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Bipartite knots

by

Sergei Duzhin (St. Petersburg) and **Mikhail Shkolnikov** (Genève)

Abstract. We give a solution to a part of Problem 1.60 in Kirby's list of open problems in topology, thus answering in the positive the question raised in 1987 by J. Przytycki.

1. Problem. We will call *bipartite* a knot that can be represented by a *matched* diagram, that is, a diagram whose crossings are split in pairs of the types depicted in Fig. 1. The pairs in the upper line are said to be *positive*, those in the lower line, *negative*. Note the signs of the pairs do not change when the orientation on the knot is reversed. Note, moreover, that if the crossings of an unoriented knot are split into matched unoriented pairs, then, introducing any orientation, we always get counter-directed pairs shown in Fig. 1.

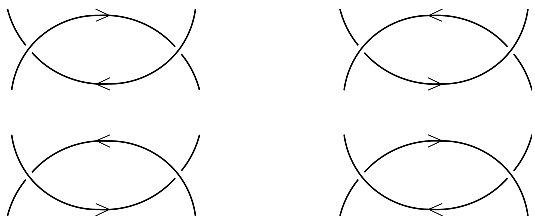

Fig. 1. Matched pairs

Examples. 1. Any rational knot has a matched diagram, because any rational number can be represented by a continued fraction with even (positive or negative) denominators (see the proof of Corollary 6 in [Prz], or [DS]).

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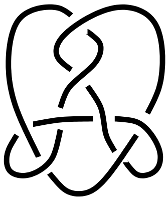
2. The standard diagram of the knot 8_{15} ([1]) (which is not rational)



can be easily transformed to a matched form:

3. We managed to find matched diagrams for all table knots with up to 8 crossings, save for the knot 8_{18}.

The problem raised by Józef Przytycki in 1987 is to investigate which knots have a matched diagram. This question appears in the well-known collection "Open problems in topology" maintained by Rob Kirby [Kir], as part of Problem 1.60. More exactly, Conjecture 1(a) therein (belonging to Przytycki [PP]) reads: "There are oriented knots without a matched diagram". As the reader understands, the word "oriented" can be here omitted without any loss of meaning. This conjecture stayed open for 24 years, notwithstanding the effort of several excellent mathematicians, including its author and J. H. Conway [APR]. We give a positive solution to the conjecture, that is, demonstrate that some knots, e.g. pretzel knot with parameters $(3, 3, -3)$, are not bipartite. In the next section, we introduce our main construction, which also explains the meaning of the word "bipartite" in this context.

2. Chord diagram of a bipartite knot. Consider a matched diagram of a knot K. Replace every matched pair of crossings by two parallel segments, directed as the knot and joined by a common perpendicular; see Fig. 2.

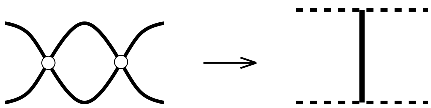


Fig. 2. Local transformation of a matched diagram

([1]) Here and below, we use Rolfsen's numbering of knots, which differs from that adopted in Knotscape [HT].

The parallel segments are then joined by the remaining fragments of the knot diagram into a simple closed line on the plane, straightenable into a circle, whereas the common perpendiculars become *chords*. An example for the matched diagram of the knot 8_{15} mentioned above is given in Fig. 3, where the mutual position of the inner and outer chords is changed: this leads to turning the knot diagram inside out with respect to some point and does not alter the isotopy type of the knot.

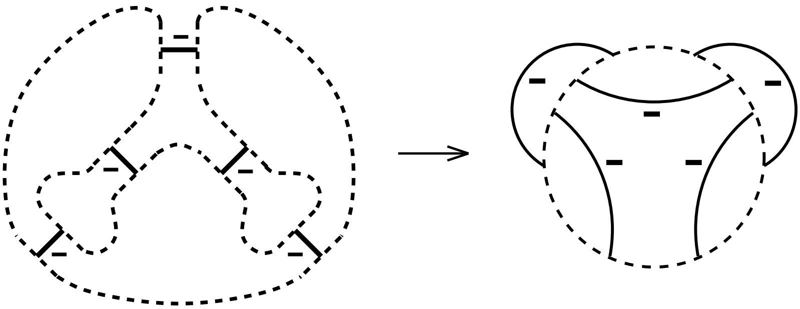


Fig. 3. Matched diagram → chord diagram

The chord diagrams obtained in this way are rather special: the set of all chords is split into two parts (inner chords and outer chords), so that the chords in each part do not intersect one another, and the intersection graph [CDM] of the whole diagram is bipartite.

