Functional Analysis and Its Applications, Vol. 41, *No.* 3, *pp.* ??–??, 2007
Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 41, *No.* 3, *pp.* 48–59, 2007
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Detecting the Orientation of String Links by Finite Type Invariants*

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Received October 11, 2005

Abstract. We prove the existence of a degree 7 Vassiliev invariant of long (string) links with two numbered components which is not preserved under orientation reversal. The proof is based on the study of a weight system with values in the tensor square of the universal enveloping algebra for the Lie algebra $\mathfrak{gl}_N$.

Key words: link, knot, Vassiliev invariant, invertibility.

1. Introduction

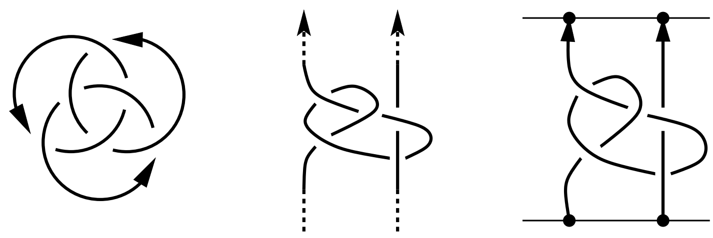
It is well known that classical knot polynomials (Jones polynomials, HOMFLY polynomials, etc.), as well as quantum knot invariants in general, take equal values on a knot and its inverse. The class of Vassiliev invariants is strictly wider [16], and the problem as to whether they can be used to tell a knot from its inverse is so far open. In the present paper, we discuss the counterpart of this problem for links with more than one component.

Let S^1_p be the disjoint union of p numbered copies of an oriented circle, $\mathbb{R}^1_p$ the disjoint union of p numbered copies of an oriented real line, and I^1_p the disjoint union of p numbered copies of an oriented interval $[0,1]$.

Definition. A *p-component closed link* is a smooth embedding of S^1_p in oriented 3-space $\mathbb{R}^3$, considered up to a component-preserving isotopy.

A *p-component long link* is a smooth embedding of R^1_p in oriented 3-space $\mathbb{R}^3$ with fixed asymptotics $x_i(t) = [i,0,t]$ for $|t| > C$ at infinity, considered up to isotopies identical outside sufficiently large balls and Euclidean motions of space.

A *p-component string link* is a smooth embedding of I^1_p in the strip $\mathbb{R}^2 \times [0,1] \subset \mathbb{R}^3$ with fixed endpoints $x_i(t) = [i,0,t]$, $t = 0,1$, considered up to isotopies identical at the boundary and Euclidean motions of the strip.



A closed link   A long link   A string link

It is immediately clear that the theories of long and string links are equivalent. In the case of knots ($p = 1$), they are also equivalent to the theory of closed links; this is not true for $p > 1$.

The *invertibility problem* is stated as follows. Given a link L, let L' be the inverse link, i.e., the same link with orientation of all components reversed. L is said to be invertible if L' is equivalent to L. Do noninvertible links exist? If yes, what invariants can be used to detect noninvertibility?

For knots, the problem had been open for a long time until Trotter [14] proved in 1964 that some knots (e.g., the $(3,5,7)$ pretzel knot) are noninvertible. The simplest noninvertible knot is 8_{17} [8]. The invariants used in [14], [8], etc. when proving the nonequivalence of a knot to the inverse

*Supported by grants RFBF 05-01–1-12, RFBR 05-01-00899, NSh 1972.2003.1, NCNIL 06-01-72551.

knot are rather complicated, and it is unknown yet if some pair of mutually inverse knots can be distinguished by finite type invariants.

In the case of links, the only published result is a theorem of Lin [11] claiming that Vassiliev invariants distinguish closed links with at least 6 components from their inverses. There are also several papers having only indirect relation to this problem. For example, Bar-Natan [2] studies *homotopy* invariants of string links, while Lin [10] and Fiedler [7] use classes of invariants different from classical finite type invariants.

