the corresponding sign at the midpoint of each chord.

From such a knot diagram (in which the special pairs are divided into interior and exterior) we construct a Seifert surface in the following way (rather than by the standard method involving chess coloring and Seifert disks): we paste the circle of the chord diagram by a disk, cut off narrow strips along the interior chords from this disk, and join the pieces thus obtained by a twice twisted ribbon of suitable orientation at each place where a strip was cut out. The external chords are processed in a different way: we replace each of them by a strip to which a perpendicular narrow ribbon, also twice twisted according to the sign, is attached somewhere in the middle. For the chord diagram of the knot 8_{15} given above, the result is as follows:



Here, the solid line is the given knot, and the dashed line shows the fragments of the visible contour of the Seifert surface not contained in its boundary. Different shades of gray correspond to the two sides of the surface.

3. THE SPECIAL ALEXANDER MATRIX

Lemma 1. *The Seifert matrix for the surface constructed above can be written in the form* $\begin{pmatrix} E & 0 \\ I & F \end{pmatrix}$, *where* I, 0, E, *and* F *are square matrices of the same size; moreover,* I *is the identity matrix,* 0 *is the zero matrix, and* E *and* F *are some symmetric integer matrices.*

Proof. To obtain such a matrix, it is sufficient to take the set of e_i and f_i for basis cycles, where e_i is the cycle traversing the ribbon attached to the exterior chord number i and f_i is the cycle going along this exterior chord and closing inside the inner disk.

Recall that one of the Alexander matrices of the knot under consideration can be constructed from the Seifert matrix S as $A = tS - S^T$, where T denotes transposition. Each knot has a whole class of equivalent Alexander matrices (the equivalence is determined by a certain set of elementary transformations; see, e.g., [4]); the determinant of an Alexander matrix (the Alexander polynomial) and the ideals of the ring $\mathbb{Z}[t, t^{-1}]$ generated by its minors of fixed order (Alexander ideals) are invariants of the knot determined up to multiplication by an invertible element of this ring, i.e., by $\pm t^k$. As is known, the Alexander polynomial can always be expressed in terms of the Conway variable $z^2 = t + \frac{1}{t} - 2$. For knots having a matched diagram, the following more precise assertion is valid.

Lemma 2. *Any knot possessing a matched diagram has an Alexander matrix in which all elements can be expressed in terms of the Conway variable; to be more precise, such a knot has an Alexander matrix of the form* $I + z^2B$, *where* I *is the identity matrix and* B *is an integer matrix.*

Proof. It is sufficient to take an Alexander matrix of the form $tS - S^T$, where S is the Seifert matrix described in Lemma 1, and perform a number of elementary transformations.

This lemma immediately implies that any Alexander ideal of a knot having a matched diagram has a system of generators being polynomials in z^2. As is known (see [4]), the second Alexander ideal of the pretzel knot with parameters (3, 3, −3) is generated in the ring $\mathbb{Z}[t, t^{-1}]$ by the elements 3 and $t + 1$. A simple argument proves that this ideal cannot be generated by polynomials in $z^2 = t + \frac{1}{t} - 2$, which proves the theorem.

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