This procedure is reversible: from a bipartite signed chord diagram one can reconstruct the knot diagram in a unique way (see Fig. 4).

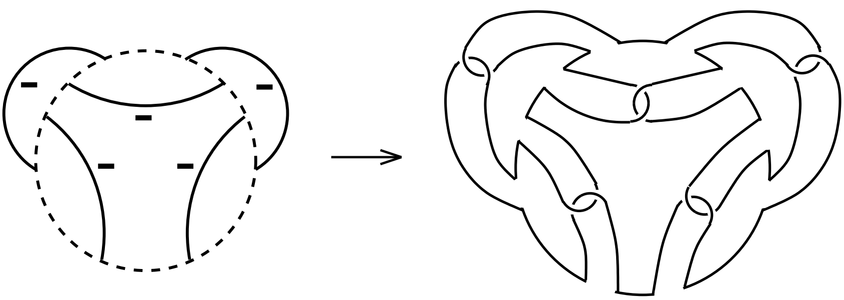


Fig. 4. Chord diagram → matched diagram

3. Seifert surfaces. A *Seifert surface* S of a knot is a compact oriented surface embedded in $\mathbb{R}^3$ so that its boundary is the given knot. Choosing a basis in $H_1(S)$, one can construct a matrix of the bilinear form $\mathrm{lk} \circ (\mathrm{id}, \alpha)$, where lk is the linking number, and α is a small shift in the positive direction along the normal of S. This matrix is called a *Seifert matrix* of the given knot.

There is a standard procedure to construct a Seifert surface from any diagram, using Seifert circles. For matched diagrams, there exists a different construction, which is crucial for our needs: it yields a Seifert matrix of a special type, which, in turn, produces an Alexander matrix with extraordinary properties.

LEMMA 1. *Any bipartite knot has a Seifert surface such that its Seifert matrix has the form* $\left(\begin{smallmatrix} E & 0 \\ I & F \end{smallmatrix}\right)$, *where* I, 0, E, F *are square matrices of the same size,* I *is the unit matrix,* 0 *is the zero matrix, and* E *and* F *are both symmetric integer matrices.*

Proof. Consider a bipartite knot K and its plane diagram, construct the chord diagram as indicated above. Start constructing the Seifert surface from the inner circle of the chord diagram, out of which we cut out every chord together with a small open neighborhood and glue instead a double twisted band, so that the direction of the twists corresponds to the sign of the chord (see Fig. 5).

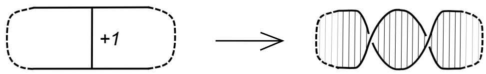


Fig. 5. Construction of a Seifert surface: inner chords

So far the surface remains orientable, and its boundary follows the knot as much as it can. Now we must add the bands along the outer chords. Here one must be cautious, because simply connecting the ends of the two half-chords by two half-twisted bands results in an unorientable surface. We will proceed as follows: first we attach a band along each outer chord, then, around the middle of that band, we attach a perpendicular small band which is twice twisted according to the sign of the chord.

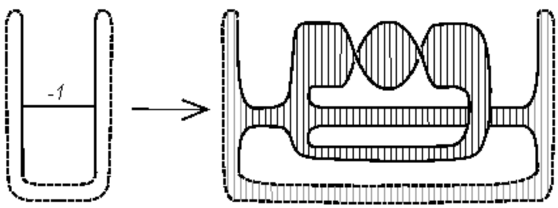


Fig. 6. Construction of a Seifert surface: outer chords

We show this procedure in Fig. 6 for one chord on big scale and in Fig. 7 for the whole Seifert surface of a certain knot. In the latter picture, the narrow twice-twisted bands on the left and on the right should be thought of as lying above the surface of the corresponding perpendicular wide bands. The thick solid lines indicate the boundary of the Seifert surface; the four small dashed segments show parts of the visible contour of the surface which do not belong to its boundary. The two sides of the Seifert surface (which is two-sided by definition) are indicated by different shades of gray.