The present text deals with the problem of detecting the orientation by finite type invariants for *string* (or *long*) *links*. The restatement of the problem in terms of chord diagrams readily shows that this is possible for $p > 2$. In the case of 2-component links ($p = 2$), the problem is nontrivial; below we give a proof of the following theorem.

Theorem. *There exists a Vassiliev invariant f of degree $\leqslant 7$ and a 2-component string link L such that $f(L') \neq f(L)$.*

We give two proofs of this theorem, both of which are based on straightforward computations. The first proof (Proposition 1) uses chord diagrams and requires computer calculations. The second one (Proposition 2) uses Jacobi diagrams and can be done by hand. Frankly speaking, both proofs refer to invariants of *framed* links. In Sec. 6 we explain why they also imply the same result in the unframed case.

The invertibility problem for p-component string links is closely related to the question as to whether the algebra $\mathscr{A}(p)$ of chord diagrams on p strings is commutative. Namely, the noncommutativity of $\mathscr{A}(p)$ implies the existence of elements that are not invariant under orientation reversal. Indeed, the operation τ of orientation reversal (see below) is an antiautomorphism. If all elements of $\mathscr{A}(p)$ were symmetric, then we would have $xy = \tau(xy) = \tau(y)\tau(x) = yx$ for all $x, y \in \mathscr{A}(p)$. (We thank the referee, who has drawn our attention to this argument.)

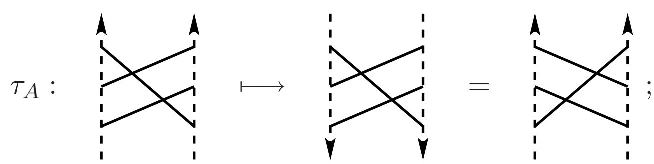
The noncommutativity of the algebra $\mathscr{A}(p)$ is obvious for $p > 2$ and folklore for $p = 2$, although it was never proved for the latter case in any published text. Proposition 1 of the present paper in particular contains the first rigorous proof of this fact.

2. Reduction to Chord Diagrams

Finite type invariants for various types of links are defined in the same way as in the classical case of (closed) knots; see [1], [6].

Let $\mathbb{F}$ be a field of characteristic 0, e.g., $\mathbb{Q}$ or $\mathbb{C}$. Denote the vector space of $\mathbb{F}$-valued Vassiliev invariants for p-component long links of degree no greater than n by $V_n(p)$, the space spanned by chord diagrams of degree n on p strings modulo 4-term relations ([6], [13]) by $\mathscr{A}_n(p)$, and the corresponding space of *weight systems* by $W_n(p) = \mathscr{A}_n^*(p) = \mathrm{Hom}_{\mathbb{Q}}(\mathscr{A}_n(p), \mathbb{F})$. Then there exists a linear map $\sigma_n^p \colon V_n(p) \to W_n(p)$ (taking the *symbol* of a Vassiliev invariant) whose kernel coincides with $V_{n-1}(p)$ and whose image consists of all weight systems that vanish on chord diagrams with isolated chords. (For *framed* links, the image is equal to the entire $W_n(p)$.) Denote the direct sum of vector spaces $\mathscr{A}_n(p)$ for all $n \geqslant 0$ by $\mathscr{A}(p)$; this is a graded algebra with multiplication defined as concatenation of chord diagrams.

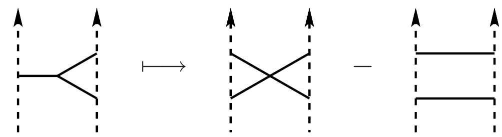
As before, we denote the inverse of a link L by L'. By setting $\tau_V(f)(L) = f(L')$ for $f \in V$, we obtain an involution τ_V on the space of Vassiliev invariants. In terms of chord diagrams, the corresponding operation τ_A acts as follows:



i.e. it changes the orientation of all the strings of a chord diagram, or, which is the same, reflects the plane picture of the diagram in a horizontal line, assuming that the support is drawn vertically (and preserves its orientation).