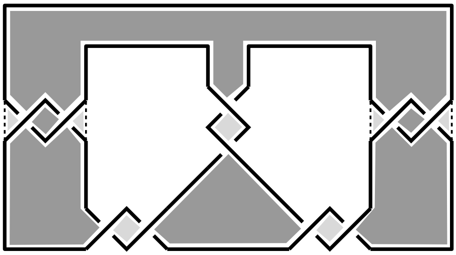


Fig. 7. Seifert surface for a matched diagram of the knot 8_{15}

Let n be the number of outer chords (if this number is greater than the number of inner chords, we can turn the chord diagram inside out to simplify computations). Then, as a basis of $H_1(S)$, one can take the set $e_1, \dots, e_n, f_1, \dots, f_n$ corresponding to the outer chords and shown in Fig. 8.

Fig. 8. Cycles e_k (left) and f_k (right)

It follows that the Seifert matrix for this Seifert surface (see [Lik]) has the form

$$M = \begin{pmatrix} E & 0 \\ I & F \end{pmatrix},$$

where 0, I, E, F are matrices of size $n \times n$, I is the unit matrix, 0 is the zero matrix, and $F_{i,j} = \operatorname{lk}(f_i, f_j^+)$, $E_{i,j} = \operatorname{lk}(e_i, e_j^+)$. It is clear that E is a diagonal matrix with numbers ± 1 on the diagonal (the sign is inverse to the sign of the outer chord number k). The cycles f_i and f_j can be chosen not to have common points if $i \neq j$, therefore $\operatorname{lk}(f_i, f_j^+) = \operatorname{lk}(f_i, f_j) = \operatorname{lk}(f_j, f_i) = \operatorname{lk}(f_j, f_i^+)$ and thus the matrix F is symmetric.

This construction shows that, on a practical side, it is advisable to turn the diagram inside out if the number of outer chords is greater than that of inner chords—as we did before for the example knot 8_{15}. ∎

4. Alexander matrices. The determinant of the Alexander matrix $A = tM - M^\top$ is equal to the Alexander polynomial of the knot K; it is an element of the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$, determined up to multiplication by invertible elements of the ring, that is, monomials $\pm t^m$. The Alexander matrix is not determined by the knot uniquely; in fact, to any knot there corresponds a big family of Alexander matrices related to one another by a set of equivalence transformations which is well known (see [Lik]). In particular, even the size of the matrix A is not invariant. What is invariant, however, is the sequence of Alexander ideals, the mth ideal being defined as the ideal in $\mathbb{Z}[t, t^{-1}]$ generated by all minors of an arbitrary Alexander matrix of size $n-m+1$, where n is the smallest among the number of columns and rows in A (see [Lik]).

It is well known that the Alexander polynomial can be rewritten in terms of the Conway variable $z^2 = t + t^{-1} - 2$; in general, this is no longer true about the generators of all Alexander ideals. In the case of bipartite knots, we can prove a stronger assertion.

LEMMA 2. *If the knot K is bipartite, then there exists a square integer matrix B such that $I + z^2B$ is an Alexander matrix for K (here I is the unit matrix).*

Proof. Consider the Seifert matrix M from Lemma 1. Put $A = tM - M^\top$, multiply the left block column by t^{-1}, the second by -1, then interchange both columns. Using the symmetry of E and F, we get

$$A = tM - M^\top \sim \begin{pmatrix} (t-1)E & -I \\ tI & (t-1)F \end{pmatrix} \sim \begin{pmatrix} I & (1-t^{-1})E \\ (1-t)F & I \end{pmatrix}.$$

By a sequence of elementary transformations, we can make zero the upper right block of this matrix, using its lower right block: In doing so, we will be always adding polynomials $(1-t)a(1-t^{-1})b = -z^2ab$ to the elements of the upper left block. In the end, the matrix will become

$$\begin{pmatrix} I + z^2B & 0 \\ (1-t)F & I \end{pmatrix} \sim \begin{pmatrix} I + z^2B & 0 \\ 0 & I \end{pmatrix} \sim I + z^2B. \ \blacksquare$$

To achieve our goal, it suffices to prove one technical proposition.

LEMMA 3. *Let $p_1(x), \ldots, p_n(x) \in \mathbb{Z}[x]$ be a set of ordinary polynomials, and $I = \langle p_1(z^2), \ldots, p_n(z^2) \rangle$ be the corresponding ideal in $\mathbb{Z}[t, t^{-1}]$. Suppose that I contains the binomial $1 + t$. Then the ideal I is trivial: $I = \mathbb{Z}[t, t^{-1}]$.*

Proof. It is clear that $(t+1)(1+t^{-1}) = z^2 + 4 \in I$. Then division gives $p_k(z^2) = p_k^0(z^2)(z^2+4) + a_k$, where $a_k \in \mathbb{Z}$ are some integers. So our ideal coincides with $I = \langle z^2 + 4, a \rangle$, where $a = (a_1, \ldots, a_n)$ is the greatest common divisor of all a_i's. Expand the element $1 + t$ in the new generators:

$1+t=t^{-1}(t+1)^2q_1(t)+aq_2(t)$. Then $aq_2(t)$ is divisible by $1+t$. Therefore,

$$1=(t+1)t^{-1}q_1(t)+a\frac{q_2(t)}{t+1}\in I,$$

and the ideal is trivial. ∎

5. Main result. The last lemmas show that the Alexander ideals of bipartite knots cannot be arbitrary. In particular, they are always generated by polynomials in z^2.

THEOREM. *Let K be a bipartite knot. If the Alexander ideal $I_m(K)$ is nontrivial, then it cannot contain the polynomial $1+t$.*

Proof. This is a direct consequence of Lemmas 2 and 3. ∎

This condition immediately gives a series of knots which are not bipartite.

COROLLARY. *The Rolfsen table knots $9_{35}, 9_{37}, 9_{41}, 9_{46}, 9_{47}, 9_{48}, 9_{49}, 10_{74}, 10_{75}, 10_{103}, 10_{155}, 10_{157}$ are not bipartite.*

Proof. For the knot 9_{46}, also known as the pretzel knot with parameters $(3,3,-3)$, a detailed calculation of the second Alexander ideal is available from [Lik]. For the other knots from the given list, we borrowed the result from computer generated tables of the Knot Atlas [KnA]. ∎

6. From under the carpet. Contrary to the universal tradition, we allow ourselves to raise the carpet and explain how we actually arrived at this solution.

It was clear to us from the beginning that the rational knots are all bipartite. Then we designed a procedure to very quickly compute the Conway polynomial of a bipartite graph, starting from the corresponding signed intersection graph (see [Du]). Looking through the table of all knots with ≤ 8 crossings, we managed to find the bipartite graphs that would give the same Conway polynomials, and after one or two tries, using Knotscape [HT] and Knotinfo [Liv], we obtained a bipartite representation for the corresponding knots. This worked for all knots, save for 8_{18}. Now, this is the only knot up to 8 crossings with nontrivial second Alexander ideal. We looked at other knots with nontrivial second Alexander ideal and found some that cannot be expressed through the Conway variable z^2. On the other hand, we devised a procedure to represent the Alexander matrix of a bipartite graph in terms of z^2.

After this work was finished, the second author (M. Sh.) invented another argument showing that a knot with second Alexander ideal $\langle 3, t^2+1\rangle$, e.g. the table knot 10_{122}, cannot be bipartite. A separate publication is being prepared in this connection.

To summarize, we have presented a sufficient condition for a knot not to have any matched diagram. We do not know, however, of any reasonable necessary condition in terms of Alexander ideals. The simplest knot which still stands our efforts is 8_{18}.

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Sergei Duzhin
POMI RAN
Fontanka 27
191023, St. Petersburg, Russia
E-mail: duzhin@pdmi.ras.ru

Mikhail Shkolnikov
Université de Genève
Section de mathématiques
Villa Battelle 7, route de Drize
1227 Carouge, Suisse
E-mail: mikhail.shkolnikov@gmail.com

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