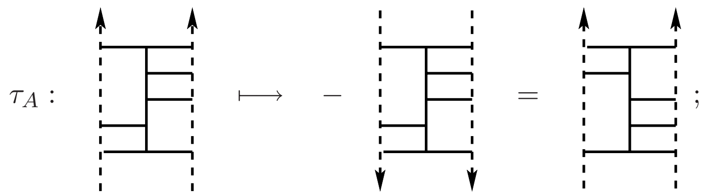
The same space $\mathscr{A}(p)$ can be spanned by all generalized chord diagrams, i.e. 1-3-valent graphs whose univalent vertices are identified with points on the manifold $\mathbb{R}^1_p$, with a cyclic order of edges specified at every 3-valent vertex.

Generalized chord diagrams can be treated as linear combinations of ordinary chord diagrams. These linear combinations are obtained by iterative application of the STU relations of the following form (see [1] for more detail):



(Here and below, we use the *blackboard convention*: edges around trivalent vertices for every diagram drawn on paper are always ordered counterclockwise.)

For generalized diagrams, one should be more cautious with the definition of the involution τ_A. In fact, the extension of τ_A to generalized chord diagrams by STU relations results in the following rule:



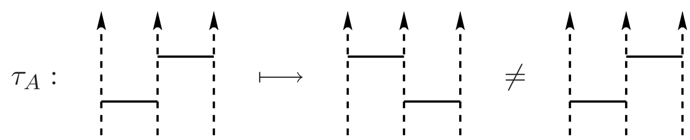
i.e., one should either change the orientation of all strings *and multiply the result by* ± 1 depending on the parity of the number of trivalent vertices or, which is the same, simply reflect the plane picture of the diagram in a horizontal line, preserving the orientations.

Lemma 1. *The involution τ_A is the graded counterpart of the involution τ_V; i.e., the following square commutes*:

$$\begin{array}{ccc} V_n(p) & \xrightarrow{\sigma_n^p} & W_n(p) \\ \big\downarrow{\scriptstyle \tau_V} & & \big\downarrow{\scriptstyle \tau_A^*} \\ V_n(p) & \xrightarrow{\sigma_n^p} & W_n(p) \end{array}$$

Proof. For usual chord diagrams, this readily follows from the definition of σ ([1], [6]). The extension to generalized chord diagrams is carried out as was explained above. □

By using finite type invariants, one can reduce the problem of the existence of noninvertible string links to the following problem: *does there exist a chord diagram on p strings nonequivalent to its inverse modulo 4-terms relations*? If $p \geqslant 3$, then there is a straightforward example:



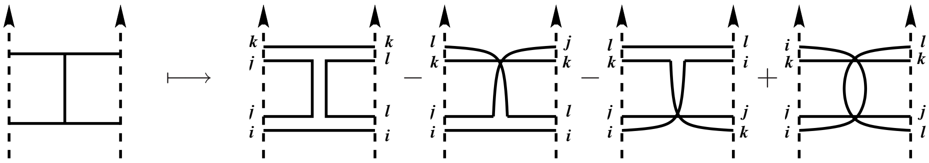
For $n = 2$, the problem is nontrivial, because, as one can check, chord diagrams of small degrees (e.g., the diagrams in the first map in this section) are all τ_A-invariant. In the next section, we indicate a diagram that is not τ_A-invariant.

3. First Proof of the Theorem

To prove that some element of the space $\mathscr{A}(p)$ is nonzero, one can use *weight systems*, i.e., linear functionals on these spaces. A powerful weight system is provided by Kontsevich's homomorphism $\varphi = \varphi_{\mathfrak{g}} \colon \mathscr{A}(p) \to U(\mathfrak{g})^{\otimes p}$ for a metrized Lie algebra $\mathfrak{g}$ (see [9], [6]). In fact, φ ranges in the $\mathfrak{g}$-invariant subalgebra $U(p) = [U(\mathfrak{g})^{\otimes p}]^{\mathfrak{g}}$. We give an explicit description for the case $\mathfrak{g} = \mathfrak{gl}_N$, using the basis e_{ij} (matrix with 1 at position (i,j) and 0 elsewhere) and the metric defined by the conjugation rule $e_{ij}^* = e_{ji}$.

Lemma 2. *Let $D \in \mathscr{A}(p)$ be a (generalized) chord diagram on p strings. The element $\varphi(D) \in U(\mathfrak{gl}_N)^{\otimes p}$ can be obtained as follows. Consider the alternating sum of all resolutions* [1] *of the inner triple points of the diagram D. For each resolution, label the connected components of the resulting diagram by distinct independent indices; then replace each pair of adjacent indices by e_{ij} and take the sum over all indices from 1 to N. (Closed components, if any, turn into multiplication by N.)*

The proof reproduces that for the special case $p = 1$, which can be found in [1] and [6]. We only give an example:



Therefore, the image of this diagram under φ is

$$\sum_{i,j,k,l=1}^{N} (e_{ij}e_{jk} \otimes e_{li}e_{kl} - e_{ij}e_{kl} \otimes e_{li}e_{jk} - e_{ij}e_{kl} \otimes e_{jk}e_{li} + e_{ij}e_{ki} \otimes e_{jl}e_{lk}).$$

Note that the order of the factors e_{ij} agrees with the orientation of each component of the support, but the order of the two subscripts (i,j) in each factor follows a go-round pattern, in our case bottom-to-top for the left line and top-to-bottom for the right line.

Let τ_U be the operator in $U(\mathfrak{gl}_N)^{\otimes p}$ that rewrites each monomial backwards and transposes the subscripts on each generator e_{ij}; e.g., $\tau_U(e_{12}e_{23} \otimes e_{13}e_{24}) = e_{32}e_{21} \otimes e_{42}e_{31}$. This is an involution that preserves the ad-invariant subspace $U(p) = [U(\mathfrak{gl}_N)^{\otimes p}]^{\mathfrak{g}}$.

The construction of φ shows that the following assertion holds.

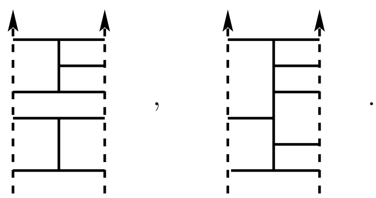
Lemma 3. *The square*

$$\begin{array}{ccc} \mathscr{A}(p) & \xrightarrow{\ \varphi\ } & U(p) \\ \downarrow{\scriptstyle \tau_A} & & \downarrow{\scriptstyle \tau_U} \\ \mathscr{A}(p) & \xrightarrow{\ \varphi\ } & U(p) \end{array}$$

commutes.

It follows that the noninvertibility of a chord diagram can be checked on the level of the universal enveloping algebra: if the φ-image of a diagram is not τ_U-invariant, then the diagram itself is not invariant under τ_A. Therefore, the following proposition gives the first proof of the theorem stated in Sec. 1.

Proposition 1. *Each of the following diagrams is not equivalent to its image under the involution τ_A:*



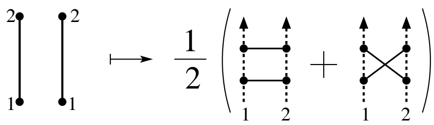
Proof. A computer calculation of the φ-images of each diagram and its inverse shows that they are distinct for the Lie algebra $\mathfrak{gl}_4$ (and hence for any $\mathfrak{gl}_N$ with $n \geqslant 4$). The calculations are organized as follows: we fix a lexicographical ordering of the basic elements e_{ij} and transform the expression for $\varphi(D)$ obtained by the above algorithm using the commutator relations between the generators in order to rewrite each monomial lexicographically. The programs, as well as input and output files, are available at [5]. For example, the result for the left diagram is a polynomial consisting of 58378 terms; its calculation requires several hours on a reasonably fast modern PC. Since this diagram is the product of two τ-symmetric diagrams, we have also proved the noncommutativity of the algebra $\mathscr{A}(2)$. $\square$

4. Reduction to Jacobi Diagrams

Another, yet simpler restatement of the invertibility problem can be given in terms of *colored Jacobi diagrams*. A colored Jacobi diagram is the same thing as a *Chinese character* as defined in [1] except that its univalent vertices are labeled by p colors. The *space* of colored Jacobi diagrams $\mathscr{B}(p)$ is defined as the vector space formally generated by all p-colored Jacobi diagrams modulo two sets of relations, antisymmetry and IHX (see [1], [6]).

By analogy with Theorem 8 in [1], one can prove that the symmetrization map $\chi\colon \mathscr{B}(p) \to \mathscr{A}(p)$ is a linear isomorphism between the two spaces (see also [12]).

The map χ is defined as follows—we explain that in the case $p = 2$. Let D be a Jacobi diagram with k "legs" of color 1 and l "legs" of color 2; then $\chi(D)$ is the average of all the $k!l!$ ways to attach 1-colored legs to the first support line and 2-colored legs to the second support line; e.g.,



Colored Jacobi diagrams can be viewed as symmetric elements of the space $\mathscr{A}(p)$, much in the same way as commutative polynomials over a Lie algebra $\mathfrak{g}$ can be identified with symmetric elements of the universal enveloping algebra $U(\mathfrak{g})$ by the PBW theorem.

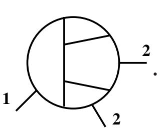
The isomorphism χ is very important for our needs, because the involution τ_A takes an especially simple form when transferred to $\mathscr{B}(p)$ via this isomorphism. In fact, the following assertion holds.

Lemma 4. *Let $\tau_B\colon \mathscr{B}(p) \to \mathscr{B}(p)$ be the linear operator defined as identity on each Jacobi diagram with even number of legs and as multiplication by -1 on each Jacobi diagram with odd number of legs. Then the square*

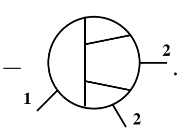
$$\begin{array}{ccc} \mathscr{B}(p) & \xrightarrow{\ \chi\ } & \mathscr{A}(p) \\ \downarrow{\scriptstyle \tau_B} & & \downarrow{\scriptstyle \tau_A} \\ \mathscr{B}(p) & \xrightarrow{\ \chi\ } & \mathscr{A}(p) \end{array}$$

commutes.

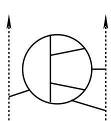
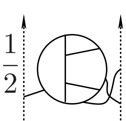
Proof. The proof obviously follows from the definitions of τ_A and τ_B given above. A simple illustration might be helpful to understand how it goes. Let

$D =$ .

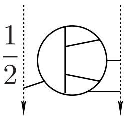
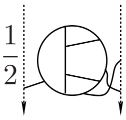
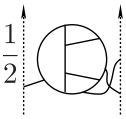
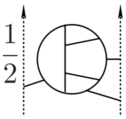
Then, by definition,

$\tau_B(D) = -$ .

Now

$\chi(D) \ = \ \frac{1}{2}$  $+ \ \frac{1}{2}$ 

and hence

$$\tau_A(\chi(D)) \;=\; -\frac{1}{2}$$  $$-\frac{1}{2}$$  $$=\; -\frac{1}{2}$$  $$-\frac{1}{2}$$  $$=\; \chi(\tau_B(D)). \qquad \square$$

The invertibility problem for 2-component string links can now be restated as follows: *Is there a nonzero* 2*-colored Jacobi diagram with odd number of legs*? In the next section, we give an example of such a diagram.

5. Second Proof of the Theorem

Take a metrized Lie algebra $\mathfrak{g}$ and denote the symmetric algebra of the vector space $\mathfrak{g}$ by $S(\mathfrak{g})$. The weight system $\mathscr{B}(1) \to S(\mathfrak{g})$ described in [1], [6] has an immediate generalization to an arbitrary value of p giving a homomorphism $\psi\colon \mathscr{B}(p) \to S(\mathfrak{g})^{\otimes p}$ whose image lies in the $\mathfrak{g}$-invariant subalgebra $S(p) = [S(\mathfrak{g})^{\otimes p}]^{\mathfrak{g}}$.

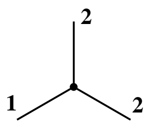
Lemma 5. *The mapping* ψ *fits into the commutative diagram*

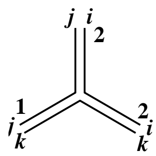
$$\begin{array}{ccc} \mathscr{B}(p) & \xrightarrow{\ \psi\ } & S(p) \\ \big\downarrow \chi & & \big\downarrow \pi \\ \mathscr{A}(p) & \xrightarrow{\ \varphi\ } & U(p) \end{array}$$

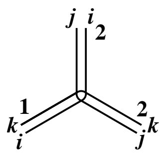
where χ *is the isomorphism defined in the previous section,* φ *is the Kontsevich weight system for the algebra* $\mathscr{A}(p)$*, and* π *is the Poincaré–Birkhoff–Witt isomorphism raised to the* p*th tensor power.*

Proof. The proof reproduces that for the case $p = 1$ word for word. $\square$

For the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N$, there exists a pictorial algorithm for the calculation of ψ, similar to the procedure for φ described in Sec. 3. The only difference is that now the basic elements e_{ij} commute, so we do not have to bother about the order of factors in the monomials. Here is a simple example:

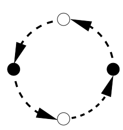
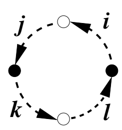
ψ : 

$\longmapsto$ 

$-$ 

$$\longmapsto \sum_{i,j,k=1}^{N} (e_{jk} \otimes e_{ij}e_{ki} - e_{ki} \otimes e_{ij}e_{jk}) \;=\; 0.$$

An advantage of ψ is that the $\mathfrak{g}$-invariant part of $S(\mathfrak{gl}_N)^{\otimes p}$ has a more transparent structure than $U(\mathfrak{gl}_N)^{\otimes p}$. A p-colored *necklace* of degree n is a combinatorial object defined as a sequence of n numbers between 1 and p considered up to cyclic permutations; it can be best viewed as a circular arrangement of n p-colored beads. To every necklace one can assign an ad-invariant element of $S(\mathfrak{gl}_N)^{\otimes p}$ as follows. Assign distinct variable indices (i, j, etc.) to all arcs of the necklace; every bead transforms into e_{ij} where i is the index on the incoming arc and j is the index on the outgoing arc; put this element e_{ij} into the tensor factor whose number is the color of the bead and take the sum over all indices from 1 to N. For example,

$[1212] \;=$  $\longmapsto$ 

$$\longmapsto \sum_{i,j,k,l=1}^{N} (e_{ij}e_{kl} \otimes e_{jk}e_{li}) \;=:\; x_{1212}\,.$$

We denote such *necklace elements* of $S(\mathfrak{gl}_N)^{\otimes p}$ by x_μ, where μ is the lexicographically smallest color sequence corresponding to the given necklace.

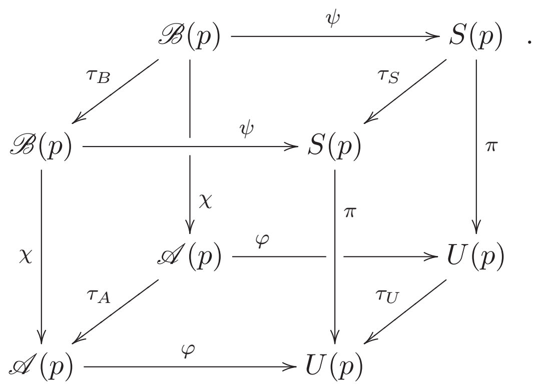
Lemma 6. *For the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N$, the $\mathfrak{g}$-invariant part of $S(\mathfrak{g})^{\otimes p}$ coincides with algebra generated by all necklace elements. Algebraic relations between necklace elements of a given degree may exist for small N but disappear as $N \to \infty$.*

Proof. Statements close to this lemma are proved in [17] and [15]. Our assertion is an easy consequence. □

Remark. A weight system ranging in the necklace algebra can be defined directly, without resorting to Lie algebras. This will be discussed elsewhere.

In the algebra generated by necklaces, there is an involution τ_S that reverses the orientation of each necklace. For $p = 2$, in small degrees, up to 5, it is identical; the minimal necklace that is not invariant under τ_S is x_{112122}. It turns out that the operation τ_S agrees with all other inversions denoted by τ with various subscripts in this paper.

Lemma 7. *The orientation-reversing involutions in the spaces $\mathscr{A}(p)$, $\mathscr{B}(p)$, $S(p)$, $U(p)$ commute with the four arrows of the diagram in Lemma 5. More precisely, one has the commutative cube*



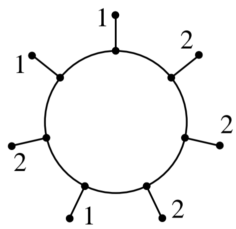
(*Recall that $\mathfrak{g} = \mathfrak{gl}_N$, $S(p) = [S(\mathfrak{g})^{\otimes p}]^{\mathfrak{g}}$, and $U(p) = [U(\mathfrak{g})^{\otimes p}]^{\mathfrak{g}}$. All maps were defined in the text above; in particular, π is the Poincaré–Birkhoff–Witt isomorphism.*)

Proof. The only thing not proved yet is the commutativity of the top face of the cube. Since all vertical arrows are isomorphisms, this fact follows from the commutativity of the remaining five faces, which has been proved earlier at various places of this paper. □

The commutativity of the upper face of the cube implies that to prove the non-τ_B-invariance of a Jacobi diagram (an element of $\mathscr{B}(p)$) it suffices to prove the non-τ_S-invariance of its image under ψ in the necklace algebra. Since the minimal degree of a noninvertible necklace is 6 and we need to find a noninvertible diagram of an odd degree, the smallest example should be sought for in degree 7. And indeed it exists, thus providing a second proof of the Theorem.

Proposition 2. *The following diagram is nonzero as an element of the space $B(2)$:*

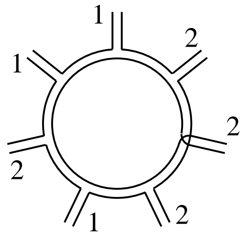
H =



Proof. A straightforward computation shows that the image of this diagram in the algebra $S(2)$ (for sufficiently large N), when expressed via necklaces, is equal to

$$N(x_{1121222} - x_{1122212}) + 3x_2(x_{112212} - x_{112122}),$$

which is different from zero. The entire expression for $\psi(H)$ involves 128 terms (corresponding to the 2^7 resolutions of trivalent vertices), of which only 8 contain nonsymmetric necklaces, e.g.



$$\longmapsto \quad -x_2 x_{112122}.$$

The same result can also be obtained by a computer program available at [5]. □

Remark 1. The diagram H was first discovered by Bar-Natan [3]: it can be found in the file `table.m` posted on his web site and containing bases of various spaces of colored diagrams. However, the preprint [3], which contains comments on this table, indicates that the program used to obtain the table is not ready for publication owing to its awkwardness and some drawbacks in the algorithm.

Remark 2. Bar-Natan's table also shows that 7 is the smallest degree of a Vassiliev invariant that can detect the orientation of two-component string links. The computations required to check this fact can easily be done by hand.

6. Deframing

As was mentioned in the introduction, the proofs given in Secs. 3 and 5 actually refer to the case of *framed* links, because 1-term relations were not taken into consideration. Generally speaking, detecting the orientation of a framed link is easier, because it contains an additional structure, which, in principle, may not be preserved by the inversion. In this section, we prove that the main theorem remains valid for conventional (unframed) links.

Indeed, let $\mathscr{A}'(2)$ be the quotient of the space $\mathscr{A}(2)$ modulo 1-term relations, i.e., by the ideal generated by the diagrams a_1 (with 1 chord on the first component of the support) and a_2 (with 1 chord on the second component of the support): $\mathscr{A}'(2) = \mathscr{A}(2)/\langle a_1, a_2\rangle$, where angle brackets denote the 2-sided ideal with given generators. The quotient algebra $\mathscr{A}'(2)$ can also be considered as a subalgebra of $\mathscr{A}(2)$: by the structure theorem for cocommutative Hopf algebras, $\mathscr{A}(2)$ is the
Q1 universal enveloping algebra of the Lie algebra P of its primitive elements P, so if we take the subspace of P spanned by all connected diagrams except for a_1 and a_2, we obtain an embedding $\mathscr{A}'(2) \subset \mathscr{A}(2)$. Since the diagrams displayed in Proposition 1 belong to $\mathscr{A}'(2)$, this means that noninvertibility holds also for unframed string links.

Now let us show that the second proof (Sec. 5) is valid in the unframed case as well. Let $\chi\colon \mathscr{A}(2) \to \mathscr{B}(2)$ be the isomorphism defined above.

Lemma 8. *The subspace $\chi^{-1}(\mathscr{A}'(2)) = \mathscr{B}'(2)$ coincides with the subalgebra of $\mathscr{B}(2)$ generated by all connected Jacobi diagrams except for $b_1 = \overset{1\ \ \ 1}{\bullet\!\!-\!\!-\!\!\bullet}$ and $b_2 = \overset{2\ \ \ 2}{\bullet\!\!-\!\!-\!\!\bullet}$.*

Proof. The assertion is nontrivial, because χ does not preserve the multiplication and the subalgebras $\mathscr{A}'(2)$ and $\mathscr{B}'(2)$ are described by their generators in the sense of different multiplications. However, it is easily seen by a straightforward argument that the image of a Jacobi diagram different from b_1 and b_2 under χ is a linear combination of products of connected chord diagrams different from a_1 and a_2. □

The lemma shows that the seven-leg diagram H occurring in Proposition 2 belongs to the subalgebra $\mathscr{B}'(2)$ responsible for the Vassiliev invariants of unframed 2-component long links. Since it is nonzero, the existence of a degree 7 invariant that can distinguish the orientation again follows.

7. Open Problems

Question 1. Is there a homomorphism $\mathscr{B}(1) \to \mathscr{B}(2)$ whose image is not contained in the τ-invariant subspace of $\mathscr{B}(2)$?

Comment. Should such a mapping exist, one could try to apply the technique of this paper to the invertibility problem for *knots*. Unfortunately, it is easily seen that the standard doubling operator $\Delta\colon \mathscr{B}(1) \to \mathscr{B}(2)$ defined in [1] and [12] is no good to this end.

Question 2. Do Vassiliev invariants detect invertibility of closed 2-component links?

Comment. The combinatorial object responsible for finite type invariants of closed links is the space $\mathscr{A}(S^1_2)$ of chord diagrams on two circles. There is an obvious epimorphism $\mathscr{A}(\mathbb{R}^1_2) \to \mathscr{A}(S^1_2)$, whose kernel is spanned by the so called link relations (see [4]). To solve the problem, one should study the interplay between link relations and orientation reversal.

Acknowledgements

We are obliged A. Khoroshkin, S. Loktev and E. B. Vinberg for useful comments on invariant theory. We thank S. Lando and the referee, who read the text and made a number of useful remarks.

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Вопросы к авторам

Q1. Так правильно?